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Exact computation of loop averages in two-dimensional Yang-Mills theory*

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We present an explicit algorithm that allows the exact computation of all nonlocal gauge-invariant correlation functions in two-dimensional Yang-Mills theory. Explicit expressions are given for the expectation value of entangled loop operators and their products in the U(N) theory with arbitrary N. Using these results we show that the Schwinger-Dyson equation for loops holds without singularities in the continuum, and we verify the factorizability of the correlation functions in the $N\to\infty$ limit. A non-Abelian version of Stokes's theorem which is used in the calculations is derived for an arbitrary number of dimensions.

I. INTRODUCTION

In recent years there has been a continued effort to understand the nonperturbative structure of non-Abelian gauge theories in the description of which the expectation value of nonlocal gauge-invariant operators play an important role. Exact results for these expectation values are of special interest since they may offer new insights and can serve to test new developments and approximation schemes.

In this paper we consider the case of the two-dimensional continuum theory where we construct an explicit algorithm for the exact computation of the vacuum expectation value of arbitrary loops and products of loop operators. Although this model represents a drastic simplification as compared to the physical four-dimensional theory, it still has a structure rich enough to illustrate properties that are believed to be common, to some extent, to all non-Abelian theories.

Explicit computations will be done for the U(N) theory and, although we concentrate here on the exact finite-N results, it is shown how the algorithm simplifies for large-N calculations by means of an explicit power series in $1/N$ whose coefficients are easy to compute to any order.

In Sec. II we review the derivation of the Abelian results whereas Secs. III and IV deal with the general non-Abelian calculation. In Sec. V we apply the general results to some explicit examples of interest with a brief discussion of the results. A non-Abelian version of Stokes's theorem, used in the calculations of Secs. III and IV is derived in the Appendix for arbitrary dimensionality of space-time.

II. ABELIAN CASE

In this section we give a brief derivation of the vacuum expectation value of arbitrary loops in the two-dimensional Abelian theory. Although the final results are very simple and well known, it is still interesting to recast the calculation in a way that, with appropriate modifications, we will be able to generalize to the non-Abelian case.

Let γ be a loop in two-dimensional Euclidean space-time, parametrized by s, $0\le s\le 1$. We denote by $U[\gamma]$ the gauge-invariant Abelian phase factor

$$U[\gamma]=\exp\left[ie\int_0^1 ds\,\dot\gamma^\mu(s)A_\mu(\gamma(s))\right]. \tag{2.1}$$

The current associated with γ is given by

$$j^\mu(\gamma;x)=\int_0^1 ds\,\dot\gamma^\mu(s)\,\delta^{(2)}(\gamma(s)-x)\,, \tag{2.2}$$

and we can write

$$U[\gamma]=\exp\left[ie\int d^2x\,j^\mu(\gamma;x)A_\mu(x)\right]. \tag{2.3}$$

The field strength in two dimensions has the form

$$F_{\mu\nu}=F\epsilon_{\mu\nu}\,, \tag{2.4}$$

and if γ is a loop that does not intersect itself Stokes's theorem gives

$$\int_0^1 ds\,\dot\gamma^\mu(s)A_\mu(\gamma(s))=\pm\int_{\Sigma_\gamma} d^2x\,F(x)\,,$$

where the surface integral extends over the area enclosed by the loop and the $\pm$ sign is determined by its orientation. In the case of arbitrary loops that intersect and overlap with themselves an appropriately modified version of Stokes's theorem also holds. Let $\theta(\gamma;x)$ denote the winding number of a closed loop γ around the point x. It can be written as

$$\theta(\gamma;x)=\int_0^1 ds\,\dot\gamma^\mu(s)b_\mu(\gamma(s);x)\,, \tag{2.5a}$$

where $b_\mu(x;x_0)$ can be thought of as the "gauge potential" at x due to a magnetic vortex of unit strength located at x_0. b_μ can be written in different gauges although θ is independent of this choice, and in the Lorentz gauge it is given by

$$b_\mu(x;x_0) = -\frac{1}{2\pi}\epsilon_{\mu\nu}\frac{(x-x_0)^\nu}{(x-x_0)^2}$$
$$= -\frac{1}{4\pi}\epsilon_{\mu\nu}\partial^\nu \ln m^2(x-x_0)^2\,, \qquad (2.5b)$$

