Hidden symmetry in Yang-Mills theory: A path-dependent gauge transformation

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The search for new additional symmetry in Yang-Mills theory is described. A reparametrization-invariant path-dependent Lagrangian is given whose fundamental variables are the functional fields $\psi[\xi] = P\exp(\oint A\cdot d\xi)$. The Lagrangian is used to construct path-dependent Noether currents corresponding to translations, SO(4) rotations, and scale transformations on the path ξ_μ. An internal-symmetry current is interpreted to signal the existence of an infinitesimal functional field transformation $\psi \to \psi + \Delta\psi$, where $\Delta\psi$ is a certain specific path-dependent gauge transformation. A second new hidden-symmetry current is derived explicitly. Also discussed is the gauge-invariant formulation of scalar electrodynamics. It is a functional field theory in which space-time transformations on the path-dependent fields give rise to internal-symmetry Noether currents of the local variables in contrast to Yang-Mills theory.

I. INTRODUCTION

This paper discusses the search for an extra additional symmetry of Yang-Mills theory. In lieu of solving the theory exactly, such symmetry may lead to important information about the nonperturbative sector. One example of this is that the new symmetry implies complete integrability of the classical system. Another is that it gives rise to new conserved quantities.

It is unlikely that there exist further continuous symmetry currents which are polynomials in the local field variables and their derivatives.[1] Of course, new discrete symmetries and nonlocal conserved currents may occur. In this work, the new invariance is described in Yang-Mills theory reformulated in terms of functional fields, a formalism where the role of the fundamental local variables $A_\mu^a(x)$ is taken on by the path-dependent field $\psi = P\exp(\oint A\cdot d\xi)$.

The results of this paper are the following. In Sec. II a reparametrization-invariant path-dependent Lagrangian is presented and its relation to Yang-Mills theory is discussed. This Lagrangian is then used in Sec. III as a tool to derive path-dependent Noether currents associated with familiar space-time transformations on the paths: translations, SO(4) rotations, and scale transformations. Functionally conserved currents computed in this manner often involve no singular expressions in the derivation of their conservation laws.

In Sec. IV, the functional Noether current associated with internal-symmetry local gauge transformations is given. A hidden-symmetry current[2] is interpreted to signal the existence of an infinitesimal functional field transformation $\psi \to \psi + \Delta\psi$, where $\Delta\psi$ is a certain specific path-dependent gauge transformation. A second new hidden-symmetry current is derived in direct analogy with the Noether-current analysis of the nonlocal charges in the two-dimensional chiral models.[3]

In the Appendix, the gauge-invariant path-dependent formulation of scalar electrodynamics[4] is shown to be a functional field theory in which a path symmetry gives rise to an internal-symmetry current. This is in sharp contrast with the Yang-Mills case.

II. LAGRANGIAN FORMULATION

The fundamental variables of a functional formulation of Yang-Mills theory are path-dependent fields $\psi(x,C)$. These are expressed in terms of the local gauge potentials $A_\mu(x) \equiv A_\mu^a \sigma^a/2i$ as

$$\psi(x,C) = \psi[\xi] = P\exp\left(\oint A\cdot d\xi\right) = P\exp\left[\int_{s_1}^{s_2} ds\, \dot{\xi}_\mu(s) A_\mu(\xi(s))\right]. \tag{2.1}$$

The line integration in (2.1) is done around a closed path in four-dimensional Euclidean space parametrized by four functions $\xi_\mu(s)$, where $s_1 \leq s \leq s_2$ and $\xi_\mu(s_1) = \xi_\mu(s_2) = x_\mu$. The path ordering P in (2.1) is defined by

$$P\exp\left(\oint A\cdot d\xi\right) = \lim_{N\to\infty} P\exp\left[\sum_{i=1}^{N+1}(\xi_\mu^i - \xi_\mu^{i-1})A_\mu(\xi^{i-1})\right]$$
$$= \lim_{N\to\infty} e^{(\xi^1-x)\cdot A(x)}e^{(\xi^2-x)\cdot A(\xi^1)}\cdots e^{(x-\xi^N)\cdot A(\xi^N)}$$
$$= \lim_{N\to\infty}[1+(\xi^1-x)\cdot A(x)][1 + (\xi^2 - \xi^1)\cdot A(\xi^1)]\cdots[1+(x-\xi^N)\cdot A(\xi^N)]\,. \tag{2.2}$$

In (2.2), the path described by $\xi_\mu(s)$ has been made discrete in order to exhibit that path ordering is necessary to express the product of exponentials of noncommuting classical matrix fields $A_\mu(x)$ as the exponential of the sum of these fields.

From (2.2) we see that the functional field $\psi(x,C) = P\exp(\oint A\cdot d\xi)$ depends on the starting point x_μ. In (2.1), $\psi(x,C)$ is also written as $\psi[\xi]$ to stress that just as the local field $A_\mu(x)$ is a function of the point x, the field $\psi[\xi]$ is a functional of the functions $\xi_\mu(s)$ used to describe the path. If we reparametrize any given path $\xi_\mu(s)$ by a new path $\tilde\xi_\mu(t): s=s(t),\ \xi_\mu(s)=\xi_\mu(s(t))\equiv\tilde\xi_\mu(t)$, then

$$\psi = P\exp\left[\int ds\ \dot\xi_\mu(s)A_\mu(\xi(s))\right]$$
$$= P\exp\left[\int dt(d\tilde\xi_\mu/dt)A_\mu(\tilde\xi(t))\right]$$

remains invariant.[5] Thus to be precise, the functional field $\psi[\xi]$ depends only on the path C and the starting point x: $\psi[\xi]=\psi(x,C)$.

