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Grand unification groups and charges of quarks and leptons

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The ansatz which demands that there exists a four-charge (2/3, − 1/3, 0, − 1) multiplet yields a strong restriction on the search for a grand unification group, $G = G_1 \otimes G_2 \otimes \cdots \otimes G_n \otimes U(1)$. If the group is simple or semisimple, we have only two possibilities, SU(n) or SU(n) ⊗ SU(m).

Recently there have been a considerable number of attempts to unify electromagnetic, weak, and strong interactions under a single gauge group G, stimulated by the success of the Weinberg-Salam $SU(2)_L \otimes U(1)$ model.[1] The search for the right group encompasses simple groups and semisimple groups: SU(n) ($n = 5, 6, 7, 8, 9, 10, 11, 16$),[2] SO($4n+2$) ($n = 2, 3, 4, 5$),[3] E_n ($n = 6, 7, 8$),[4] SU(n) ⊗ SU(n) ($n = 4, 5, 6, 8, 12, 16$),[5] SU(n) ⊗ SU(n) ⊗ SU(n) ⊗ SU(n).[6] Of course, G has to be a compact group, otherwise nontrivial unitary representations are infinite dimensional.[7] But, the fact that G is compact only tells us that G is in the form of $G_1 \otimes G_2 \otimes \cdots \otimes G_n \otimes U(1) \otimes U(1) \otimes \cdots \otimes U(1)$ (Ref. 8), where G_j are simple compact groups. Where do we go from here? So far, people have worked on simple or semisimple groups as mentioned. However, we have to remember the fact that the unifying group for the electroweak theory is $SU(2)_L \otimes U(1)$.[1] It is not, for example, SU(3), whose predictions contradict neutral-current data. So we should keep in mind that there is always a possibility of having U(1).

Even working in simple groups, we still have many from which to choose. Gell-Mann, Ramond, and Slansky[9] require the color restriction which demands that representations should contain only color singlet and triplet under the decomposition of G into $G^{\text{flavor}} \otimes SU(3)^{\text{color}}$. They have shown which representations should be used for a particular simple group, but they could only exclude G_2 and E_8. The rest of the simple groups can have representations which satisfy the color restriction. Hacinliyan and Saclioglu[10] have put on more restrictions in addition to the color restriction: one being lepton-quark universality and the other being that the number of quark flavors is less than 16. They have not excluded semisimple groups, but they could not find any which satisfy their conditions. The results are similar to Gell-Mann *et al.*,[9] though their method differs from that of the latter authors. Natural cancellation of the triangle anomaly will not tell us much either: the only simple gauge group with a possible anomaly is SU(n) ($n \geq 3$).[11] The rest of the simple groups are safe.

Georgi and Glashow[11] demand that representations should be complex. Then, we have only few groups available: E_6, SU(n) ($n \geq 3$), and SO($4n+2$) ($n \geq 2$). Okubo[12] argues that the nonexactness of the Okubo-Zweig-Iizuka rule[13] implies that G has to be one of E_6, SU(n) ($n \geq 3$), or SO($4n+2$) ($n \geq 2$), which are the same as the Georgi and Glashow groups.

In this paper, we assume the simplest principle in selecting the unifying group: *There should exist a four-charge multiplet*, i.e., up-type ($Q_u = \frac{2}{3}$), down-type ($Q_d = -\frac{1}{3}$), electron-type ($Q_e = -1$), and neutrino-type ($Q_\nu = 0$) in the same multiplet. We do not put any restrictions on the number of each type charge. (Since we are trying to unify electromagnetic, weak, and strong interactions, it is natural to assume that four fundamental constituents should be in the same multiplet.) Note that in SU(5) (Ref. 2), the $\underline{5}$ contains only three charges ($\overline{Q}_d$, Q_e, Q_ν), while the $\underline{10}$ does have four distinct charges but its content is (Q_u, Q_d, $\overline{Q}_u$, $\overline{Q}_e$). (They are really strange multiplets.) In SO (10) (Ref. 3), it becomes more natural, since the $\underline{16}$ contains (Q_u, Q_d, Q_e, Q_ν, $\overline{Q}_u$, $\overline{Q}_d$, $\overline{Q}_e$, $\overline{Q}_\nu$). As can be guessed, the bar over Q_j implies the charge of an antiparticle. From these examples, we might guess that our four-charge principle leads to a different group, other than SU(5) or SO(10), but we have no reason to exclude our possibility. One may first think that the four-charge ansatz is too weak to choose a special group, but it is, in fact, strong.