where m is an arbitrary mass parameter. Using this representation for b_μ it is easy to show that the current in Eq. (2.2) is given by

$$j^\mu(\gamma;x) = \epsilon^{\mu\nu}\partial_\nu\theta(\gamma;x) \qquad (2.6)$$

and using this in the right-hand side of (2.3) we get

$$\int_0^1 ds\, \dot\gamma^\mu(s)A_\mu(\gamma(s)) = \int d^2x\,\theta(\gamma;x)F(x)\,, \qquad (2.7)$$

which is the general version of Stokes's theorem valid for arbitrary loops.

The functional integral for the vacuum expectation value of $U[\gamma]$ can now be written as

$$\langle U[\gamma]\rangle = \frac{1}{Z}\int \mathfrak{D}A_\mu \times \exp\left\{-\int d^2x\left[\tfrac{1}{2}F^2(x) - ie\theta(\gamma;x)F(x)\right]\right\},$$

where a gauge-fixing condition is implicit in the measure $\mathfrak{D}A_\mu$. Choosing the axial gauge

$$A_1(x_1,x_2) = 0\,,$$
$$A_2(0,x_2) = 0\,,$$

it is easy to show that, since in two dimensions there is no Bianchi identity, the change of integration variables from the gauge potential to the field strength is linear:

$$\prod_x dA_\mu(x)\,\delta(A_1(x_1,x_2))\,\delta(A_2(0,x_2)) = C\prod_x dF(x)\,, \qquad (2.8)$$

where C is a constant that cancels with the corresponding factor arising from the same change of variables in the normalization factor Z. Then we are left with

$$\langle U[\gamma]\rangle = \frac{1}{Z}\int\prod_x dF(x)\exp\left\{-\int d^2x\left[\tfrac{1}{2}F^2(x) - ie\theta(\gamma;x)F(x)\right]\right\},$$

which is a Gaussian integral giving

$$\langle U[\gamma]\rangle = \exp\left(-\frac{e^2}{2}A[\gamma]\right), \qquad (2.9)$$

where $A[\gamma]$ is the area enclosed by the loop γ as defined by

$$A[\gamma] = \int d^2x\,\theta^2(\gamma;x)\,. \qquad (2.10)$$

Similarly, for the product of many loops one finds

$$\langle U[\gamma_1]\cdots U[\gamma_n]\rangle = \exp\left\{-\frac{e^2}{2}\left(\sum_{i=1}^n A[\gamma_i] + 2\sum_{i<j}A[\gamma_i,\gamma_j]\right)\right\}, \qquad (2.11)$$

where

$$A[\gamma_i,\gamma_j] = \int d^2x\,\theta(\gamma_i;x)\theta(\gamma_j;x)\,. \qquad (2.12)$$

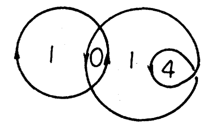


FIG. 1. Weights assigned to the different portions of the area of overlapping loops in the Abelian theory.

Notice that the quadratic composition law for the areas of overlapping loops (Fig. 1) is just what should be expected from a Hamiltonian description. Regions where loops overlap describe the propagation of more than one e^+e^- pair having their strings of electric flux falling on top of each other. In the Abelian theory these fluxes add up algebraically giving rise to the winding number θ, but the energy of the configuration depends on the square of the total flux thus giving rise to the above results.

Using Eqs. (2.5) it is easy to show that the result in (2.9) can be rewritten in its more usual form

$$\langle U[\gamma]\rangle = \exp\left\{+\frac{e^2}{8\pi}\int_0^1 ds\int_0^1 ds'\,\dot\gamma^\mu(s)\dot\gamma_\mu(s')\ln m^2[\gamma(s)-\gamma(s')]^2\right\}$$

with a similar expression holding for Eq. (2.11).

III. EXPECTATION VALUE OF NON-ABELIAN LOOPS

In order to extend the calculation of Sec. II to the non-Abelian theory we need an appropriate generalization of Stokes's theorem. This generalization, valid in any number of dimensions, is derived in the Appendix and here we present only the result.

