New gravitational instantons and their interactions

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We explore some of the possibilities for constructing new gravitational instantons. We find a new type, a rotating analog of the Taub-"bolt" situation, and we also rule out the possibility of various multi-instanton configurations.

I. INTRODUCTION

Recently, considerable advances in quantum field theory have been achieved by studying classical solutions of field equations and their geometric and topological properties. Solutions in real time which have finite energy and persist in time are loosely termed *solitons*. Solutions in imaginary time which are nonsingular and have finite action are loosely referred to as instantons. Instantons arise most naturally in the functional-integral approach to quantum field theory, in which the functional integral is Wick rotated and expressed as an integral over Euclidean field configurations. Instantons are used in the steepest-descent method of evaluating the functional integral.

These ideas are admirably suited to the study of gravity—a theory rich in geometric and topological structure—and form the basis of the path-integral approach.[1,2] The relevant classical field equations are the vacuum Einstein equations

$$R_{ab}=0\,. \tag{1.1}$$

As far as solitons are concerned, one obviously thinks of black holes and the time-independent Kerr solution. However, in addition to black holes, which may be thought of as electric-type gravitational monopoles (which may rotate), there exist solitons of a different sort which may be thought of as magnetic-type gravitational monopoles—the Taub-NUT (Newman-Unti-Tamburino) solutions.[3,4] Magnetic gravitational monopoles do not play a role in the classical theory because of their acausal behavior, and the necessity of Dirac-type singular strings emanating from them. These can only be eliminated at the expense of identifying the time coordinate; this means that in the Lorentzian regime they cannot be thought of as isolated systems, although a pole-antipole pair might be isolated, the string from one pole terminating on the other.

In the quantum theory it is possible that such pole-antipole pairs could spontaneously form in the vacuum, interact with matter, and then annihilate. This would be in addition to interactions with virtual black holes which should swallow particles and then emit them via the Hawking effect. Since these monopoles have a built-in twist in handedness, they may give rise to a number of interesting parity-violating effects.[5]

In the Euclidean regime one may define a gravitational instanton as a complete nonsingular solution of (1.1) with signature $(++++)$. Two sorts of instantons have been discovered: asymptotically locally flat[6,7] (ALF) and asymptotically locally Euclidean[8] (ALE). The ALE class of solutions are flat at infinity in the four-dimensional sense, except that identifications must be made so that near infinity they tend to E^4/Γ, where E^4 is flat Euclidean space, and Γ is a discrete subgroup of SO(4) with free action on S^3. If Γ is the identity, the space is asymptotically Euclidean. By the positive-action theorem,[9,10] the only such solution is flat space. If $\Gamma=Z_k$, we have the multi-instanton solutions,[8] $k=2$ being the Eguchi-Hanson solution. ALE solutions corresponding to more complicated groups have been constructed by Hitchin.[11]

ALF spaces are asymptotically flat in the three-dimensional sense, the fourth, imaginary-time, direction being periodic. Surfaces of large radii can thus be thought of as an S^1 bundle over S^2. The product bundle corresponds to the asymptotically flat (AF) solutions which include the Euclidean Schwarzchild and Euclidean Kerr solutions.[2,12] The twisted bundles correspond to the multi-Taub-NUT solution,[6] and the Taub-"bolt" solution discovered by Page.[13]

Physically, one expects the ALE solution to correspond to some sort of twisted vacuum state, and the ALF solution to the finite-temperature state with the same boundary conditions, the temperature being inversely proportional to the period.[2,12] If we concentrate on the Z_k sequence, we have the scheme

AE/ALE	AF/ALF
Flat space	Flat space–Schwarzschild–Kerr
Eguchi-Hanson	Taub-NUT–Taub-bolt–Kerr-Taub-bolt
Multi-instanton	Multi-Taub-NUT.

Thus, we think of the Taub-bolt solution as an excitation of the Taub-NUT solution, rather as Schwarzschild space is an excitation of flat space (with time identified, i.e., on $R^3 \times S^1$). In a heat bath at temperature $T = \beta^{-1}$ with a Taub-NUT boundary condition, the Taub-bolt solution might condense out of equilibrium just like the black-hole phase transition discussed in Ref. 12. The solution to the right of the Taub-bolt solution in this scheme is a new solution we have found, and it is described in Sec. III of this paper. It bears the same relation to the Taub-bolt solution as the Kerr solution does to the Schwarzschild solution. The spaces marked with question marks in the scheme are possible further generalizations. The results of this paper tend to suggest that such generalizations do not exist.