We will discuss the cases where $G = G_1 \otimes G_2 \otimes \cdots \otimes G_n \otimes U(1)$, since the U(1) factor only adds some constant to the charge operator and thus it is enough to consider only one U(1) factor. First we prove the following lemmas:

Lemma 1. The existence of a four-charge multiplet requires that G is one of the three possibilities: $G = G_1 \otimes U(1)$, $G = G_1 \otimes G_2 \otimes U(1)$, or $G = G_1 \otimes G_2 \otimes G_3 \otimes U(1)$. [The U(1) factor may not be present, depending upon G_j.]

Proof. Note that the charge operator Q can be expressed in terms of Cartan subalgebra elements $h_j^{(i)}$ of each group G_i:

$$Q = \sum_{i,j} a_j^{(i)} h_j^{(i)} + c = \sum_i Q_i + c, \qquad (1)$$

where $Q_i = \Sigma_j a_j^{(i)} h_j^{(i)}$ and c is some numerical constant, coming from the U(1) factor. Since $\mathrm{Tr}\, h_j^{(i)} = 0$ (Ref. 14) and thus $\mathrm{Tr}\, Q_j = 0$, each Q_j must have at least two eigenvalues. Let us assume that the group G is $G_1 \otimes G_2 \otimes G_3 \otimes G_4 \otimes \mathrm{U}(1)$. Then, the least number of eigenvalues of the charge operator Q can be obtained for the case where each Q_j ($j = 1, 2, 3, 4$) has two eigenvalues. We denote these eigenvalues as $b_i^{(\pm)}$ ($i = 1, 2, 3, 4$), where b_i^+ (b_i^-) is positive (negative). Then, we obtain at least five different eigenvalues:

$$(b_1^+ + b_2^+ + b_3^+ + b_4^+) > (b_1^+ + b_2^+ + b_3^+ + b_4^-) > (b_1^+ + b_2^+ + b_3^- + b_4^-) > (b_1^+ + b_2^- + b_3^- + b_4^-) > (b_1^- + b_2^- + b_3^- + b_4^-).$$

Q.E.D.

Lemma 2. It is necessary to have a U(1) for $G = G_1 \otimes G_2 \otimes G_3 \otimes \mathrm{U}(1)$. None of the G_j's are exceptional groups (G_2, E_6, E_7, E_8, or F_4). The charge structure of a multiplet is $(1, \frac{2}{3}, \frac{1}{3}, 0)$.

Lemma 3. For $G = G_1 \otimes G_2 \otimes \mathrm{U}(1)$, only one of the G_j ($j = 1, 2$) can be an exceptional group. If G_1 is an exceptional group, then the U(1) is necessary, and the charge structure of a multiplet is $(1, \frac{2}{3}, \frac{1}{3}, 0)$. If G_j ($j = 1, 2$) are not exceptional groups, we have three possible charge structures: $(\frac{2}{3}, 0, -\frac{1}{3}, -1)$, $(1, \frac{2}{3}, 0, -\frac{1}{3})$, or $(1, \frac{2}{3}, \frac{1}{3}, 0)$.

For the proof of *Lemma* 2, we note that for $G = G_1 \otimes G_2 \otimes G_3 \otimes \mathrm{U}(1)$, each Q_j has only two eigenvalues, otherwise we have five or more eigenvalues for $Q = \Sigma\, Q_j + c$. Okubo's theorem,[8] which states that none of the exceptional groups can have only two eigenvalues for Q_j, now yields the result that none of G_j can be exceptional groups. We still have to impose some restriction on these eigenvalues, since the possibility remains of them being eight in number (see Fig. 1). Since we demand that there exist only four eigenvalues, we must have