The non-Abelian group element $U[\gamma]$ associated with a path γ, open or closed, is given now by the path-ordered exponential

$$U[\gamma]=P\exp\left[ig\int_0^1 ds\,\dot\gamma^{\mu}(s)A_{\mu}(\gamma(s))\right]$$
$$=\sum_{n=0}^{\infty}(ig)^n\int_0^1 ds_1\cdots\int_0^{s_{n-1}}ds_n\dot\gamma^{\mu_1}(s_1)\cdots\dot\gamma^{\mu_n}(s_n)A_{\mu_1}(\gamma(s_1))\cdots A_{\mu_n}(\gamma(s_n))\,, \tag{3.1}$$

where $A_\mu(x)=A_\mu^a(x)T_a$ is the Lie-algebra-valued gauged potential. For brevity we will refer to $U[\gamma]$ as a "loop variable" if γ is a closed path or a "string variable" if it is an open path.

If $0\leq s_1,s_2\leq 1$ let us denote by ${}_{s_2}\gamma_{s_1}$ the segment of the path connecting the points $\gamma(s_1)$ and $\gamma(s_2)$, given explicitly in terms of γ by

$${}_{s_2}\gamma_{s_1}(s)=\gamma(s_1+(s_2-s_1)s) \tag{3.2a}$$

so ${}_{s_2}\gamma_{s_1}(0)=\gamma(s_1)$ and ${}_{s_2}\gamma_{s_1}(1)=\gamma(s_2)$. In particular,

$${}_1\gamma_0=\gamma \tag{3.2b}$$

and we write ${}_0\gamma_1=\gamma^{-1}$. Let ξ now be a sheet, that is, a one-parameter family of paths parametrized by τ, $0\leq\tau\leq1$. Then, for each τ, $\xi(\tau)$ is a path, itself parametrized as before by s, $0\leq s\leq1$. Let us require the sheet ξ to be such that for all τ the paths $\xi(\tau)$, which are in general open, have fixed starting and ending points so

$$\frac{\partial\xi^{\mu}}{\partial\tau}(\tau,s)=0\quad\text{if } s=0,1\,. \tag{3.3}$$

Otherwise, the sheet ξ is arbitrary and is allowed to fold on itself. Then with $\partial\xi$ denoting the boundary of ξ, oriented according to the initial path $\xi(0)$ (Fig. 2), we have

$$U[\partial\xi]=U^{-1}[\xi(1)]U[\xi(0)]=U[\xi^{-1}(1)]U[\xi(0)] \tag{3.4}$$

and the non-Abelian version of Stokes's theorem gives

$$U[\partial\xi]=S[\xi]$$
$$=\tau\exp\left\{ig\int_0^1 d\tau\int_0^1 ds\,\frac{\partial\xi^{\mu}}{\partial s}\,\frac{\partial\xi^{\nu}}{\partial\tau}\,U[{}_0\xi(\tau)_s]\times F_{\mu\nu}(\xi(\tau,s))U[{}_s\xi(\tau)_0]\right\}. \tag{3.5}$$

Here τ exp indicates ordering of the τ integrals in the exponent [as Eq. (3.1) was ordered in the s integrals] whereas the s integrals are not ordered, and in the string variables $U[{}_s\xi(\tau)_0]$ and $U[{}_0\xi(\tau)_s]=U^{-1}[{}_s\xi(\tau)_0]$ we have used for the path $\xi(\tau)$ the notation in Eq. (3.2). The explicit dependence on the gauge potential that these string variables introduce will be removed later by exploiting the freedom of parametrization of the sheet ξ.

We now turn to the problem of computing the vacuum expectation value of arbitrary loops and products of loops in the two-dimensional non-Abelian theory. As in the Abelian case, the functional integral will be computed in the generalized axial gauge with $A_1^a=0$ everywhere and A_2^a vanishing along a vertical line that can be located anywhere but, for definiteness, we will always choose to be at the left of all the loops. In this section we will prove that this general problem can be reduced to that of computing expectation values of the form

$$\langle(U_{k+1}U_1)^{\alpha_1}_{\beta_1}\cdots(U_{2k}U_k)^{\alpha_k}_{\beta_k}\rangle\,, \tag{3.6}$$

where the string variables U_i correspond to $2k$ open paths located in some arbitrary order along a horizontal strip of constant width and the brackets stand for a functional integral in the generalized axial gauge described above. The explicit evaluation of these expectation values will be the subject of Sec. IV.