One can describe these admittedly speculative ideas in terms of the classification of gravitational instanton symmetries developed in Ref. 14. Since these solutions possess a (generally unique) Killing vector $k^a\partial/\partial x^a = \partial/\partial t$, that is, they are independent of imaginary time, we can describe the fixed-point sets of k. This will consist of points or "nuts" and two-surfaces or "bolts". The multi-Taub-NUT solution has k nuts. The Taub-bolt solution has the same boundary as the single Taub-NUT solution, but the single nut has become a bolt of nonvanishing self-intersection number. In the Kerr-Taub-bolt solution, this configuration is made to rotate. The question this paper addresses is whether the k nuts in the multi-Taub-NUT solution can be converted to bolts, or perhaps one large bolt, while maintaining the same twisted boundary conditions at infinity. Our results indicate that the answer is negative.

The plan of the paper is as follows. In Sec. II, we discuss the topology of solutions of Einstein's equations possessing a Killing vector. In Sec. III, we present our new Kerr-Taub-bolt solution. In Sec. IV, we rule out some multi-Schwarzschild solutions—i.e., solutions which are AF with more than one bolt. In Sec. V, we discuss the multi-Taub-NUT and multi-Taub-bolt solutions, and the forces between the nuts or bolts. This rules out the existence of all but the multi-Taub-Nut form. In Sec. VI, we discuss the possibility of accelerating solutions, and again we find none.

II. TOPOLOGY

We shall consider in this paper solutions (M, g_{ab}) of the Einstein equations which are either Lorentzian or Riemannian. Each of these solutions will be assumed to be invariant under a one-parameter isometry group G of "time translations". The action of G is given by μ_t: $M \to M$, where t is the group parameter which may be thought of as real or imaginary time, depending on whether one is Lorentzian or Riemannian. The infinitesimal generator of G is the Killing-vector field $k = k^a\partial/\partial x^a = \partial/\partial t$. If we remove from the manifold M, the fixed-point set C of μ_t, one obtains a fibration π: $M - C \to B$, where B is the space of nontrivial orbits of G. If $M - C$ is a trivial fiber bundle we can find a nonzero section which may be thought of as "space" and represent this as a regular spacelike surface t = constant. An example of this situation is the region exterior to the event horizon associated with a black hole—the domain of outer communication.[15] In general, however, the bundle will not be trivial and no such regular surface t = constant will exist. An example of this more generic situation occurs in the Lorentzian section of the Taub-NUT solution. The metric is

$$ds^2 = -\left(1 - \frac{2(Mr + l^2)}{r^2 + l^2}\right)(dt + 2l\cos\theta d\phi)^2 + dr^2\left(1 - \frac{2(Mr + l^2)}{r^2 + l^2}\right)^{-1} + (r^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.1)$$

As pointed out by Misner,[16] in order to remove the apparent singularities at $\theta = 0$ and $\theta = \pi$, t must be identified modulo $8\pi l$. Provided that $r^2 + l^2 \neq 2(Mr + l^2)$, $\partial/\partial t$ has no fixed points and the manifold M is, topologically, an S^1 bundle. The surfaces r = constant are then homeomorphic to S^3 and the fibration is the standard Hopf fibration. The Dirac stringlike singularities at $\theta = 0$ and $\theta = \pi$ simply reflect the impossibility of finding a regular section. Because of the obvious similarities with the Dirac magnetic monopole,[17] we can regard the Taub-NUT solution as the simplest case of a gravitational magnetic monopole, a point made also by Demianski and Newman[4] and by Dowker.[3]

The analogy may be taken further by introducing a $3+1$ split of the metric. We set

$$ds^2 = -V(dt + w_i dx^i)^2 + V^{-1}\gamma_{ij}dx^i dx^j \quad (2.2)$$

in the Lorentzian case, and

$$ds^2 = V(d\tau + \omega_i dx^i)^2 + V^{-1}\gamma_{ij}dx^i dx^j \quad (2.3)$$

in the Riemannian case. All quantities are independent of t or τ. The Wick rotation that transforms from one case to the other is

$$t \to i\tau\,, \tag{2.4}$$

$$w_i \to i\omega_i\,, \tag{2.5}$$

solutions of the Lorentzian Einstein equations being mapped into solutions of the Riemannian Einstein equations. V may be thought of as an electric-type potential, and ω_i (or w_i) as a magnetic-type vector potential. The associated magnetic field

$$H_{ij} = \partial_i \omega_j - \partial_j \omega_i \tag{2.6}$$

may be used to define a magnetic monopole moment called the "nut charge" N.[14] If $N \neq 0$, the fibration cannot be trivial. The fixed-point sets of μ_t form boundaries of B which act as sources for the magnetic field H_{ij}. In the Lorentzian case, these fixed-point sets are the two-dimensional Boyer bifurcation sets of event horizons.[15,18] In the Riemannian case, these fixed-point sets are of two types: points or nuts, and two-surfaces or bolts.[14] A nut possesses a pair of surface gravities κ_1 and κ_2. p and q are a pair of coprime integers such that $\kappa_1\kappa_2^{-1} = pq^{-1}$. If $\kappa_1\kappa_2^{-1}$ is irrational, $p = q = 1$. A nut of type (p, q) has a nut charge of