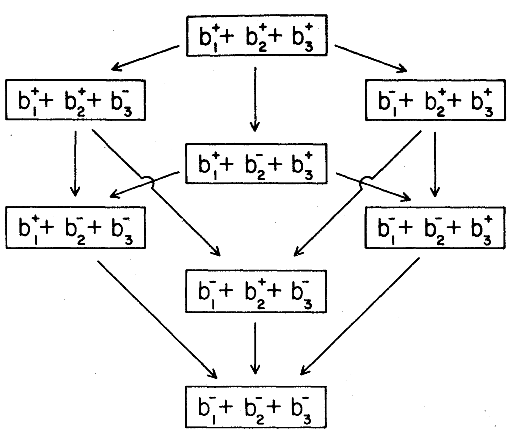

FIG. 1. Relations among possible eigenvalues. The arrow shows which one is larger.

$$b_1^+ - b_1^- = b_2^+ - b_2^- = b_3^+ - b_3^-. \qquad (2)$$

This condition implies that the charge structure of a multiplet is $(1, \frac{2}{3}, \frac{1}{3}, 0)$ and it is impossible to have, e.g., $(\frac{2}{3}, 0, -\frac{1}{3}, -1)$. Since the b_j^- are negative, we must have U(1) in order to make $Q_\nu = 0$.

For the proof of *Lemma* 3, we first note that both groups G_1 and G_2 cannot have three or more eigenvalues simultaneously, since we then have at least five different eigenvalues. Hence, we have two possibilities: one is that one of the groups has three different eigenvalues, $b_1^{\pm,0}$, while the other has only two, $b_2^{\pm}$. In this case, we have to impose some relation between eigenvalues, since we may have six different eigenvalues (see Fig. 2). We obtain

$$b_1^+ - b_1^0 = b_1^0 - b_1^- = b_2^+ - b_2^-. \qquad (3)$$

This condition implies that the charge structure of a multiplet is $(1, \frac{2}{3}, \frac{1}{3}, 0)$ only. Since both b_1^- and b_2^- are negative, a U(1) factor is necessary in order to have $Q_\nu = 0$. Okubo's theorem[8] again tells us that only one of the G_j can be an exceptional group.

The second possibility is that each group has two different eigenvalues, $b_j^{\pm}$ ($j = 1, 2$) (see Fig.3). In order to get four different charges, we have

$$b_1^+ - b_1^- \neq b_2^+ - b_2^-. \qquad (4)$$

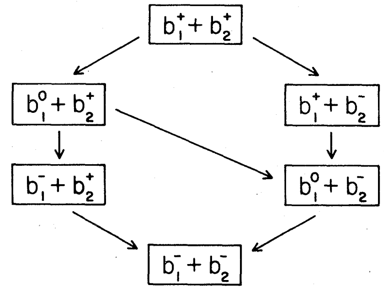

FIG. 2. Relations among possible eigenvalues.

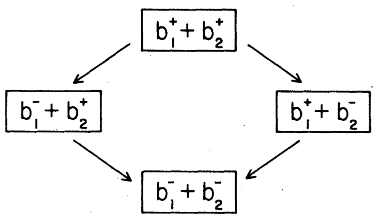

FIG. 3. Relations among possible eigenvalues.

It is possible to have a multiplet with $(\frac{2}{3}, -\frac{1}{3}, 0, -1)$. Moreover, we have two other possibilities: $(1, \frac{2}{3}, 0, -\frac{1}{3})$ or $(1, \frac{2}{3}, \frac{1}{3}, 0)$. [We do not count charge-conjugate states as independent. Thus, our four-charge condition implies at most four cases; one missing case in the above is $(\frac{2}{3}, \frac{1}{3}, 0, -1)$]. Okubo's theorem tells us again that in this case none of the G_j are exceptional groups.

We can get stronger statements if we use the following theorem proved by Okubo[15]:

Theorem. For any X in a simple Lie algebra, we have

$$\mathrm{Tr}X^3=0 \quad \text{for } \mathrm{E}_6 \text{ and } \mathrm{SO}(10),$$

$$\mathrm{Tr}X^3=\mathrm{Tr}X^5=0 \quad \text{for } \mathrm{SO}(4m+2) \quad (m\geq 3),$$

$$\mathrm{Tr}X^p=0 \ (p=1,3,5,\cdots) \quad \text{for } \mathrm{SO}(4m),\ B_n[=\mathrm{SO}(2n+1)],$$

$$C_n[=\mathrm{Sp}(2n)],\ \mathrm{G}_2,\ \mathrm{F}_4,\ \mathrm{E}_7, \text{ and } \mathrm{E}_8,$$

where SO(10), $\mathrm{SO}(4m+2)$ $(m\geq 3)$, and $\mathrm{SO}(4m)$ make $D_n[=\mathrm{SO}(2n)]$. We can prove the following lemmas, using this theorem.