Consider one or more arbitrary loops and let us divide two-dimensional space-time into horizontal strips chosen so as to have in each strip an even number of paths, half of them going up and half going down, and any horizontal portions of the

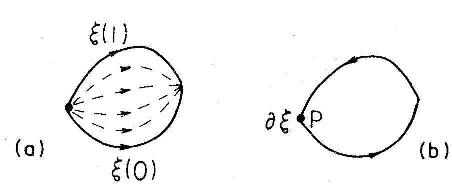


FIG. 2. (a) A one-parameter family of paths used in the non-Abelian Stokes's theorem. (b) Boundary of the sheet ξ in (a) oriented according to the initial path $\xi(0)$. The point P from where all the paths $\xi(\tau)$ emerge is the base point for the corresponding loop variable $U[\partial\xi]$.

loops falling on the boundaries of the strips (Fig. 3). Each of the loop variables in the expectation value to be computed can now be written as a product of string variables U_i corresponding to the open paths γ_i contained in each of the strips. Given the choice of gauge we have made, each of these string variables can be thought of as the only nontrivial portion of an untraced loop extending to the left of the corresponding path γ_i up to the line along which $A_2=0$ (Fig. 4). For each of these loops we use Stokes's theorem with the corresponding sheets ξ_i chosen as shown in Fig. 5(a) if the path γ_i travels upwards, or Fig. 5(b) if it travels downwards. In either case the sheets ξ_i satisfies the condition of fixed end points, Eq. (3.3), and its initial and final paths, $\xi_i(0)$ and $\xi_i(1)$, are properly oriented as indicated by Eq. (3.4). But, given the choice of gauge, for each τ the string variables $U[{}_s\xi_i(\tau)_0]$ entering Eq. (3.5) are the identity matrix unless s is large enough so the point $\xi_i^\mu(\tau, s)$ lies on the path γ_i. However, with our choice of ξ_i, on the path γ_i the two tangents $\partial\xi_i^\mu/\partial s$ and $\partial\xi_i^\mu/\partial\tau$ are parallel[1] and therefore do not contribute to the integral in (3.5) which then reduces to

$$U_i = \tau \exp\left[ig\int_0^1 d\tau \int_0^1 ds\, \epsilon_{\mu\nu}\, \frac{\partial\xi_i^\mu}{\partial s}\ \frac{\partial\xi_i^\nu}{\partial\tau}\, F(\xi_i(\tau, s))\right], \tag{3.7}$$

thus eliminating the explicit dependence on the gauge potential.

We can now change the integration variable in the functional integral from the gauge potential in the generalized axial gauge to the field strength in the same way as in the Abelian case [Eq. (2.8)] and we are left to compute an integral of the form

$$\frac{1}{Z}\int \prod_{a,x} dF^a(x) \times \exp\left[-\tfrac{1}{2}\int d^2x\, F^a(x)F^a(x)\right]\left(\operatorname{tr}\prod_i U_i\right)\cdots, \tag{3.8}$$

where the factors U_i are given by Eq. (3.7) and therefore depend only on the field strength $F(x)$.

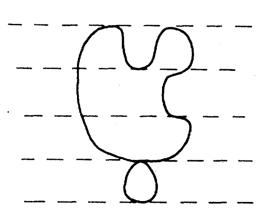


FIG. 3. Division of a loop into strips, chosen so the configuration of paths does not change within each strip.

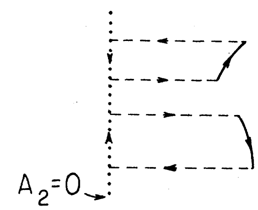


FIG. 4. Open paths as borders of closed loops in the $A_1=0$ gauge, with one vertical border running along the line where $A_2=0$.

Since the action in (3.8) contains no derivatives there are no "nearest-neighbor interactions" and the functional integral over all $F^a(x)$ with x outside of all the sheets ξ_i factors out and cancels with the corresponding factor in the denominator. The remaining integral factors as well into integrals over $F^a(x)$ with x in each of the strips into which we have divided the loops.

