Comments on an approximation scheme for strong-coupling expansions. II

R. J. Rivers
Physics Department, Imperial College, London SW7 2AZ, England
(Received 25 April 1980)

We show that a recently proposed improved approximation scheme for deriving the continuum limit of strong-coupling lattice theories is as likely to be in error as its precursor.

Recently there has been a renewal of activity in strong-coupling lattice theories. As always, the main difficulty is that of performing the continuum limit of setting the lattice spacing a to zero. Characteristically, dynamical quantities are expressed in terms of series of the type

$$s(x)=\frac{x}{\sum\limits_{p=0}^{\infty} a_p x^p}, \quad a_0 \neq 0 \tag{1}$$

where x is the dimensionless quantity

$$x=g^{-1/2}a^{(d-4)/2} \tag{2}$$

for a theory with coupling constant g in d space-time dimensions.

The strong-coupling limit on the lattice corresponds to the $x \to 0$ limit of such functions $s(x)$. The coefficients a_p have a diagrammatic interpretation and, as in any perturbation expansion, only the first few a_p can be readily calculated.

On the other hand, in $d<4$ dimensions the continuum limit corresponds to taking $x\to\infty$ and the problem becomes that of inferring the $x\to\infty$ limit of $s(x)$ given the knowledge of only the first several a_p.

In Ref. 1 an approximation scheme was proposed in which the nth extrapolant for $s(x)$ was taken to be

$$s_n=\left\{\lim_{x\to\infty}\frac{x^n}{[(\sum\limits_{p=0}^{\infty} a_p x^p)^n]_n}\right\}^{1/n}, \tag{3}$$

where

$$\left(\sum_{p=0}^{\infty} A_p x^p\right)_n=\sum_{p=0}^{n} A_p x^p . \tag{4}$$

At *best*, this series of approximants s_n seems to behave empirically like the partial sums of an asymptotic series: The correct limit is approached, but before attaining it the s_n diverge away.

At *worst*, as we observed in a recent article[2] (henceforth known as I), it is very easy to find a series

$$f(x)=\sum_{p=0}^{\infty} a_p x^p , \tag{5}$$

for which the approximants s_p *converge* to the *incorrect* answer.

In particular, we showed in I that if $s(x)$ has an extremum at finite $x=x_s$ we should proceed with care. For example, we are necessarily in error if

$$|s(x_s)/s(\infty)|<1 \tag{6}$$

provided $s(\infty)$ exists, which we shall assume throughout this work.

Condition (6) is not a sufficient condition for obtaining incorrect results, or we would always be correct for $s(\infty)=0$, to which the case $f(x)=e^x$ provides a counterexample. If $s(\infty)=0$ it follows that the presence of any extremum of $s(x)$ guarantees an incorrect result if the only singularities of $s(x)$ are at infinite $|x|$.

We shall not attempt to tabulate the circumstances for which the s_n converge to the incorrect limit (see I for examples), but we shall assume that the cause is always that of the existence of a dominating saddle point at $x=x_s$. We merely observe that the fact that the above approximation scheme is often in error is not devastating since no approximation scheme always gives correct results.

A competing scheme of approximants to the s_n of (3) is that of Padé approximants. It was observed[3] that the class of functions $f(x)$ for which Padé approximants $P_n^{n+k}(x)$ (k fixed, $n=1,2,3,\ldots$) converge to the correct $x\to\infty$ limit is *enlarged* if $P_n^{n+k}(x)$ is evaluated at $x=x_n$, for some suitably chosen monotonic sequence $\{x_n\}$, rather than at infinite x. In a recent paper[4] the authors of Ref. 1 proposed a similar modification to the approximation scheme of Eq. (3). Instead of (3), they consider the sequence of approximants s'_n given by

$$s'_n=\left(\frac{(x_n)^n}{\{[\sum\limits_{p=0}^{\infty} a_p(x_n)^p]^n\}_n}\right)^{1/n}, \tag{7}$$

where the x_n are chosen to increase monotonically with n.[5]

There is no doubt that this modified approximation scheme is a great improvement over the original for those series $s(x)$ for which it is successful. For suitably chosen x_n the s'_n display monotonic convergence.

It is the purpose of this short note to show that, unlike the case of the Padé approximants mentioned previously, the modified approximants s'_n are no improvement over the original s_n if the s_n converge to the incorrect result due to the existence of a saddle point behaving as mentioned earlier. In such cases, s_n and s'_n will each converge to the same incorrect limit.