Lemma 4. For $G=G_1\otimes G_2\otimes G_3\otimes \mathrm{U}(1)$, if each G_j $(j=1,2,3)$ is one of $B_n[=\mathrm{SO}(2n+1)]$, $C_n[=\mathrm{Sp}(2n)]$, or $D_n[=\mathrm{SO}(2n)]$, we have the relation that No. up = No. down = 3 No.e = 3 No.ν in a multiplet.

Proof. In Okubo's theorem above, put $X=Q_j$. Then, we have $\mathrm{Tr}\,Q_j=\mathrm{Tr}\,Q_j{}^3=0$ for B_n, C_n, and D_n. We do not have the possibility of G_j being exceptional, by Lemma 2. Each Q_j has only two eigenvalues, and thus

$$n_j^+b_j^+ + n_j^-b_j^- = n_j^+(b_j^+)^3 + n_j^-(b_j^-)^3 = 0, \tag{5}$$

where $j=1,2,3$ and $n_j^\pm$ denote multiplicities of states with $b_j^\pm$ in a multiplet. From Eq. (2), we have

$$b_j^+ - b_j^- = \frac{1}{3}. \tag{6}$$

The solutions are

$$b_j^+ = -b_j^- = \frac{1}{6} \text{ and } n_j^+ = n_j^- \ (j=1,2,3), \tag{7}$$

The number of up-type quarks can be given by the combinations $(b_1^+ + b_2^+ + b_3^-)$, $(b_1^+ + b_2^- + b_3^+)$, or $(b_1^- + b_2^+ + b_3^+)$, while the number of charged leptons is given by $(b_1^+ + b_2^+ + b_3^+)$. Since $n_j^+ = n_j^-$ $(j=1,2,3)$, we have No. up = 3 No.e, etc.

Lemma 5. For $G=G_1\otimes G_2\otimes \mathrm{U}(1)$, if each G_j $(j=1,2)$ is one of B_n, C_n, and D_n, it is necessary to have a U(1). If each Q_j has only two eigenvalues for the case where none of the G_j $(j=1,2)$ is SU(n), we have the relation No. up = No. down = No.e = No.ν in a multiplet.

The proof is similar to Lemma 4. We obtain the corollary from Lemmas 3 and 5.

Corollary. For $G=G_1\otimes G_2\otimes \mathrm{U}(1)$, it is necessary to have a U(1), if none of the G_j $(j=1,2)$ is A_n $[=\mathrm{SU}(n+1)]$.

In Lemma 5, for the case where G_1 has three eigenvalues (G_2 has to have two) and G_2 is not A_n $[=\mathrm{SU}(n+1)]$, the number relations are

$$\frac{\text{No. } e}{\text{No. } \nu}=\frac{n^+}{n^-},\quad \frac{\text{No. up}}{\text{No. down}}=\frac{n^+ + n^0}{n^- + n^0},\quad \frac{\text{No. up}}{\text{No. } e}=\frac{n^+ + n^0}{n^+},$$

where $n^{\pm,0}$ denote multiplicities of the $b_1^{\pm,0}$ in a given multiplet. Thus, if No. up = No. down, then we necessarily have No. e = No. ν in a multiplet.

Lemma 6. For $G=G_1\otimes \mathrm{U}(1)$, if G_1 is one of B_n, C_n, G_2, F_4, E_7, E_8, or D_n $(n\neq 5)$, it is necessary to have a U(1).

Proof. Assume that there is no U(1) factor. Then, $\mathrm{Tr}\,Q=\mathrm{Tr}\,Q^3=\mathrm{Tr}\,Q^5=0$ for the groups tabulated above. It is easy to see that it is impossible to satisfy the three equations at the same time for charges $(\frac{2}{3}, -\frac{1}{3}, 0, -1)$ or its variations.

For the rest of the simple groups, $A_n[=\mathrm{SU}(n+1)]$, E_6, and SO(10), the following lemma holds, using $\mathrm{Tr}\,Q^3=\mathrm{Tr}\,Q=0$.