We now define a reparametrization-invariant Lagrangian. It is

$$\mathcal{L}(C)=\mathcal{L}[\xi]=\int\frac{ds}{[\dot\xi^2(s)]^{1/2}}\,\mathrm{tr}\,\frac{\delta\psi}{\delta\xi_\mu(s)}\,\frac{\delta\psi^{-1}}{\delta\xi_\mu(s)}\,. \tag{2.3}$$

The gauge-covariant functional derivative $\delta/\delta\xi_\mu(s)$ is given as follows[6]:

$$\psi[\xi+\delta\xi]-\psi[\xi] = -\delta x_\mu[A_\mu(x),\psi[\xi]] + \int ds\,\delta\xi_\mu(s)\,\psi_{x:\xi(s)}F_{\mu\beta}(\xi(s))\psi_{\xi(s):x}\dot\xi_\beta(s)$$
$$\equiv -\delta x_\mu[A_\mu(x),\psi] + \int ds\,\delta\xi_\mu(s)\frac{\delta\psi}{\delta\xi_\mu(s)}\,. \tag{2.4}$$

Therefore,

$$\frac{\delta\psi}{\delta\xi_\mu(s)} = \psi_{x:\xi(s)}F_{\mu\beta}(\xi(s))\psi_{\xi(s):x}\dot\xi_\beta(s)\,. \tag{2.5}$$

In (2.5), $\psi_{x:\xi(s)}\equiv P\exp(\int_x^{\xi(s)}A\cdot d\xi')$ and in (2.3)

$$\psi^{-1}[\xi] = \lim_{N\to\infty}P\exp\left(-\sum_{i=1}^{N+1}(\xi_\mu^i-\xi_\mu^{i-1})A_\mu(\xi^{i-1})\right)$$
$$=\lim_{N\to\infty}e^{-(x-\xi^N)\cdot A(\xi^N)}\cdots e^{-(\xi^1-x)\cdot A(x)}. \tag{2.6}$$

ψ^{-1} is defined by going the opposite direction around the path C. $\psi^{-1}[\xi]\psi[\xi]=I$.

We observe from (2.4) and (2.5) that the covariant functional derivative commutes[7] for s and t not equal to s_1 or s_2:

$$\frac{\delta}{\delta\xi_\mu(s)}\frac{\delta}{\delta\xi_\nu(t)}\psi = \psi_{x:s}F_{\mu\alpha}(s)\psi_{s:t}F_{\nu\beta}(t)\psi_{t:x}\dot\xi_\alpha(s)\dot\xi_\beta(t)\theta(t-s)$$
$$+\psi_{x:t}F_{\nu\beta}(t)\psi_{t:s}F_{\mu\alpha}(s)\psi_{s:x}\dot\xi_\alpha(s)\dot\xi_\beta(t)\theta(s-t)$$
$$+\psi_{x:t}D_\mu F_{\nu\beta}(t)\psi_{t:x}\dot\xi_\beta(t)\delta(s-t)+\psi_{x:t}F_{\nu\mu}(t)\psi_{t:x}\frac{d}{dt}\delta(s-t)\,, \tag{2.7a}$$

$$\left[\frac{\delta}{\delta\xi_\mu(s)},\frac{\delta}{\delta\xi_\nu(t)}\right]\psi = \psi_{x:t}F_{\nu\mu}(t)\psi_{t:x}\left(\frac{d}{dt}\delta(s-t)+\frac{d}{ds}\delta(s-t)\right)$$
$$=0. \tag{2.7b}$$

For simplification, the notation has been changed slightly from $\psi_{x:\xi(s)}$ to $\psi_{x:s}$ and from $F_{\nu\mu}(\xi(s))$ to $F_{\nu\mu}(s)$.

Expressed in terms of the local field variables, the path-dependent Lagrangian (2.3) is

$$\mathcal{L}[\xi] = -\int\frac{ds}{[\dot\xi^2(s)]^{1/2}}\dot\xi_\alpha(s)\dot\xi_\beta(s)\,\mathrm{tr}\,F_{\mu\alpha}(s)F_{\mu\beta}(s)\,. \tag{2.8}$$

Unlike the functional formulation of scalar electrodynamics discussed in the Appendix, the La-

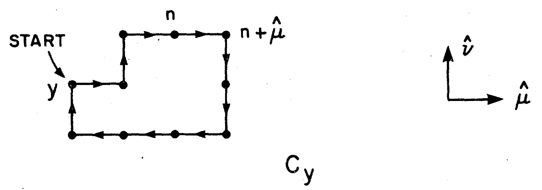

FIG. 1. The path C_y.

grangian functional $\mathcal{L}[\xi]$ itself also depends on the path. Thus to define the action one has to integrate over all paths with some measure:

$$I=\int D^4\xi\,\mathcal{L}[\xi] \tag{2.9}$$

The measure $D^4\xi$ must be reparametrization invariant, include some weighting factor for convergence, and be defined such that (2.9) is equivalent to the Yang-Mills action:

$$I=\int d^4x\,[-\tfrac{1}{2}\operatorname{tr}F_{\mu\nu}(x)F_{\mu\nu}(x)]\ . \tag{2.10}$$

On the lattice,[8] the classical Euclidean gauge theory action is given by

$$I_{\text{lat}}=-\sum_{n,\mu,\nu}\operatorname{tr}U_{n\mu}U_{n+\hat{\mu}\nu}U^{\dagger}_{n+\hat{\nu}\mu}U^{\dagger}_{n\nu}\,. \tag{2.11}$$

In (2.11) $U_{n\mu}=e^{aA_\mu(\vec{n}a)}$ is a 2×2 special unitary matrix $U^{\dagger}_{n\mu}=U_{n\mu}^{-1}=e^{-aA_\mu(\vec{n}a)}$, a is the lattice spacing, and $a^4\sum_n$ becomes the continuum integral $\int d^4x$ with $x=na$ for $a\to0$. Now define a lattice matrix field on a set of links:

$$\psi(C_y)=U_{y\mu}U_{y+\hat{\mu}\nu}U_{y+\hat{\mu}+\hat{\nu}\mu}U_{n\mu}\times U_{n+\hat{\mu}-\hat{\nu}}^{-1}\cdots U_{y-\hat{\nu}\nu}\,,$$

where C_y is given in Fig. 1. Following Polyakov,[2] then define $\psi(C_y+\pi_{n\mu\nu})$ as the matrix field on a path where the link n to $n+\hat{\mu}$ is replaced by $\Pi_{n\mu\nu}$, i.e., the links n to $n+\hat{\nu}$ to $n+\hat{\nu}+\hat{\mu}$ to $n+\hat{\mu}$ (see Fig. 2). Thus Eq. (2.11) can be rewritten as

$$\begin{aligned}I_{\text{lat}}=-&\sum_{n,\mu,\nu}\operatorname{tr}I\\&+\tfrac{1}{2}\sum_{n,\mu,\nu}\operatorname{tr}\{[\psi(C_y+\Pi_{n\mu\nu})-\psi(C_y)]\\&\qquad\times[\psi^{-1}(C_y+\Pi_{n\mu\nu})-\psi^{-1}(C_y)]\}\ .\end{aligned} \tag{2.12}$$