$$N = \frac{\beta}{8\pi p q}\,. \tag{2.7}$$

Further,

$$N = \frac{Y\beta}{8\pi} \tag{2.8}$$

for a bolt of self-intersection number Y. β is the period of the imaginary time coordinate. In the Riemannian case, the number of nuts and bolts is related to the Euler number χ and the Hirzebruch signature τ of the manifold M by[14]

$$\chi = \sum_{\text{bolts}} \chi_i + \sum_{\text{nuts}} 1\,, \tag{2.9}$$

where χ_i is the Euler number for the ith bolt, and

$$\tau = \sum_{\text{bolts}} Y_i \csc^2\theta - \sum_{\text{nuts}} \cot p_i\theta \cot q_i\theta + \eta(0,\theta)\,. \tag{2.10}$$

Equation (2.10) is valid for arbitrary θ. Y_i is the self-intersection number of the ith bolt, the ith nut is of type (p_i, q_i). We assume that $\partial/\partial t$ is either tangential or transverse to the boundary of the manifold M for Eq. (2.9) to be valid. In (2.10), we assume $\partial/\partial t$ is parallel to the boundary and $\eta(0,\theta)$ is a correction term which depends solely on the boundary. For both ALE and ALF boundary conditions, for which ∂M is the cyclic lens space S^3/Z_k, Z_k acting on S^3 as right translations, $\eta(0,\theta)$ may be evaluated by reference to the special self-dual case for which (2.10) must hold. Since the multi-Taub-NUT and multi-instanton solutions have k nuts of type $(1,1)$ and $\tau = k-1$, it follows that

$$\tau = \sum_{\text{bolts}} Y_i \csc^2\theta - \sum_{\text{nuts}} \cot p_i\theta \cot q_i\theta + k\csc^2\theta - 1 \tag{2.11}$$

for the above boundary conditions. Note that for AF boundary conditions, $\eta(0,\theta) = 0$. There are also formulas giving the Euler number and Hirzebruch signature as integrals of the curvature. For ALE and ALF boundary conditions,[7] these are

$$\chi = \frac{1}{32\pi^2}\int_M (R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2) g^{1/2} d^4x + \frac{1-\epsilon}{k} \tag{2.12}$$

and

$$\tau = \frac{1}{96\pi^2}\int_M R_{abcd}\epsilon^{cdef}R^{ab}{}_{ef}\, g^{1/2} d^4x - \frac{2\epsilon}{3k} + \frac{(k-1)(k-2)}{3k}\,, \tag{2.13}$$

where $\epsilon = 0$ for ALE boundary conditions and $\epsilon = 1$ for ALF boundary conditions. We are concerned with solutions of the field equations $R_{ab} = 0$, so we have the Hitchin-type inequality[9,19]

$$2\left(\chi - \frac{1-\epsilon}{k}\right) \geq 3\left|\tau + \frac{2\epsilon}{3k} - \frac{(k-1)(k-2)}{3k}\right|\,, \tag{2.14}$$

Equality in (2.14) is obtained in the case of half-flat solutions. Hitchin's inequality implies that one cannot find a small perturbation of a solution to give a nearby solution with a different topology. For instance, in the self-dual case there are no nearby solutions in which a nut is changed into a bolt. Furthermore, Eqs. (2.9), (2.11), and (2.14) show that some types of solutions cannot exist. For instance, one might have thought that Page's Taub-bolt solution which has $\chi = 2$ and $\tau = -1$, a single spherical bolt of self-intersection number -1, and an ALF boundary, could be generalized to a solution with a single spherical bolt of self-intersection number $-k$ with the same boundary. Such a solution would have, by (2.9) and (2.11), a $\chi = 2$ and $\tau = -1$. For ALE boundary conditions, compatibility with (2.14) requires $k = 2$ and the metric being self-dual (or anti-self-dual). This is the Eguchi-Hanson metric described using the Killing vector corresponding to rotations about the line of centers in the Gibbons-Hawking form of the metric.[8] For ALF boundary conditions we require $k \leq 4$, with equality in the half-flat case. $k = 1$ corresponds to the Page Taub-bolt solution. Presumably, if it exists, $k = 4$ could be obtained from the general self-dual ALF form. However, we have not been able to do so. Nor have we been able to construct examples with $k=2$ or 3 which, of course, could not be self-dual. The important point is that there can be no extra series of solutions of this type indexed by $k>4$. These general arguments, however, cannot rule out the possibility of k bolts, each with self-intersection number -1. This would be a multi-Taub-bolt family. In fact, we shall see in Sec. V by a detailed examination of the field equations that such a family does not exist. Similarly, the general topological arguments do not rule out the possibility of a multi-Schwarzschild solution. This would have l spherical bolts with $Y=0$, hence $\chi=2l$, $\tau=0$. The boundary terms in (2.11)–(2.13) are absent, and the Hitchin inequality is just $2\chi \geq 3|\tau|$ which is obviously satisfied. In Sec. IV we show that no multi-Schwarzschild solution can exist.