Lemma 7. If $G=G_1$ and G_1 is E_6 or SO(10), we have, in a multiplet,

$$\frac{\text{No. up}}{\text{No. down}}=\frac{4}{5} \text{ and } \frac{\text{No. down}}{\text{No. } e}=5,$$

and the charge structure is only $(\frac{2}{3}, 0, -\frac{1}{3}, -1)$.

Summarizing the foregoing lemmas, we obtain the following.

Proposition 1. In order to have a four-charge multiplet, the grand unification group is one of the three possibilities: $G=G_1\otimes G_2\otimes G_3\otimes \mathrm{U}(1)$, $G=G_1\otimes G_2\otimes \mathrm{U}(1)$, or $G=G_1\otimes \mathrm{U}(1)$. For $G=G_1\otimes G_2\otimes G_3\otimes \mathrm{U}(1)$, a U(1) is necessary and none of the G_j $(j=1,2,3)$ are exceptional groups. If any one of the G_j $(j=1,2,3)$ is not SU(n), we necessarily have the relation that No. up = No. down = 3 No. e = 3 No. ν in a multiplet. For $G=G_1\otimes G_2\otimes \mathrm{U}(1)$, a U(1) is necessary if none of the G_j $(j=1,2)$ is SU(n). Only one of the G_j can be an exceptional group. If each Q_j has only two eigenvalues for the case where G_j $(j=1,2)$ are not SU(n), we necessarily have No. up = No. down = No. e = No. ν in a multiplet. For $G=G_1\otimes \mathrm{U}(1)$, a U(1) is necessary if G_1 is not one of E_6, SO(10), or SU(n). For the case where $G=G_1$ is E_6 or SO(10), we necessarily have No. up/No. down = 4/5 and No. down/No. e = 5 in a multiplet.

Hence, only two possibilities are left out if we do not want to have a U(1) and unusual number relations among quarks and leptons: SU(n) or SU(n) $\otimes$ SU(m). It is surprising to obtain this strong result from such weak requirements. Amazingly, we can prove another proposition for $G = \mathrm{SU}(n) \otimes \mathrm{SU}(m)$.

Proposition 2. For $\mathrm{SU}(n) \otimes \mathrm{SU}(m)$, the four-charge ansatz yields that we have either

$$\frac{\text{No. }\nu}{\text{No. up}} = \frac{\text{No. }e}{\text{No. down}} = \frac{1}{3}\left(2\frac{\text{No. up}}{\text{No. down}} - 1\right)$$

for the charge structure $(\frac{2}{3}, 0, -\frac{1}{3}, -1)$, or

$$\frac{\text{No. }\nu}{\text{No. down}} = \frac{\text{No. }e}{\text{No. up}} = \frac{1}{3}\left(\frac{\text{No. down}}{\text{No. up}} - 2\right)$$

for the charge structure $(1, \frac{2}{3}, 0, -\frac{1}{3})$. Only these two cases are allowed.

The proof can be done easily, using $\mathrm{Tr}\,Q_i = 0$ and Lemma 3. If we demand either No. up = No. down or No. e = No. ν in a multiplet with the four-charge ansatz, then we have automatically No. e = No. ν $= \frac{1}{3}$ No. up $= \frac{1}{3}$ No. down with the unique charge structure $(\frac{2}{3}, 0, -\frac{1}{3}, -1)$. The charge operators Q_1 and Q_2 take the following forms for this case:

$$Q_1 = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \frac{1}{6} & \\ & -\frac{1}{2} \end{pmatrix}.$$
$$\quad N \;\; N \qquad\qquad 3M \;\; M$$

From the form of charge operators, we must have at least $n \geq 2$ and $m \geq 4$.

For $G = \mathrm{SU}(n)$, we can have any charge structure as long as n is sufficiently large. For example, the fundamental representation of SU(8) can accommodate $(\frac{2}{3}, -\frac{1}{3}, 0, -1)$. Since SU($n$) is the only group which can have the triangle anomaly, we should be careful about model construction.