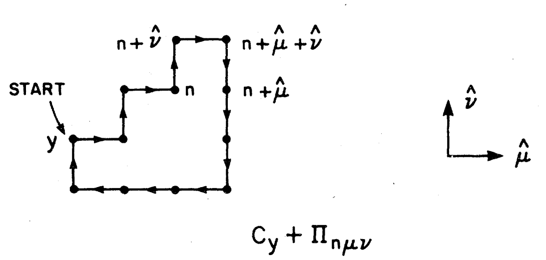

FIG. 2. The path $C_y+\Pi_{n\mu\nu}$.

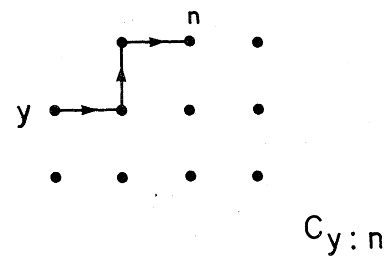

FIG. 3. The path $C_{y:n}$.

Also, the equations of motion on the lattice can be written as

$$\begin{aligned}0=2\sum_{\nu}(&U_{n\mu}U_{n+\hat{\mu}-\hat{\nu},\nu}^{-1}U_{n-\hat{\nu}\mu}^{-1}U_{n-\hat{\nu}\nu}\\&-U_{n\nu}U_{n+\hat{\nu}\mu}U_{n+\hat{\mu}\nu}^{-1}U_{n\mu}^{-1})\end{aligned} \tag{2.13}$$

or equivalently[2]

$$\begin{aligned}0&=\psi(C_{y:n})\sum_{\nu}(U_{n\mu}U_{n+\hat{\mu}-\hat{\nu},\nu}^{-1}U_{n-\hat{\nu},\mu}^{-1}U_{n-\hat{\nu}\nu}\\&\qquad-U_{n\nu}U_{n+\hat{\nu},\mu}U_{n+\hat{\mu}\nu}^{-1}U_{n\mu}^{-1})\psi^{-1}(C_{y:n})\\&=\sum_{\nu}[\psi(C_y)\psi^{-1}(C_y-\Pi_{n\mu\nu})-\psi(C_y+\Pi_{n\mu\nu})\psi^{-1}(C_y)],\end{aligned} \tag{2.14}$$

where the paths $C_{y:n}$ and $C_y-\Pi_{n\mu\nu}$ are defined in Figs. 3 and 4.

The classical continuum limit $(a\to0)$ of (2.11) is unambiguous[8]:

$$\lim_{a\to0}I_{\text{lat}}=\int d^4x\,[-\tfrac{1}{2}\operatorname{tr}F_{\mu\nu}(x)F_{\mu\nu}(x)]+\text{constant}. \tag{2.15}$$

Using the fact that

$$\begin{aligned}\lim_{a\to0}\,&[\psi(C_y+\Pi_{n\mu\nu})-\psi(C_y)]\\&=\psi_{y:an}\exp[F_{\alpha\beta}(an)\,\delta\xi^n_\alpha(\xi^{n+1}-\xi^n)_\beta]\psi_{an:y}\\&\quad-\psi_{y:an}\psi_{an:y}\,,\end{aligned} \tag{2.16}$$

where $\delta\xi^n_\alpha=a\delta_{\alpha\nu}$ and $(\xi^{n+1}-\xi^n)_\beta=a\delta_{\beta\mu}$, we have for fixed $an=\xi$

$$\begin{aligned}\lim_{a\to0}\,&[\psi(C_y+\Pi_{n\mu\nu})-\psi(C_y)]\\&=a^2\psi_{y:\xi}F_{\nu\mu}(\xi)\psi_{\xi:y}+O(a^3)\,.\end{aligned} \tag{2.17}$$

The continuum limit of the equations of motion (2.14) is therefore[9]

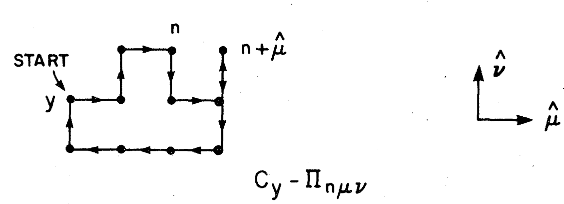

FIG. 4. The path $C_y-\Pi_{n\mu\nu}$.

$$\lim_{a\to 0}\sum_{\nu}[\psi(C_y)\psi^{-1}(C_y-\Pi_{n\mu\nu})-\psi(C_y+\Pi_{n\mu\nu})\psi^{-1}(C_y)]\frac{1}{a^3}=\lim_{a\to 0}\sum_{\nu}\{[\psi(C_y+\Pi_{n\mu\nu})-\psi(C_y)][\psi^{-1}(C_y+\Pi_{n\mu\nu})-\psi^{-1}(C_y)]$$
$$+\psi(C_y)[\psi^{-1}(C_y+\Pi_{n\mu\nu})+\psi^{-1}(C_y-\Pi_{n\mu\nu})-2\psi^{-1}(C_y)]\}\frac{1}{a^3}$$
$$=-2\psi_{y;\xi}D_\nu F_{\nu\mu}(\xi)\psi_{\xi;y}{}^{-1}$$
$$=0\,. \tag{2.18}$$

From (2.17) we see that

$$\lim_{a\to 0}\frac{1}{a^2}[\psi(C_y+\Pi_{n\mu\nu})-\psi(C_y)]\neq\frac{\delta\psi}{\delta\xi_\mu}\,.$$