III. THE KERR-TAUB-BOLT INSTANTION

In this section we present a new gravitational instanton which generalizes the Taub-bolt instanton found by Page[13] in the same way that the Kerr solution generalizes the Schwarzschild solution. Consider the metric

$$ds^2 = \Xi\left(\frac{dr^2}{\Delta}+d\theta^2\right)+\frac{\sin^2\theta}{\Xi}(\alpha dt+P_r d\phi)^2 + \frac{\Delta}{\Xi}(dt+P_\theta d\phi)^2\,, \tag{3.1}$$

where

$$\Delta = r^2 - 2Mr + N^2 - \alpha^2\,, \tag{3.2}$$

$$P_\theta = -\alpha\sin^2\theta + 2N\cos\theta - \frac{\alpha N^2}{N^2-\alpha^2}\,, \tag{3.3}$$

$$P_r = r^2 - \alpha^2 - \frac{N^4}{N^2-\alpha^2}\,, \tag{3.4}$$

$$\Xi = P_r - \alpha P_\theta = r^2 - (\alpha\cos\theta + N)^2\,, \tag{3.5}$$

where α, N, M are constants.

This Ricci-flat metric is obtained from Carter's[20] ten-parameter family of solutions to the Einstein-Maxwell equations by a simple change of variables and relabeling of parameters.

If $\alpha=0$, we obtain the general Riemannian Taub-NUT metric. If $N=0$, we obtain the Reimannian Kerr metric. In order to remove the Dirac string-like singularities at $\theta=0$ and $\theta=\pi$, t must be identified modulo $8\pi N$, and ϕ identified modulo 2π. In order to remove the apparent singularity at values of r for which $\Delta=0$, we must identify (t,ϕ,θ,r) and $(t+2\pi/\kappa,\ \phi-2\pi\Omega/\kappa,\ \theta,\ r)$ where

$$\kappa = \frac{r_+ - r_-}{2{r_0}^2}\,, \tag{3.6}$$

$$\Omega = \frac{\alpha}{{r_0}^2}\,, \tag{3.7}$$

$$r_\pm = M \pm (M^2 - N^2 + \alpha^2)^{1/2}\,, \tag{3.8}$$

$${r_0}^2 = {r_+}^2 - \alpha^2 - \frac{N^4}{N^2-\alpha^2}\,. \tag{3.9}$$

This leads to the conditions

$$\kappa = \frac{1}{4|N|} \tag{3.10}$$

and

$$\Xi \geq 0\ \forall\ r > r_+,\quad 0 \leq \theta \leq \pi\,, \tag{3.11}$$

which are necessary and sufficient for $r=r_+$ to be a regular bolt in a nonsingular instanton on the manifold $CP^2-\{0\}$. This manifold is also diffeomorphic to the line bundle over S^2, with Chern number $c_1=-1$, i.e., the spin bundle of S^2. Condition (3.11) is equivalent to

$$M > |N|\,. \tag{3.12}$$

If $\alpha=0$, these are satisfied for $M=\frac{5}{4}|N|$, and the resulting solution is called Taub-bolt. If $|N|>0$, the Hirzebruch signature equals -1, and the Euler number equals 2. $r=r_+$ is a two-sphere of area $A=4\pi {r_+}^2$ with self-intersection number -1.

If $\alpha\neq 0$, the solution of (3.10) is quite complicated. Removing surds in (3.10), one obtains a cubic equation in M.

$$(-20N^8+52N^6\alpha^2-49N^4\alpha^4+16N^2\alpha^6) + (16N^7-48N^5\alpha^2+48N^3\alpha^4-16N\alpha^6)M + (20N^6-32N^4\alpha^2+8N^2\alpha^4+4\alpha^6)M^2 + (-16N^5+32N^3\alpha^2-16N\alpha^4)M^3 = 0\,. \tag{3.13}$$

For each α (normalizing $N=1$), (3.13) has at most three roots. These may not, in fact, be roots of (3.10) and a separate check must be made. In Fig. (1) we plot the results of a numerical determination of M against α. The dotted lines show the solutions of (3.13). The heavy lines I and II are the two possible disjoint families of regular instanton solutions satisfying (3.10) and (3.12). The action for both branches is $I=4\pi|N|M$.