Let us consider first the functional integral over the top strip [Fig. 6(a)]. Since all the paths running upwards across this strip must turn around to emerge at the bottom, the expectation value to be computed is of the form (3.6) with the string variables U_i, $i=1,\ldots,k$ corresponding to upgoing paths and U_{k+i}, $i=1,\ldots,k$ corresponding to paths oriented downwards. Since the generalized axial-gauge condition is preserved by arbitrary global gauge transformations and the integration measure is gauge invariant, it is easy to see that this expectation value is invariant under all global rotations and therefore it must be of the form[2]

$$\langle (U_{k+1}U_1)^{\alpha_1}_{\beta_1}\cdots(U_{2k}U_k)^{\alpha_k}_{\beta_k}\rangle = \sum_P L_P\,\delta^{\alpha_1}_{\beta_{P(1)}}\cdots\delta^{\alpha_k}_{\beta_{P(k)}}, \tag{3.9}$$

where the sum runs over all permutations of the indices $1, 2, \ldots, k$.

In general, some of the group indices in this expression must be contracted when the corres-

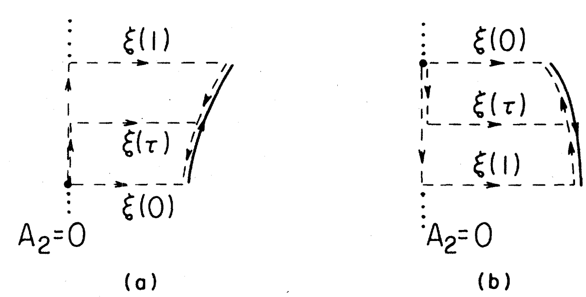


FIG. 5. Sheets used to write string variables (corresponding to open paths) in terms of Stokes's theorem. (a) for a path running upwards, (b) for a path oriented downwards.

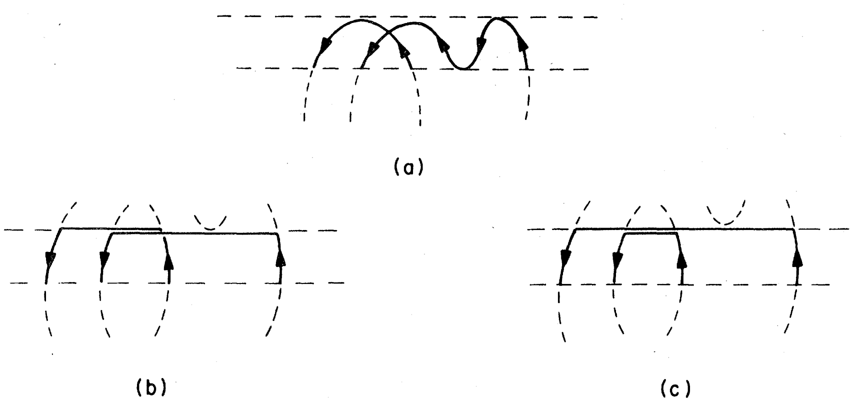


FIG. 6. Computing a loop average one strip at a time. After removing the upper strip in (a) we are left with two loops to evaluate, whose upper strips are shown in (b) and (c).

ponding paths join with each other at the bottom of the strip, whereas the free, uncontracted indices correspond to the paths proceeding in their way down or coming in their way up from the lower portions of the loop. Then, after contracting the appropriate indices and inserting this expression in the original functional integral in Eq. (3.8), we obtain a sum of integrals each of the same general form as the one we started with but extending over a smaller region of space-time since one of the horizontal strips has been removed [Figs. 6(b) and 6(c)].

We can then proceed in this way to perform the integration over each of the horizontal strips, one at a time, until all of them have been contracted thus leaving a gauge-invariant answer. The problem of computing all gauge-invariant correlation functions of the theory reduces in this way to that of evaluating the coefficients L_P in Eq. (3.9).

IV. EXPECTATION VALUE OF STRING VARIABLES

We must now compute the expectation values of the form given in Eq. (3.9). Except in a particular example worked out at the end of this section, the gauge group of the theory is taken to be U(N) and the string variables to be in the fundamental representation. Other gauge groups and representations can be discussed in a similar way.