Thus, although (2.18) implies

$$\frac{\delta}{\delta\xi_\mu(s)}\left(\psi\frac{\delta}{\delta\xi_\mu(s)}\psi^{-1}\right)=0 \tag{2.19}$$

or equivalently

$$\frac{\delta}{\delta\xi_\mu(s)}\left(\psi^{-1}\frac{\delta}{\delta\xi_\mu(s)}\psi\right)=0\,, \tag{2.20}$$

the continuum limit of the lattice functional Lagrangian in (2.12) is the local Yang-Mills Lagrangian density and not the path-dependent Lagrangian (2.3):

$$\lim_{a\to 0}\frac{1}{2a^4}\sum_{\nu,\mu}\mathrm{tr}\,[\psi(C_y+\Pi_{n\mu\nu})-\psi(C_y)][\psi^{-1}(C_y+\Pi_{n\mu\nu})-\psi^{-1}(C_y)]=-\tfrac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}(z)F_{\mu\nu}(z)\,. \tag{2.21}$$

The Lagrangian functional (2.3) is presented in this paper as a tool to construct conserved quantities, i.e., conserved by the Yang-Mills equations of motion. $\mathcal{L}[\xi]$ is not equal to the Yang-Mills Lagrangian density. But when ψ is expressed in terms of the local gauge potentials $A_\mu^a(z)$, $D_\mu F_{\mu\nu}(z)=0$ implies solutions of (2.20).

III. SPACE-TIME SYMMETRIES

The functional Lagrangian (2.3) is covariant under translations, SO(4) rotations, and scale transformations. In analogy with local field theory, choose the infinitesimal transformations on the paths as follows:

$$\xi_\mu(s)\to\bar{\xi}_\mu(s)=\xi_\mu(s)-\delta\xi_\mu(s)\,,$$

where

$$\delta\xi_\mu(s)=a_\mu\,, \tag{3.1a}$$
$$\delta\xi_\mu(s)=\omega_{\mu\nu}\xi_\nu(s)\,, \tag{3.1b}$$
$$\delta\xi_\mu(s)=-\rho\xi_\mu(s) \tag{3.1c}$$

for translations, SO(4) transformations, and scale transformations, respectively.

Let the matrix functional field ψ transform as a scalar under the transformations (3.1). For (3.1a) and (3.1b),

$$\psi[\xi]\to\psi'[\xi']\equiv\psi[\xi]+\Delta\psi[\xi]\,, \tag{3.2}$$

where $\Delta\psi$ in (3.2) is given by

$$\Delta\psi=\int_{s_1}^{s_2}ds\,\delta\xi_\mu(s)\frac{\delta\psi}{\delta\xi_\mu(s)}\,. \tag{3.3}$$

Now use the functional Lagrangian (2.3) to construct path-dependent Noether currents. For a transformation on the field $\psi\to\psi+\Delta\psi$, $\psi^{-1}\to\psi^{-1}-\psi^{-1}\Delta\psi\psi^{-1}$, the Lagrangian $\mathcal{L}[\xi]$ transforms as $\mathcal{L}\to\mathcal{L}+\Delta\mathcal{L}$, where

$$\Delta\mathcal{L}=\int\frac{ds}{[\dot{\xi}^2(s)]^{1/2}}\mathrm{tr}\left[\frac{\delta\Delta\psi}{\delta\xi_\mu(s)}\frac{\delta\psi^{-1}}{\delta\xi_\mu(s)}-\frac{\delta\psi}{\delta\xi_\mu(s)}\frac{\delta}{\delta\xi_\mu(s)}(\psi^{-1}\Delta\psi\psi^{-1})\right] \tag{3.4a}$$
$$=\int\frac{ds}{[\dot{\xi}^2(s)]^{1/2}}\mathrm{tr}\,2\left[\frac{\delta}{\delta\xi_\mu(s)}\left(\Delta\psi\frac{\delta\psi^{-1}}{\delta\xi_\mu(s)}\right)-\Delta\psi\psi^{-1}\frac{\delta}{\delta\xi_\mu(s)}\left(\psi\frac{\delta}{\delta\xi_\mu(s)}\psi^{-1}\right)\right]. \tag{3.4b}$$

When ψ is a solution to the equations of motion

(2.19), we have from (3.4b)

$$\Delta\mathcal{L} = \int \frac{ds}{[\xi^2(s)]^{1/2}}\, 2\,\mathrm{tr}\,\frac{\delta}{\delta\xi_\mu(s)}\left(\Delta\psi\frac{\delta\psi^{-1}}{\delta\xi_\mu(s)}\right) \quad (3.5a)$$

$$= \int ds\frac{\delta}{\delta\xi_\mu(s)}\, 2\,\mathrm{tr}\,\Delta\psi\, \frac{1}{[\xi^2(s)]^{1/2}}\,\frac{\delta}{\delta\xi_\mu(s)}\,\psi^{-1}. \quad (3.5b)$$

Equation (3.5b) follows from the reparametrization invariance of ψ:

$$\dot{\xi}_\mu(s)\frac{\delta\psi}{\delta\xi_\mu(s)} = 0\,.$$

Thus if for a given $\Delta\psi$ one can show that

$$\Delta\mathcal{L} = \int ds\frac{\delta}{\delta\xi_\mu(s)}\,\Lambda_\mu(s,C)$$

without using the equations of motion, a functionally conserved quantity can be constructed:

$$J_\mu(s,C) = 2\,\mathrm{tr}\,\Delta\psi\,\frac{1}{[\xi^2(s)]^{1/2}}\,\frac{\delta}{\delta\xi_\mu(s)}\,\psi^{-1} - \Lambda_\mu(s,C)\,, \quad (3.6)$$

where

$$\int ds\frac{\delta}{\delta\xi_\mu(s)}\,J_\mu(s,C) = 0\,.$$

These are path-dependent Noether currents. For the space-time transformations given in (3.1) we now show that the field transformations (3.2) shift $\mathcal{L}[\xi]$ by a total functional divergence and we construct path-dependent currents (3.6).