Branch I starts at $\alpha=0$, $M=\frac{5}{4}$ with Page's Taub-bolt solution, and continues to the point a, $\alpha=0.693$, $M=1.147$. Note that the action increases at first, but starts to decrease at $\alpha=0.552$, $M=1.333$ and becomes less than that of the Page solution at $\alpha=0.674$.

Branch II starts at the point b, $\alpha=(1+\sqrt{17})/4$ and $M=1$, and tends asymptotically to $M=2$ as $\alpha\to\infty$. As α gets larger and larger, the metric

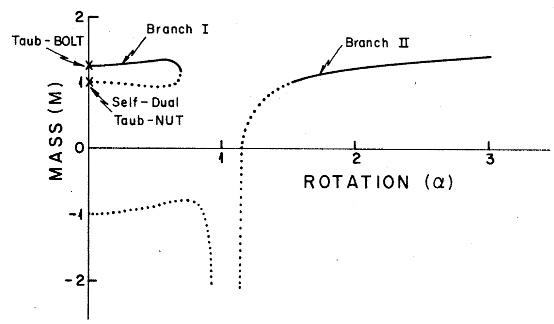

FIG. 1. A graph of the permitted values of M against α for $|N|=1$ in the Kerr-Taub-bolt solution. The solid line is the permitted region. The dotted line corresponds to configurations that satisfy (3.13), but either (3.10) or (3.12) is not satisfied. × marks the self-dual Taub-NUT solution.

tends to the same metric as the Kerr form for $\alpha^2 \gg M^2$. In the limit $\alpha \to \infty$, both metrics become locally flat.

The only other instanton which appears in Fig. 1 corresponds to the self-dual Taub-NUT solution. This instanton has a nut rather than a bolt at the origin, so it of course does not satisfy the conditions (3.10) and (3.12). Note that the bolt solutions all have larger action than the self-dual Taub-NUT instanton.

IV. MULTI-SCHWARZSCHILD CONFIGURATIONS

In this section we shall consider a metric invariant under a time translation isometry group where nontrivial orbits are orthogonal to a family of spacelike hypersurfaces. Thus, all the magnetic-type fields and nut charges vanish.

The space of orbits B will have as boundary a number l of Boyer bifurcation fixed-point sets each having mass M_n, surface gravity κ_n, and area A_n related by a Smarr-type formula[14]

$$M_n = \frac{\kappa_n A_n}{4\pi}. \tag{4.1}$$

If the metric is asymptotically flat (AF) the total mass M will then be

$$M = \sum_{i=1}^{l} M_i . \tag{4.2}$$

If the metric is Riemannian, one must identify the imaginary time coordinate modulo $2\pi/\kappa_n$ to complete the metric. This requires all κ_n to be equal. This is the geometrical analog of the zeroth law of thermodynamics. Because of the attractive nature of gravity the following seems plausible.

Conjecture. The only asymptotically flat static Ricci-flat instanton or soliton is the Schwarzschild solution. If we assume only one black hole, we can appeal to the Israel theorem.[21] The proofs in Refs. 21 and 22 will only work for the case with a single horizon. The arguments in Ref. 23 may be extended to the case of many black holes yielding the inequalities

$$\sum_{i=1}^{l} \kappa_i \geq 4 \sum_{i=1}^{l} \kappa_i{}^2 M_i , \tag{4.3}$$

and

$$\left(\sum_{i=1}^{l} M_i\right)\left(\sum_{i=1}^{l} \kappa_i{}^2 M_i\right) \geq \tfrac{1}{16}. \tag{4.4}$$

Equality is obtained only for the Schwarzschild case. If $l=1$, they yield the result $4\kappa M \geq 1$ and $4\kappa M \leq 1$ which proves the result. However, if $l>1$, it is easily seen that (4.3) and (4.4) are too weak to establish the result.