In order to simplify the discussion we will assume that all paths entering Eq. (3.9) are vertical, although this is in no way essential to the calculation and the results will not depend on it. We have then an even number of paths of equal length located along a horizontal strip of constant width. Let us denote by $\gamma_1, \gamma_2, \ldots, \gamma_k$ the paths oriented upwards, labeled from left to right in order of increasing subindex, and by $\gamma_{k+1}, \ldots, \gamma_{2k}$ the paths oriented downwards labeled also in the same order. The relative position of these two sets of paths is arbitrary and we must compute

$$\langle (U_{k+1}U_{P(1)})^{\alpha_1}_{\beta_{P(1)}} \cdots (U_{2k}U_{P(k)})^{\alpha_k}_{\beta_{P(k)}} \rangle = \sum_Q L_{P,Q}\,\delta^{\alpha_1}_{\beta_{Q(1)}} \cdots \delta^{\alpha_1}_{\beta_{Q(k)}}\,, \tag{4.1}$$

where P and Q are permutations of the indices $1,2,\ldots,k$ and the quantity in brackets is evaluated in the generalized axial gauge with $A_1=0$ everywhere and A_2 vanishing along a vertical line which we choose at the left of all the paths. [The permutation P was not included in Eq. (3.9) since there the paths were not labeled in any particular order.]

Let us divide each of the paths γ_i and γ_{k+i}, $i=1,\ldots,k$, into n segments of equal length, and denote by $U_{i,l}$ and $U_{k+i,l}$ the strings variables corresponding to the lth segment of these paths, with the segments being labeled from bottom to top (Fig. 7). Set

$$\left.\begin{aligned} U_i(l) &= U_{i,l}\cdots U_{i,2}U_{i,1} \\ U_{k+i}(l) &= U_{k+i,1}U_{k+i,2}\cdots U_{k+i,l} \end{aligned}\right\} \begin{aligned} &(i=1,\ldots,k)\,, \\ &(l=1,2,\ldots,n)\,, \end{aligned} \tag{4.2a}$$

so

$$\begin{aligned} U_i &= U_i(n)\,, \\ U_{k+i} &= U_{k+i}(n)\,. \end{aligned} \tag{4.2b}$$

For each of the string variables $U_{i,l}$ and $U_{k+i,l}$ we use Stokes's theorem with the corresponding sheets chosen in an analogous way as before (Fig. 5). Then the functional integral, written in terms of the field strength, factorizes again into n hori-

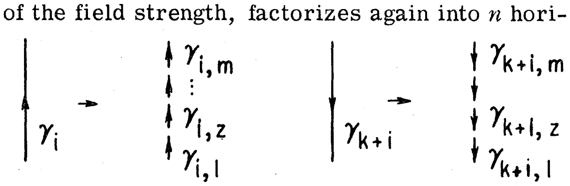


FIG. 7. Labeling of the segments of paths [Eq. (4.2a)].

zontal strips and in each of them we have, as in Eq. (4.1),

$$\langle (U_{k+1,l}U_{P(1),l})^{\rho_1}_{\rho_{P(1)}} \cdots (U_{2k,l}U_{P(k),l})^{\rho_k}_{\rho_{P(k)}} \rangle = \sum_Q B_{P,Q} \delta^{\rho_1}_{\rho_{Q(1)}} \cdots \delta^{\rho_k}_{\rho_{Q(k)}}, \quad (4.3)$$

where, by translational invariance, the coefficients $B_{P,Q}$ do not depend on l. Then, performing first the integration over the field strengths defined over the nth horizontal strip we have, using Eqs. (4.2) and (4.3),

$$\begin{aligned}\langle (U_{k+1}U_{P(1)})^{\alpha_1}_{\beta_{P(1)}} \cdots (U_{2k}U_{P(k)})^{\alpha_k}_{\beta_{P(k)}} \rangle &= \sum_Q B_{P,Q}\delta^{\rho_1}_{\rho_{Q(1)}} \cdots \delta^{\rho_k}_{\rho_{Q(k)}} \langle (U_{k+1}(n-1))^{\alpha_1}_{\rho_1} (U_{P(1)}(n-1))^{\rho_{P(1)}}_{\beta_{P(1)}} \cdots (U_{2k}(n-1))^{\alpha_k}_{\rho_k} (U_{P(k)}(n-1))^{\rho_{P(k)}}_{\beta_{P(k)}} \rangle \\ &= \sum_Q B_{P,Q} \langle (U_{k+1}(n-1)U_{Q(1)}(n-1))^{\alpha_1}_{\beta_{Q(1)}} \cdots (U_{2k}(n-1)U_{Q(k)}(n-1))^{\alpha_k}_{\beta_{Q(k)}} \rangle .\end{aligned}$$