The field transformations $\psi \to \psi + \Delta\psi$ are found from (3.1) and (3.2) and in analogy with local scalar field theory to be

$$\Delta\psi = \int dt\, a_\alpha\,\frac{\delta\psi}{\delta\xi_\alpha(t)}\,, \quad (3.7a)$$

$$\Delta\psi = \int dt\, \omega_{\alpha\beta}\,\xi_\beta(t)\frac{\delta\psi}{\delta\xi_\alpha(t)}\,, \quad (3.7b)$$

$$\Delta\psi = d\psi + \int dt\;\xi_\alpha(t)\frac{\delta\psi}{\delta\xi_\alpha(t)}\,. \quad (3.7c)$$

Here d is the scale dimension of the field ψ. That ψ has zero scale dimension is consistent with its definition in terms of the local gauge potential; $\psi = P\exp(\oint A\cdot d\xi)$ is a dimensionless quantity. The constant $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. The field transformations (3.7) shift Lagrangian density without use of equations of motion by the following total divergences, respectively:

$$\Delta\mathcal{L} = \int ds\,\frac{\delta}{\delta\xi_\mu(s)}\;\delta_{\mu\alpha}\,a_\alpha\,\mathcal{L}[\xi], \quad (3.8a)$$

$$\Delta\mathcal{L} = \omega_{\alpha\beta}\int ds\frac{\delta}{\delta\xi_\alpha(s)}\,(\xi_\beta(s)\mathcal{L}[\xi]), \quad (3.8b)$$

$$\Delta\mathcal{L} = \int ds\;\frac{\delta}{\delta\xi_\mu(s)}\int dt\,\frac{\xi_\mu(t)}{[\xi^2(t)]^{1/2}}\;\mathrm{tr}\;\frac{\delta\psi}{\delta\xi_\nu(t)}\;\frac{\delta\psi^{-1}}{\delta\xi_\nu(t)}\,. \quad (3.8c)$$

Therefore, the associated functionally conserved Noether currents are from (3.6) and (3.8)

$$J^{\mathrm{transl}}_{\mu\alpha}(s,C) = 2\,\mathrm{tr}\int dt\;\frac{\delta\psi}{\delta\xi_\alpha(t)}\;\frac{1}{[\xi^2(s)]^{1/2}}\;\frac{\delta\psi^{-1}}{\delta\xi_\mu(s)} - \delta_{\mu\alpha}\mathcal{L}[\xi], \quad (3.9a)$$

$$\omega_{\alpha\beta}J^{SO(4)}_{\mu\alpha\beta}(s,C) = \left\{2\,\mathrm{tr}\int dt\;\xi_\beta(t)\,\frac{\delta\psi}{\delta\xi_\alpha(t)}\;\frac{\delta\psi^{-1}}{\delta\xi_\mu(s)} \times\frac{1}{[\xi^2(s)]^{1/2}} - \delta_{\mu\alpha}\,\xi_\beta(s)\,\mathcal{L}[\xi]\right\}\omega_{\alpha\beta}\,, \quad (3.9b)$$

$$J^{\mathrm{scale}}_\mu(s,C) = 2\,\mathrm{tr}\int dt\;\xi_\alpha(t)\;\frac{\delta\psi}{\delta\xi_\alpha(t)}\;\frac{\delta\psi^{-1}}{\delta\xi_\mu(s)}\;\frac{1}{[\xi^2(s)]^{1/2}} + 2\;\mathrm{tr}\;d\psi\,\frac{\delta\psi^{-1}}{\delta\xi_\mu(s)}\;\frac{1}{[\xi^2(s)]^{1/2}} - \int dt\;\frac{\xi_\mu(t)}{[\xi^2(t)]^{1/2}}\,\mathrm{tr}\,\frac{\delta\psi}{\delta\xi_\nu(t)}\;\frac{\delta\psi^{-1}}{\delta\xi_\nu(t)}\,. \quad (3.9c)$$

Note that the canonical form (3.9a) is not symmetric in μ and α and that (3.9b) and (3.9c) cannot be written in terms of (3.9a). Possibly this could be achieved by the addition of the divergence of an antisymmetric tensor to the canonical currents. Also, remark that the term in the scale current (3.9c) proportional to d is identically zero. In contrast to the internal-symmetry currents discussed in Sec. IV, the currents given by (3.9) are not matrix currents. In contrast to the Abelian model treated in the Appendix, the Noether currents derived from the space-time symmetries above do not imply the existence of an internal symmetry in terms of the local gauge potential fields $A^a_\mu(x)$. Lastly, the naive choice

$$\Delta\psi = \int dt[2\,c\cdot\xi(t)\,\xi_\alpha(t) - c_\alpha\,\xi^2(t)]\,\frac{\delta\psi}{\delta\xi_\alpha(t)}$$

corresponding to ψ transforming as a scalar under special conformal transformations on the paths

$$\delta\xi_\mu(s) = 2\,\xi_\mu(s)\,c\cdot\xi(s) - c_\mu\,\xi^2(s)$$

does not leave the equations of motion (2.20) invariant.

IV. INTERNAL SYMMETRIES

Internal-symmetry transformations on the field ψ are now considered. Local gauge transformations lead to a Noether current whose conservation is just the equation of motion. The first nontrivial hidden-symmetry current is seen to be consistent with invariance under a certain specific path-dependent gauge tranformation. Arguing in direct analogy with the two-dimensional chiral models we are then able to write down a second, new, hidden-symmetry current.

Under local gauge transformations

$$A_\mu(x) \to U(x)A_\mu(x)U^{-1}(x) + U(x)\partial_\mu U^{-1}(x)\,, \tag{4.1}$$

where $A_\mu(x) \equiv A^a_\mu(\alpha^a/2i)$ is a traceless anti-Hermitian 2×2 matrix and $U(x) = e^{S^a(x)\sigma^a/2i}$ is an element of SU(2). The corresponding transformation on ψ is

$$\psi(x, C) \to \psi'(x, C) = U(x)\psi(x, C)U^{-1}(x)\,. \tag{4.2}$$

Infinitesimally (4.2) is

$$\begin{aligned} &\psi' = \psi + \Delta\psi\,, \\ &\Delta\psi = [S(x), \psi]\,, \end{aligned} \tag{4.3}$$

where $S(x) = S^a(x)\sigma^a/2i$ depends only on the origin of the closed path. The field transformation (4.3) leaves $\mathcal{L}[\xi]$ invariant since the covariant derivatives $\delta/\delta\xi\mu(s)$ only vary the path away from the starting point x_μ. Alternatively, from (2.8), the transformation $F_{\mu\nu}(z) \to U(z)F_{\mu\nu}(z)U^{-1}(z)$ leaves $\mathcal{L}[\xi]$ invariant.