Finally, we would like to mention another possible charge structure in a multiplet, that is, $(1, \frac{2}{3}, \frac{1}{3}, 0, -\frac{1}{3}, -\frac{2}{3}, -1)$. In this case, every representation for the groups SO($4n$), Sp($2n$), G_2, F_4, E_7, E_8 has the possibility of having this charge structure automatically. The reason is that for these groups we have $\mathrm{Tr}\,Q^p = 0$ where p is an arbitrary odd integer. Thus, for every particle type we have No. particles = No. antiparticles in a multiplet for $G = G_1$; note that this relation holds for any charge structure.[15] Even for SO($4n+2$) ($n \geq 3$), we still have this relation for the particular charge structure $(1, \frac{2}{3}, \frac{1}{3}, 0, -\frac{1}{3}, -\frac{2}{3}, -1)$. For SU($n$), E_6, and SO(10), we are not guaranteed of having this structure, although we know it is possible. Therefore, unfortunately, the charge structure of a multiplet, such as $(1, \frac{2}{3}, \frac{1}{3}, 0, -\frac{1}{3}, -\frac{2}{3}, -1)$, yields no constraint on the selection of the right group for grand unification.

Note added. The conclusion that the group should be either SU(n) or $\mathrm{SU}(n) \otimes \mathrm{SU}(m)$ is derived from the nontriviality of the representation of a multiplet. Therefore, SU(n) may imply that the group for all the multiplets is, for example, $\mathrm{SU}(n)_L \otimes \mathrm{SU}(n)_R$. Similarly, $\mathrm{SU}(n) \otimes \mathrm{SU}(m)$ may imply $\mathrm{SU}(n)_L \otimes \mathrm{SU}(n)_R \otimes \mathrm{SU}(m)$ as the grand unification group. The choice, $n = 2$ and $m = 4$, leads to the Pati-Salam group $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \otimes \mathrm{SU}(4)$. Note that we have shown that $n \geq 2$ and $m \geq 4$.

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[1]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8)*, edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367.

[2]SU(5): H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974); for a review, see e.g., J. Ellis, CERN Report No. Ref. TH-2723, 1979 (unpublished); F. Wilczek, in *Proceedings of the 1979 International Symposium on Lepton and Photon Interactions at High Energies*, edited by T. B. W. Kirk and H. D. I. Abarbanel (Fermilab, Batavia, 1979); K. Kang, Brown University Report No. HET-423 TA 349, 1980 (unpublished).

SU(6): K. Inoue, A. Kakuto, and Y. Nakano, Prog. Theor. Phys. 62, 307 (1979).

SU(7): P. H. Frampton, Phys. Lett. 88B, 299 (1979).

SU(8): T. L. Curtright and P. G. O. Freund, in *Supergravity*, edited by P. van Nieuwenhuizen and D. Z. Freedman (North-Holland, Amsterdam, 1979); C. W. Kim and C. Roiesnel, Phys. Lett. 93B, 343 (1980); Y. Fujimoto, Dublin Report No. DIRS-TP-7940, 1980 (unpublished); J. Ellis, M. K. Gaillard, and B. Zumino, Phys. Lett. 94B, 343 (1980).

SU(9): P. H. Frampton and S. Nandi, Phys. Rev. Lett. 43, 1460 (1979); P. H. Frampton, Phys. Lett. 89B, 352 (1980).

SU(11): H. Georgi, Nucl. Phys. B156, 126 (1979); J. E. Kim, Phys. Rev. D 21, 1687 (1980).

SU(16): A. Salam, Nobel lecture, Trieste Report No. IC/80/28, 1980 (unpublished).

SU(n) (n = 5, 7, 8, 9, 10): J. Chakrabarti, M. Popovic, and R. N. Mohapatra, Phys. Rev. D 21, 3212 (1980).

[3]SO(10): H. Fritzsch and P. Minkowski, Ann. Phys.

(N.Y.) 93, 193 (1975); H. Georgi, in *Particles and Fields*, edited by C. E. Carlson (AIP, New York, 1975); C. E. Vayonakis, Phys. Lett. 82B, 224 (1979); 83B, 421 (1979); F. Hadjionnou, University of Athens report, 1979 (unpublished); H. Georgi and D. V. Nanopoulos, Nucl. Phys. B155, 52 (1979); A. Yu. Ignatiev, V. A. Kuzumin, and M. E. Shafoshinikov, Phys. Lett. 87B, 114 (1979); B. Barbieri, D. V. Nanopoulos, G. Morchio, and F. Strocchi, *ibid*. 90B, 91 (1980); E. Witten, *ibid*. 91B, 81 (1980).
SO(12): N. P. Chang and J. P. Mercader, Phys. Rev. D 18, 4721 (1978); R. N. Mohapatra and B. Sakita, *ibid*. 21, 1062 (1980).
SO(14): H. Fritzsch and P. Minkowski, Ann. Phys. (N.Y.) 93, 193 (1975).
SO($4n+2$) ($n=2,3,4,5$): M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, edited by P. van Niewenhuizen and D. Z. Freedman (North-Holland, Amsterdam, 1979), p. 315.
SO($10+2n$): F. Wilczek and A. Zee, Princeton report (unpublished).