An alternative approach is to consider spaces which are in addition axisymmetric, that is, invariant under a further group SO(2) generated by a Killing vector $\partial/\partial\phi$, together with the discrete operation of reversing the sense of rotation. For such spaces, the variables in (2.2) take the form[24]

$$\gamma_{ij} dx^i dx^j = e^{2k}(d\rho^2 + dz^2) + \rho^2 d\phi^2 , \tag{4.5}$$

$$w_i dx^i = w d\phi . \tag{4.6}$$

The field equations now become

$$k_\rho = \frac{\rho}{4V^2}(V_\rho{}^2 - V_z{}^2) - \frac{V^2}{4\rho}(w_\rho{}^2 - w_z{}^2)$$

$$= \frac{\rho}{4V^2}(V_\rho{}^2 - V_z{}^2 + \Omega_\rho{}^2 - \Omega_z{}^2) , \tag{4.7}$$

$$k_z = \frac{\rho}{2V^2} V_\rho V_z - \frac{V^2}{2\rho} w_\rho w_z$$

$$= \frac{\rho}{2V^2}(V_\rho V_z + \Omega_\rho \Omega_z) , \tag{4.8}$$

and

$$V\nabla^2 V = \nabla V \cdot \nabla V - \nabla\Omega \cdot \nabla\Omega , \tag{4.9}$$

$$V\nabla^2\Omega = 2\nabla V \cdot \nabla\Omega , \tag{4.10}$$

where all functions depend solely on ρ and z, and the subscripts represent partial differentiation with respect to that variable. The gradient, Laplacian, and scalar product operations in (4.9) and (4.10) are flat-space Euclidean in cylindrical polar coordinates ρ, z, ϕ. Ω is the twist potential defined by

$$\rho\Omega_z = -V^2 w\rho \tag{4.11}$$

and

$$\rho\Omega_\rho = V^2 w_z . \tag{4.12}$$

For Riemannian metrics we must set

$$w = i\omega\,, \tag{4.13}$$

$$\Omega = i\psi\,. \tag{4.14}$$

Equations (4.9) and (4.10) guarantee the integrability of (4.7) and (4.8). In the static case $w = 0$, and $\ln V$ is an axisymmetric harmonic function. In order to generate the Schwarzschild solution, one chooses $U = \frac{1}{2}\ln V$ to be the Newtonian potential for a rod of length $2M$ and mass M.[25] For l black holes lying on a common axis, one must choose U to be the sum of the potentials due to l rods, each of length $2M_n$, mass M_n centered on the points $z = z_n$. This choice is unique given the requirement that there are no other material sources, that the metric is asymptotically flat, and that we have regular horizons. Explicitly, U and k are then given by

$$U(\rho, z) = \sum_{i=1}^{l} \frac{1}{2} \ln \frac{r'_n + r''_n - 2M_n}{r'_n + r''_n + 2M_n} \tag{4.15}$$

and

$$k(\rho, z) = \frac{1}{4} \sum_{n=1}^{l} \sum_{m=1}^{l} \ln \frac{[r'_n r''_m + (z - z_n - M_n)(z - z_m + M_m) + \rho^2][r''_n r'_m + (z - z_n + M_n)(z - z_m - M_m) + \rho^2]}{[r'_n r'_m + (z - z_n - M_n)(z - z_m - M_m) + \rho^2][r''_n r''_m + (z - z_n + M_n)(z - z_m + M_m) + \rho^2]}\,. \tag{4.16}$$

Here, the r'_n's and r''_n's are the Euclidean distances from the ends of the rods to the point ρ, z,

$$r'^2_n = \rho^2 + (z - z_n - M_n)^2\,, \tag{4.17}$$

$$r''^2_n = \rho^2 + (z - z_n + M_n)^2\,. \tag{4.18}$$

At the rods the metric becomes singular. However, the metric can be extended through the apparent singularity, which is then seen to be a regular event horizon (or bolt in the Riemannian regime) with surface gravity $\kappa_n = (4M_n)^{-1}$. Superficially it appears that we have a metric that represents l black holes in equilibrium. However, this configuration is singular, as can be seen by an argument which goes back to Einstein and Rosen.[26] In order that we have a regular z axis (which is the bolt of the Killing vector $\partial/\partial\phi$), ϕ must be identified modulo $2\pi\kappa_\phi^{-1}$, where κ_ϕ is the "surface gravity" of this axial Killing vector. It is easily seen that κ_ϕ can be related to k on the z axis:

$$\kappa_\phi = \exp(-k)\big|_{\rho=0}\,, \tag{4.19}$$

where k is constant on those parts of the z axis not occupied by a rod and is zero for large $|z|$. It, however, changes linearly along the rods by a total amount of $2F_n$, where F_n is the total Newtonian force on the nth rod due to the remaining $(n-1)$ rods. Evidently, it is thus impossible to identify ϕ modulo 2π everywhere and hence obtain a nonsingular metric. The resulting metric would possess conical singularities on the unoccupied sections of the axis. This rules out an axisymmetric multi-Schwarzschild solution, as was pointed out by Gibbons[27] and by Müller zum Hagen and Seifert.[28] The same argument is immediately applicable to the Riemannian regime.