In addition, we remark that the field transformations $\Delta\psi = S(x)\psi$ and $\Delta\psi = -\psi S(x)$ separately leave $\mathcal{L}[\xi]$ invariant. The Noether current constructed from $\Delta\psi = -\psi S(x)$ is

$$J^{\text{gauge}}_{\mu(s,C)} = 2\,\mathrm{tr}\psi^{-1}\frac{1}{[\xi^2(s)]^{1/2}}\,\frac{\delta\psi}{\delta\xi_\mu(s)}\,S^a(x)\frac{\sigma^a}{2i}\,. \tag{4.4}$$

Since $\psi^{-1}\delta\psi/\delta\xi_\mu(s)$ is also a traceless anti-Hermitian 2×2 matrix, (4.4) implies a conserved matrix current

$$\mathcal{J}_\mu(s, C) = \psi^{-1}\,\frac{1}{[\xi^2(s)]^{1/2}}\,\frac{\delta\psi}{\delta\xi_\mu(s)}\,. \tag{4.5}$$

The conservation of (4.5) is just the equation of motion (2.20).

We now discuss the hidden-symmetry currents in analogy with the infinite set of nonlocal conservation laws of the general chiral models. In this theory the first two nontrivial nonlocal currents have been derived as Noether currents explicitly.[3] Furthermore, conservation of the first (trivial) nonlocal current is just the equation of motion. In this two-dimensional theory it was shown that whereas the chiral Lagrangian is invariant under arbitrary global isospin transformations $G\times G$, the symmetry giving rise to the nonlocal currents is a covariance of the Lagrangian density under certain specific local isospin transformations. Thus, the obvious extension to functional Yang-Mills theory is that the functional Lagrangian (2.3) is invariant under arbitrary local gauge transformation $U(x) = e^{S(x)}$ and is covariant under certain specific path-dependent gauge transformations $U[\xi] = e^{S[\xi]}$, where the traceless antisymmetric matrix S now depends not just on the path starting point x_μ but on the entire path $\xi_\mu(s)$ in some specific way.

For $\Delta\psi = -\psi S$, the change in $\mathcal{L}[\xi]$ is

$$\Delta\mathcal{L} = 2\int ds\;\mathrm{tr}\,\frac{\delta S}{\delta\xi_\mu(s)}\,F_\mu(s)\,, \tag{4.6}$$

where

$$F_\mu(s) \equiv \psi^{-1}\,\frac{1}{[\xi^2(s)]^{1/2}}\,\frac{\delta\psi}{\delta\xi_\mu(s)}\,.$$

Let $S = [\chi, T]$ where $T = \rho^a\sigma^a/2i$, ρ^a constant. Then Eq. (4.6) becomes

$$\Delta\mathcal{L} = 2\int ds\;\mathrm{tr}\left[F_\mu(s),\,\frac{\delta\chi}{\delta\xi_\mu(s)}\right]T\,. \tag{4.7}$$

In order to carry out the Noether analysis one must define χ such that (4.7) can be written as a total divergence without using the equations of motion. This has not been done.

Nonetheless, to construct a conserved quantity, we can proceed with an alternative definition of an infinitesimal symmetry transformation $\Delta\psi$, namely if ψ is a solution so is $\psi + \Delta\psi$. Thus for $\Delta\psi = -\psi[\chi, T^a]\rho^a$ to be a symmetry of the equations of motion, we must define χ such that if ψ is a solution so is $\psi - \psi[\chi, T^a]\rho^a$ to first order in ρ^a. This condition on χ is

$$\frac{1}{[\xi^2(s)]^{1/2}}\left[\frac{\delta^2\chi}{\delta\xi_\mu{}^2(s)}\,,T\right] + \left[F_\mu(s),\,\left[\frac{\delta\chi}{\delta\xi_\mu(s)},T\right]\right] = 0\,. \tag{4.8}$$

Let

$$F_\mu\,\frac{\delta\chi}{\delta\xi_\mu} = -\frac{\delta\chi}{\delta\xi_\mu}\,F_\mu$$

and

$$F_\mu T\,\frac{\delta\chi}{\delta\xi_\mu} = -\frac{\delta\chi}{\delta\xi_\mu}\,TF_\mu\,. \tag{4.9}$$

From (4.8) and (4.9) we construct the functionally conserved quantity

$$J_\mu(s, C) = \tfrac{1}{2}[F_\mu(s), \chi] + \frac{1}{[\xi^2(s)]^{1/2}}\,\frac{\delta\chi}{\delta\xi_\mu(s)} \tag{4.10}$$

as follows. From (4.9)

$$\left[F_\mu, \left[\frac{\delta\chi}{\delta\xi_\mu}\,,T\right]\right] = \frac{1}{2}\left[\left[F_\mu,\,\frac{\delta\chi}{\delta\xi_\mu}\right],T\right]$$

so that (4.8) implies

$$\frac{\delta}{\delta\xi_\mu(s)}\left(\frac{1}{[\xi^2(s)]^{1/2}}\,\frac{\delta\chi}{\delta\xi_\mu(s)} + \tfrac{1}{2}[F_\mu(s), \chi]\right) = 0\,. \tag{4.11}$$

(See Ref. 2.) Note that (4.10) has exactly the same form as the two-dimensional Noether currents when the chiral fields are solutions. In the two-dimensional model, the Noether currents for the first two nontrivial charges are[3]

$$\begin{aligned} \mathcal{J}_\mu^{(1)} &= [A_\mu, \chi^{(1)}] - \epsilon_{\mu\nu}A_\nu - \tfrac{1}{2}\epsilon_{\mu\nu}[\partial_\nu\chi^{(1)}, \chi^{(1)}]\,, \\ \mathcal{J}_\mu^{(2)} &= [A_\mu, \chi^{(2)}] + \tfrac{1}{2}[[A_\mu, \chi^{(1)}], \chi^{(1)}] \\ &\quad - \epsilon_{\mu\nu}[\partial_\nu\chi^{(1)}, \chi^{(2)}] - \epsilon_{\mu\nu}\tfrac{1}{6}[[\partial_\nu\chi^{(1)}, \chi^{(1)}], \chi^{(1)}] \\ &\quad - \epsilon_{\mu\nu}[A_\nu, \chi^{(1)}]\,, \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} A_\mu &\equiv g^{-1}\partial_\mu g\,, \\ \chi^{(1)}(y,t) &= \tfrac{1}{2}\int_{-\infty}^{\infty} dx\, \epsilon(y-x)A_0(x,t)\,, \qquad (4.13) \\ \chi^{(2)}(y,t) &= -\tfrac{1}{2}\int_{-\infty}^{\infty} dx\, \epsilon(y-x)A_1(x,t) \\ &\quad + \tfrac{1}{4}\int_{-\infty}^{\infty} dx\, \epsilon(y-x)[A_0(x,t), \chi^{(1)}(x,t)]\,, \end{aligned}$$