[4] E_6: F. Gürsey, P. Ramond, and P. Sikivie, Phys. Lett. 60B, 177 (1976); F. Gürsey and M. Serdaroglu, Lett. Nuovo Cimento 21, 28 (1978); Y. Achiman and B. Stech, Phys. Lett. 77B, 389 (1978); Q. Shafi, *ibid*. 79B, 301 (1978); H. Ruegg and T. Shucker, Nucl. Phys. B156, 126 (1979); P. Ramond, Caltech Report No. 68-709, 1979 (unpublished); O. K. Kalashnikov and S. E. Konstein, Nucl. Phys. B166, 507 (1980).
E_7: P. Gürsey and P. Sikivie, Phys. Rev. Lett. 36, 775 (1976); P. Ramond, Nucl. Phys. B110, 214 (1976); P. Sikivie and F. Gürsey, Phys. Rev. D 16, 816 (1977); V. Ogievetsky and V. Tzeitlin, Yad. Fiz. 28, 1616 (1978) [Sov. J. Nucl. Phys. 28, 832 (1978)]; R. Lednicky and V. Tzeitlin, Phys. Lett. 88B, 302 (1979).
E_8: N. S. Baaklini, Phys. Lett. 91B, 376 (1980); I. Bars and M. Günaydin, Phys. Rev. Lett. 45, 859 (1980).

[5] $SU(4) \otimes SU(4)$: J. C. Pati and A. Salam, Phys. Rev. Lett. 31, 661 (1973); Phys. Rev. D 8, 1240 (1973).
$SU(5) \otimes SU(5)$: A. Davidson, Phys. Lett. 90B, 87 (1980); Syracuse Report No. SU-4217, 1979 (unpublished).
$SU(6) \otimes SU(6)$: A. Salam, Nobel lecture, Trieste Report No. IC/80/28, 1980 (unpublished).
$SU(8) \otimes SU(8)$: N. G. Deshpande and P. D. Mannheim, Phys. Lett. 94B, 355 (1980).
$SU(n) \otimes SU(n)$ ($n=8,12,16$): H. Fritzsch and P. Minkowski, Ann. Phys. (N.Y.) 93, 193 (1975).

[6] $SU(4) \otimes SU(4) \otimes SU(4) \otimes SU(4)$: J. C. Pati and A. Salam, Phys. Lett. 58B, 333 (1975).

[7] For example, see, M. A. Naimark, *Normed Rings* (Gronigen, Netherland, 1959).

[8] S. Okubo, Phys. Rev. D 18, 3792 (1978).

[9] M. Gell-Mann, P. Ramond, and R. Slansky, Rev. Mod. Phys. 50, 721 (1978).

[10] A. Hacinliyan and C. Saclioglu, Phys. Rev. D 19, 2440 (1979).

[11] H. Georgi and S. L. Glashow, Phys. Rev. D 6, 429 (1972); S. Okubo, *ibid*. 16, 3528 (1977).

[12] S. Okubo, Phys. Rev. D 16, 3535 (1977).

[13] S. Okubo, Phys. Lett. 5, 165 (1963); G. Zweig, CERN Report No. Ref. TH-412, 1964 (unpublished); J. Iizuka, Prog. Theor. Phys. Suppl. 37-38, 21 (1966); for a review, see, e.g., S. Okubo, Prog. Theor. Phys. Suppl. 63, 1 (1978).

[14] This is due to the fact that any element of Cartan subalgebra can be expressed in terms of a commutator. See, e.g., N. Jacobson, *Lie Algebra* (Dover, New York, 1979).

[15] S. Okubo, Had. J. 1, 77 (1978).