To see this, we write $\{[f(x)]^n\}_n$ as the contour integral (as in I)

$$\{[f(x)]^n\}_n = \frac{1}{2\pi i}\oint \frac{dy}{y} f(y)^n \sum_{p=0}^{n}\left(\frac{x}{y}\right)^p \tag{8}$$

$$= \frac{x^n}{2\pi i}\oint \frac{dy}{y}\left[\frac{f(y)}{y}\right]^n \left[\frac{1-(y/x)^{n+1}}{1-(y/x)}\right], \tag{9}$$

where the contour is taken counterclockwise around the origin within the circle of convergence of $f(x)$. Thus

$$(s'_n)^{-n} = \frac{\{[f(x_n)]^n\}_n}{(x_n)^n} = \frac{1}{2\pi i}\oint \frac{dy}{y} s(y)^{-n}\left[\frac{1-(y/x_n)^{n+1}}{1-(y/x_n)}\right] \tag{10}$$

which is compared to[1]

$$(s_n)^{-n} = \frac{1}{2\pi i}\oint \frac{dy}{y} s(y)^{-n}. \tag{11}$$

Let us assume that the s_n converge to the *incorrect* result because of the dominance of the contour integral (11) by a single saddle point at $y = x_s$. That is,

$$\lim_{n\to\infty} s_n = s(x_s) \neq s(\infty). \tag{12}$$

We wish to show that, as a consequence of the same saddle point, we also have

$$\lim_{n\to\infty} s'_n = s(x_s). \tag{13}$$

It is convenient to separate $(s'_n)^{-n}$ into

$$(s'_n)^{-n} = A_n + B_n, \tag{14}$$

where

$$A_n = \frac{1}{2\pi i}\oint \frac{dy}{y}\frac{s(y)^{-n}}{(1-y/x_n)} \tag{15}$$

and

$$B_n = \frac{1}{2\pi i (x_n)^{n+1}}\oint dy \frac{y^n s(y)^{-n}}{(1-y/x_n)} \tag{16}$$

$$= \frac{1}{2\pi i (x_n)^{n+1}}\oint dy \frac{f(y)^n}{(1-y/x_n)}. \tag{17}$$

First, consider B_n. Let us keep the contour of integration fixed and vary n. Once n is so large that x_n lies outside the contour we have

$$B_n = 0 \tag{18}$$

from (17).

Now consider A_n. The possibility exists that, because of the singularity at $y = x_n$, the saddle-point contribution cannot be isolated cleanly. Suppose first that a finite contour can be constructed with an arc beginning and ending on a zero of $f(x)$ and passing through $x = x_s$. We only have to take n large enough for x_n to lie outside the contour for there to be no pole contribution from $x = x_s$.

The only circumstance in which difficulty could arise is if the only zeros of $f(x)$ are at $x = \infty$ such that the contour cannot be deformed to pass through these zeros without having to contain the points $x = x_n$, for all n. However, $f(x)\to 0$ as $x\to\infty$ implies $s(x)\to\infty$ as $x\to\infty$. Since we have assumed $s(x)$ to be finite we do not have to consider this possibility.[6]

Thus, provided n is large enough that

$$|x_s/x_n| \ll 1, \tag{19}$$

the slowly varying term $(1-y/x_n)^{-1}$ in A_n has negligible effect on the contribution from the saddle point x_s. That is, for large enough n

$$(s'_n)^{-n} = A_n = (s_n)^{-n}[1 + O(1/n) + O(1/x_n)], \tag{20}$$

whence

$$\lim_{n\to\infty} s'_n = \lim_{n\to\infty} s_n \neq s(\infty), \tag{21}$$

however slowly the x_n increase with n.

To summarize, we have seen that the approximation scheme proposed in Ref. 4, while a considerable improvement if the correct result is obtained, does not seem to extend the class of functions $s(x)$ for which the correct limit will be obtained.

We conclude with the remark that, while making these criticisms, we are aware that the approximation scheme works well in many applications of physical interest.[1,4] If it happens that saddle points at finite real positive x_s are most important[7] our structures would be less compelling if, for example, it was known that $s(x)$ was monotonic (i.e., if quantities of interest depend monotonically on lattice size). It seems that something like this is happening for the quantities calculated so far. This suggests that some general properties of the series $s(x)$ should be derived, if at all possible, so as to indicate whether such an approximation scheme will continue its usefulness.

The author would like to thank Professor E. R. Caianiello and the University of Salerno Physics Institute, and Professor Abdus Salam and the International Centre for Theoretical Physics, Trieste, for hospitality during the formulation and writing of this work.

[1]C. M. Bender, F. Cooper, G. S. Guralnik, and D. H. Sharp, Phys. Rev. D 19, 1865 (1979).
[2]R. J. Rivers, Phys. Rev. D 20, 3425 (1979).
[3]A. Carroll, G. A. Baker, and J. L. Gammel, Nucl. Phys. B129, 361 (1977).
[4]C. M. Bender, F. Cooper, G. S. Guralnik, R. Roskies, and D. H. Sharp, Phys. Rev. Lett. 43, 537 (1979).
[5]In Ref. 4, the choice $x_n \sim n^{(4-d)/2}$ is made. As far as this work is concerned, any monotonically increasing sequence, unbounded above, is adequate.
[6]In I it was shown that it was possible for $\lim_{n\to\infty} s_n$ to be finite even though $s(x)$ diverged as an arbitrary power of x. It may be from the above argument that, if $s(x)$ diverges as $x \to \infty$, it is possible for the s_n' to diverge correctly, even if the s_n do not. We have not been able to find an example of this.
[7]We have not required, in I and this work, that the s_n or the s_n' be positive, as happens in all cases of physical interest, merely that they converge. If, for example, we have an extremum at real positive x, it seems that the s_n and s_n' will be real and positive for large enough n.