In the Riemannian case these configurations, although singular, have finite action. The Ricci scalar has a Dirac δ-function behavior. An easy computation shows that the Euclidean action due to a two-dimensional surface of area A with a conical singularity of deficit angle δ, that is, where one identifies the polar angle with period $2\pi-\delta$ rather than 2π, is given by the formula

$$I = -A\delta/8\pi\,. \tag{4.20}$$

This is the same formula that one obtains in Regge calculus.[29] In our particular case,

$$\delta = 2\pi(1 - e^{-k}) \tag{4.21}$$

and

$$A = \Delta z \beta e^{k}\,, \tag{4.22}$$

where β is the period of identification in imaginary (Riemannian) time, Δz is the length of the appropriate sector of the z axis, and k is evaluated in that sector. The total contribution to the action is the sum of (4.20) over the $(l-1)$ interstices in the chain of rods.

For two black holes, each of mass M, $\beta = 8\pi M$. On the axis, k is given by

$$\ln \frac{(\Delta z + 2M)^2}{\Delta z(\Delta z + 4M)} \tag{4.23}$$

between the black holes. $k = 0$ elsewhere on the axis where there is no rod.

Thus, the total action is

$$I = \tfrac{1}{2}M\beta + \tfrac{1}{2}M\beta - \frac{\beta M^2}{\Delta z + 4M}\,. \tag{4.24}$$

The first two terms are the actions of the individual black holes. The third term in (4.24) arises from the canonical singularity on the axis between the two black holes. At large distances it is just what one expects for the contribution to the free energy from the gravitational attraction. As $\Delta z \to 0$ it remains bounded, in contrast to the potential energy of two point masses in the Newtonian theory.

V. MULTI-TAUB-BOLT AND MULTI-TAUB-NUT

Given a stationary solution of the vacuum Einstein equations, there is a one-parameter group of transformations, the Ehlers group,[30,31] which generates a family of solutions to the Einstein equations. If the metric has the form (2.3), we set

$$V \to \tilde{V} = V/[(1-b\psi)^2 - b^2V^2], \tag{5.1}$$

$$\psi \to \tilde{\psi} = [\psi + b(V^2 - \psi^2)]/[(1-b\psi)^2 - b^2V^2], \tag{5.2}$$

$$\gamma_{ij} \to \tilde{\gamma}_{ij} = \gamma_{ij}. \tag{5.3}$$

The half-flat metrics, for which $V^2=\psi^2$, are fixed points under the action of this group and so cannot be used to generate new solutions. If we begin with a static solution with only electric-type mass, we will obtain a stationary solution with both electric- and magnetic-type masses. If we commence with the Schwarzschild solution, we obtain in this way the general Taub-NUT metrics. The standard Schwarzschild radial coordinate r and angular coordinate θ are related to the coordinates of Sec. IV by

$$r' + r' = \lambda = r - M, \tag{5.4}$$

$$r' - r'' = M\cos\theta. \tag{5.5}$$

The Ehlers transformation now takes the Schwarzschild metric to the Taub-NUT form if we set

$$\tilde{r} = (1-b^2)^{1/2}\left(\lambda - \frac{M}{(1-b^2)^{1/2}}\right), \tag{5.6}$$

$$\tilde{t} = t(1-b^2)^{-1/2}, \tag{5.7}$$

$$\tilde{M} = M(1-b^2)(1-b^2)^{-1/2}, \tag{5.8}$$

$$\tilde{N} = 2bM(1-b^2)^{-1/2}. \tag{5.9}$$

This solution will, in general, be singular except when $|b| = \frac{1}{2}$, which is Page's Taub-Bolt solution, and $|b| = 1$ which is Hawking's self-dual Taub-NUT solution.

It is straightforward to make an Ehlers transformation on the multi-Schwarzschild solution to obtain the multi-Taub-bolt solutions. The $\omega d\phi$ is then the magnetic vector potential due to l rods, each with uniform magnetic monopole charge $2bM_n(1-b^2)^{-1/2}$. Thus,

$$\tilde{\omega} d\phi = \sum_{n=1}^{l} 4bM_n\cos\theta_n, \tag{5.10}$$

where

$$r_n' - r_n'' = M_n\cos\theta_n. \tag{5.11}$$

In order for $\tilde{V} \to 1$ at infinity, we must rescale the time coordinate and spatial coordinates