and $g(x)$ is a matrix field element of some group G. When g is a solution to the chiral equations of motion $\partial_\mu(g^{-1}\partial_\mu g) = 0$, then

$$\begin{aligned} \partial_\mu\chi^{(1)} &= -\epsilon_{\mu\nu}A_\nu\,, \\ \partial_\mu\chi^{(2)} &= -A_\mu - \tfrac{1}{2}\epsilon_{\mu\nu}[A_\nu, \chi^{(1)}]\,. \end{aligned} \tag{4.14}$$

Use of (4.14) then reduces Eqs. (4.12) to

$$\mathcal{J}_\mu^{(1)} = \partial_\mu\chi^{(1)} + \tfrac{1}{2}[A_\mu, \chi^{(1)}]\,, \tag{4.15a}$$

$$\mathcal{J}_\mu^{(2)} = [\partial_\mu\chi^{(1)}, \chi^{(1)}] + \tfrac{1}{3}[[A_\mu, \chi^{(1)}], \chi^{(1)}]\,. \tag{4.15b}$$

If we replace $\chi^{(1)}$ by χ and A_μ by $F_\mu[\xi(s)]$ then Eq. (4.15a) is (4.10) and it is now trivial to construct the next current in functional space from (4.15b). It is the new result

$$\begin{aligned} J_\mu^{(2)}(s,C) &= \frac{1}{[\dot\xi^2(s)]^{1/2}}\left[\frac{\delta\chi}{\delta\xi_\mu(s)}, \chi\right] \\ &\quad + \tfrac{1}{3}[[F_\mu(s), \chi], \chi]\,. \end{aligned} \tag{4.16}$$

V. CONCLUSION

A Lagrangian of functional Yang-Mills theory is presented. It is used as a tool to construct path-dependent Noether currents corresponding to translations, SO(4) rotations, and scale transformations of the path and internal local gauge transformations. A hidden-symmetry current is derived by requiring $\psi + \Delta\psi$ to be a solution to the equations of motion for all solutions ψ where $\Delta\psi$ is a certain specific path-dependent gauge transformation. We thus identify an additional symmetry in Yang-Mills theory to be an invariance under this gauge transformation just as in the local two-dimensional chiral models, the first nontrivial nonlocal current is associated with a specific space-time-dependent isospin transformation.[3]

Another new result is the explicit derivation of a second hidden-symmetry current.

The functional description of Yang-Mills theory is studied here in the spirit that coherent phenomena such as confinement may be more easily probed by the switch from the local variables $A_\mu^a(x)$ to the new fundamental excitations ψ. Path-dependent variables appear also in discussions of duality for Yang-Mills theory.[10] Certain two-dimensional systems such as the sine-Gordon equation exhibit both complete integrability and a dualitylike relationship (with the massive Thirring model).[11] A thorough understanding of the nature of these variables may thus prove valuable in bringing together a coherent description of nonperturbative effects for the strong interactions.

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APPENDIX

As another example of a local field theory reformulated in terms of functional fields, this appendix discusses Minkowski-space four-dimensional scalar electrodynamics. Unlike the Yang-Mills case, in this theory a space-time symmetry transformation on the paths leaves the Lagrangian density invariant; and the associated Noether current can be used to find a conserved internal-symmetry current of the original local variables.

The Lagrangian density in terms of local field variables is

$$\begin{aligned} \mathcal{L} &= (\partial_\mu + ieA_\mu)\varphi^*(\partial^\mu - ieA^\mu)\varphi \\ &\quad - m^2\varphi^*\varphi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \end{aligned} \tag{A1}$$

With the use of Mandelstam's gauge-invariant formalism,[4] (A1) can be expressed in terms of new, path-dependent fields and their functional derivatives. The nonlocal fields are defined as

$$\begin{aligned} \phi[\xi] &= \varphi(x)\exp\left[-ie\int_{-\infty}^{x} d\xi\cdot A(\xi)\right]\,, \\ \phi^*[\xi] &= \varphi^*(x)\exp\left[ie\int_{-\infty}^{x} d\xi\cdot A(\xi)\right]\,. \end{aligned} \tag{A2}$$

These functional fields are reparametrization-invariant functionals of $\xi_\mu(s)$ which parametrizes the path: $0 \le s \le s_0$; $\xi_\mu(0) = -\infty$, $\xi_\mu(s_0) = x_\mu$. Under an arbitrary variation of the path: $\xi_\mu(s) \to \xi_\mu(s) + \delta\xi_\mu(s)$, the corresponding variation of the field given below defines the ordinary functional derivative $d/d\xi_\mu(s)$,

$$\phi[\xi+\delta\xi]-\phi[\xi]=\delta x^\mu\partial_\mu\phi[\xi]+ie\delta\xi^\mu(0)A_\mu(-\infty)\phi[\xi]$$
$$+\int ds\,\delta\xi^\mu(s)(-ie)F_{\mu\beta}(\xi(s))\dot{\xi}^\beta(s)\phi[\xi]$$
$$\equiv\int ds\,\delta\xi_\mu(s)\,\frac{d}{d\xi^\mu(s)}\phi[\xi]\,, \qquad \text{(A3)}$$

where

$$\partial_\mu\phi[\xi]=(\partial_\mu-ieA_\mu(x)\varphi(x))\exp\left[-ie\int_{-\infty}^x d\xi\cdot A(\xi)\right],$$

It will be useful to consider portions of (A3) separately and to distinguish field variations and functional derivatives corresponding to different path variations $\delta\xi_\mu(s)$. That is to say, variation of the end point x_μ with the rest of the path field fixed,

$$\delta\xi_\mu(s)=\begin{cases}\delta x_\mu\,, & s=s_0\\ 0\,, & \text{otherwise}\end{cases}$$

results in

$$\phi[\xi+\delta\xi]-\phi[\xi]=\delta x^\mu\partial_\mu\phi\equiv d\phi\,. \qquad \text{(A4)}$$

If we define the corresponding functional derivative by $d\phi\equiv\int ds\,\delta\xi^\mu(s)\partial_\mu(s)\phi$, then it is

$$\partial_\mu(s)\phi\sim\delta(s-s_0)\partial_\mu\phi\,. \qquad \text{(A5)}$$

Variation of the path with both end points held fixed,

$$\delta\xi_\mu(s)=\begin{cases}0\,, & s=0,s_0\\ \delta\xi_\mu(s)\,, & \text{otherwise}\end{cases}$$

leads to

$$\phi[\xi+\delta\xi]-\phi[\xi]$$
$$=-ie\int ds\,F_{\mu\alpha}(\xi(s))\dot{\xi}^\alpha(s)\delta\xi^\mu(s)\phi[\xi]\equiv\delta\phi \qquad \text{(A6)}$$

and

$$D_\mu(s)\phi=-ieF_{\mu\alpha}(\xi(s))\dot{\xi}^\alpha(s)\phi\,. \qquad \text{(A7)}$$

The functional fields and their various derivations discussed above are all invariant under local gauge transformations $A_\mu(x)\to A_\mu(x)-(1/e)\partial_\mu\lambda(x)$ which vanish at $x_\mu=-\infty$: $\lambda(-\infty)=0$. Equation (A1) can now be written as

$$\mathcal{L}=\partial_\mu\phi^*\partial^\mu\phi-m^2\phi^*\phi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,. \qquad \text{(A8)}$$

Although $\mathcal{L}$ is expressed in terms of path-dependent fields, they combine such that $\mathcal{L}$ itself depends only on the end point x, in contrast with Yang-Mills theory. $\mathcal{L}$ is therefore invariant under arbitrary variations of the path which leave the end points fixed. This symmetry is now used to construct a functional Noether current.

In terms of the path-dependent variables the equations of motion are

$$\partial_\mu\partial^\mu\phi=-m^2\phi\,,$$
$$\partial_\mu\partial^\mu\phi^*=-m^2\phi^*\,.$$

Under an arbitrary field variation, $\phi\to\phi+\Delta\phi$ and $\phi^*\to\phi^*+\Delta\phi^*$,

$$\mathcal{L}\to\mathcal{L}+\delta\mathcal{L}\,,$$

where

$$\delta\mathcal{L}=\partial_\mu\Delta\phi^*\partial^\mu\phi+\partial_\mu\phi^*\partial^\mu\Delta\phi-m^2\phi^*\Delta\phi-m^2\phi\Delta\phi^* \qquad \text{(A9)}$$

and we have assumed that $\Delta\partial_\mu\phi=\partial_\mu\Delta\phi$. Use of the equations of motion yields

$$\delta\mathcal{L}=\partial_\mu(\partial^\mu\phi^*\Delta\phi+\partial^\mu\phi\Delta\phi^*]\,. \qquad \text{(A10)}$$

For $\Delta\phi=\delta\phi$ given in (A6) and the corresponding

$$\Delta\phi^*=ie\int ds\,F_{\alpha\beta}(\xi(s))\dot{\xi}^\beta(s)\delta\xi^\alpha(s)\phi^*[\xi]\,,$$

then from (A9) we have $\delta\mathcal{L}=0$. Therefore, the conserved current is

$$J_\mu=ie(\phi^*\partial_\mu\phi-\phi\partial_\mu\phi^*)\int ds\,F_{\alpha\beta}(\xi(s))\dot{\xi}^\beta(s)\delta\xi^\alpha(s)\,, \qquad \text{(A11)}$$

where $\delta\xi^\alpha(s)$ is arbitrary but vanishes at the end points.

J_μ is a nonlocal object whose functional derivative $\partial_\mu(s)J_\mu=0$. Since the derivative $\partial_\mu(s)$ only involves variation of the end point the factor

$$\int ds\,F_{\alpha\beta}(\xi(s))\,\frac{d\xi^\beta}{ds}\,d\xi^\alpha(s)$$

can be dropped from J_μ resulting in another conserved quantity:

$$\bar{J}_\mu=ie(\phi^*\partial_\mu\phi-\phi\partial_\mu\phi^*)\,. \qquad \text{(A12)}$$

$\bar{J}_\mu$ depends only on the end point x and is in fact the standard current obtained from global gauge invariance. In terms of local variables,

$$\bar{J}_\mu=ie(\varphi^*(\partial_\mu-ieA_\mu)\varphi-\varphi(\partial_\mu+ieA_\mu)\varphi^*)\,.$$

The distinction between the functional derivative $\partial_\mu(s)$ and the local derivative $\partial/\partial\chi^\mu$ is relevant only when the object being differentiated is nonlocal.

In this Abelian theory, the Lagrangian expressed in terms of path-dependent fields is invariant under arbitrary path variations which leave the end points fixed. The functionally conserved Noether current associated with this symmetry implies the conservation of an internal-symmetry current of the local field variables.

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$$\int_{s_1}^{s_2} ds\,\frac{d\xi_\mu(s)}{ds}A_\mu(\xi(s))=\int_{t_1}^{t_2} dt\,\frac{ds}{dt}\,\frac{d\xi_\mu(s(t))}{dt}\,\frac{dt}{ds}\times A_\mu(\xi[s(t)])=\int_{t_1}^{t_2} dt\,\frac{d\tilde{\xi}_\mu(t)}{dt}\,A_\mu(\tilde{\xi})\,.$$

The end points obey $\xi_\mu(s_1)=\xi_\mu(s_2)=x_\mu=\tilde{\xi}_\mu(t_1)=\tilde{\xi}_\mu(t_2)$. In the rest of the text the limits on the parameter integration are omitted. Note that other reparametrization-invariant quantities are $ds[\dot{\xi}^2(s)]^{1/2}$ and $\{1/[\dot{\xi}^2(s)]^{1/2}\}\delta/\delta\xi_\mu(s)$.

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