$$t \to \tilde{t} = t(1-b^2)^{-1/2}, \tag{5.12}$$

$$\rho \to \tilde{\rho} = \rho(1-b)^{1/2}, \tag{5.13}$$

$$z \to \tilde{z} = z(1-b^2)^{1/2}. \tag{5.14}$$

The new surface gravity of each bolt is given by

$$\tilde{\kappa}_n = \kappa_n = 1/4M_n. \tag{5.15}$$

The metric will now have Dirac string-type singularities along the axis between the rods and singularities on the rods, each of which now has length $M(1-b^2)^{1/2}$. Since $k \neq 0$ between the rods, it is not possible to make the necessary identifications to remove all of these singularities simultaneously, unless we take the limit $b^2 \to 1$. In this limit the rods tend to zero size, and so the difference in Newtonian potential between the ends vanishes. One can see this more physycally by noting that (5.8) and (5.9) imply

$$\tilde{M}_n\tilde{M}_m - \tilde{N}_n\tilde{N}_m = M_nM_m(1-b^2). \tag{5.16}$$

The Newtonian forces, upon which κ depends, are given by terms of the form

$$M_nM_m/(\text{distance})^2, \tag{5.17}$$

using (5.13), (5.14), and (5.16). Equation (5.17) becomes

$$\tilde{M}_n\tilde{M}_m - \tilde{N}_n\tilde{N}_m/(\widetilde{\text{distance}})^2. \tag{5.18}$$

This clearly vanishes only in the self-dual limit. That is, in the Euclidean regime the gravitational attraction is balanced by the nut-nut repulsion. In the Lorentzian regime, the nut-nut force is attractive and such a balance is not possible. If all the M_n's are equal, the resulting metric is a special case of that given by Hawking.[6] Similar results can be deduced by evaluating the contribution to the action coming from the conical singularities between the rods.

Since a pure electric and pure magnetic charge can remain in neutral equilibrium at any distance, it is tempting to speculate that a similar black-hole-pure-nut solution should also exist. We have not yet found one.

VI. ACCELERATION AND ROTATION

Carter's general Schrödinger separable family of solutions of the vacuum Einstein equations are included in the general Petrov-type-D metrics discovered by Kinnersley,[32] the C-NUT metrics. These have an additional parameter, usually referred to as an acceleration parameter. The simplest example is the C metric, discussed at length by Kinnersley and Walker.[33] The metric may be written as

$$ds^2 = (1+Ar\mu)^{-2}\left[-dt^2\left(1 - A^2r^2 - \frac{2M}{r}\right) + \frac{dr^2}{1-A^2r^2-2M/r} + r^2\left((1-\mu^2-2MA\mu^3)d\phi^2 + \frac{d\mu^2}{1-\mu^2-2MA\mu^3}\right)\right]. \tag{6.1}$$

If $A=0$, we obtain the Schwarzschild solution. If $M=0$, we obtain flat space in accelerating coordinates (Rindler space). If $M=0$, there is an acceleration horizon at $r=1/A$. If $M\neq 0$, there will in general be both an acceleration horizon and a black-hole horizon provided $27M^2A^2<1$. Kinnersley and Walker pointed out that the metric may be thought of as representing a uniformly accelerated black hole. It can be obtained as a special case of the double Schwarzschild solution described in Sec. IV, in which one of the rods is taken to be of infinite length and a suitable rescaling is carried out, as described by Israel and Khan.[25] The limiting form of the potential function giving the acceleration horizon is

$$V\sim(\rho^2+z^2)^{1/2}+z\,. \qquad (6.2)$$

The resulting metric will be the same as that described in Ref. 25. We do not have the explicit coordinate transformations linking these two forms, but since the Weyl metrics are uniquely determined by the boundary conditions at infinity and at the horizons, it is clear that they are indeed the same. Furthermore, it is apparent from the arguments presented in Refs. 25 and 26, and in Sec. IV, that these metrics must have stringlike singularities on the axial rotation axis. This is just what Kinnersley and Walker found. If μ, which is analogous to the coordinate $\cos\theta$ on the two-sphere and in fact coincides with it if $mA=0$, ranges between the two roots of the coefficient of $d\phi^2$, the identifications of ϕ necessary to complete the metric are different at the two roots (unless $mA=0$). This leads to what Kinnersley and Walker term a "nodal singularity." One might try to obtain a regular solution by allowing μ to range over an unbounded interval. However, since $R_{abcd}R^{abcd}$ is proportional to $r^{-6}(1+\mu rA)^6$, this is bound to fail. In the Riemannian case, not only must ϕ be correctly identified, but so must t. In fact, the condition that the two t periodicities are equal is the same condition that the ϕ periodicities are equal. Thus, one cannot obtain an "accelerating" instanton from the metric (6.1). The question of whether it is possible to find one from the most general class of accelerating solutions to Einstein's equations will be examined elsewhere.

There is a remaining question as to whether it is possible to have multi-instanton or multi-black-hole solutions with rotation, that is, multi-Kerr configurations. This is still an open question which will also be discussed in a future paper.

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