Each of the brackets in the sum is of the same form as the one we started with and after iterating this equation n times we get, regarding the coefficients $B_{P,Q}$ as the elements of a matrix B,

$$L_{P,Q} = (B^n)_{P,Q} . \quad (4.4)$$

However, the coefficients $L_{P,Q}$ are independent of n and therefore Eq. (4.4) requires B to be of the form

$$B = e^{-H/n} \quad (4.5)$$

and we get

$$L_{P,Q} = (e^{-H})_{P,Q} . \quad (4.6)$$

The functional integral corresponding to the bracket in Eq. (4.3) has been written in terms of the field strengths, and each of the string variables stands for a τ-ordered expansion given by Stokes's theorem as written in Eq. (3.7). This allows the computation of the coefficients $B_{P,Q}$ in a power series in $1/n$, with the coefficients of the linear order terms being the elements of the matrix H in Eq. (4.5). (n need not be large; the argument follows from n being arbitrary but at no point do we need to take the limit $n \to \infty$.) For each of the string variables let us write the τ-ordered expansion symbolically as

$$U_i = 1_i + \mathfrak{F}_i + \mathfrak{F}_i^{\,2} + \cdots ,$$

where 1_i is the identity matrix,

$$\mathfrak{F}_i = ig \int_0^1 d\tau \int_0^1 ds\, \epsilon_{\mu\nu} \frac{\partial \xi_i^\mu}{\partial s} \frac{\partial \xi_i^\nu}{\partial \tau} F^a(\xi_i(\tau, s)) T_a \quad (4.7a)$$

and

$$\begin{aligned}\mathfrak{F}_i^{\,2} = (ig)^2 \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^1 ds_1 &\times \int_0^1 ds_2 \epsilon_{\mu\nu} \frac{\partial \xi_i^\mu}{\partial s}(\tau_1 s_1) \frac{\partial \xi_i^\nu}{\partial \tau}(\tau_1 s_1) \\ &\times \epsilon_{\lambda\rho} \frac{\partial \xi_i^\lambda}{\partial s}(\tau_2 s_2) \frac{\partial \xi_i^\rho}{\partial \tau}(\tau_2 s_2) \\ &\times F^a(\xi_i(\tau_1 s_1)) F^b(\xi_i(\tau_2 s_2)) T_a T_b . \end{aligned} \quad (4.7b)$$

[In order to simplify the expressions here and in the next few equations we let the indices i,j run from 1 to $2k$. Also, at the risk of confusing the reader, in Eqs. (4.7) we have suppressed the index l. Yet, these expressions refer to the string variables in (4.3) and not (4.1).]

Let also

$$\left.\begin{aligned} &S_i = \text{area of strip between } \gamma_i \text{ and } A_2 = 0 \text{ line} \\ &S_{i,j} = \min\{S_i, S_j\} \end{aligned}\right\} (i = 1, \ldots, 2k) \quad (4.8)$$

so the sheets ξ_i in Eqs. (4.7) have areas $(1/n)S_i$ (and not S_i). The lowest-order nontrivial contributions to the brackets in Eq. (4.3) are terms of the form $\langle \mathfrak{F}_i^{\,2}\rangle$, $\langle \mathfrak{F}_i \cdot \mathfrak{F}_j\rangle$ and $\langle \mathfrak{F}_i \times \mathfrak{F}_j\rangle$, where in $\langle \mathfrak{F}_i \cdot \mathfrak{F}_j\rangle$ the U(N) generators are multiplied together whereas in $\langle \mathfrak{F}_i \times \mathfrak{F}_j\rangle$ they are not. Using

$$\frac{1}{Z} \int \prod_{a,x} F^a(x) \exp\left[-\tfrac{1}{2}\int d^2x\, F^a(x)F^a(x)\right] F^a(x_1)F^b(x_2) = \delta^{ab}\delta^{(2)}(x_1 - x_2)$$

we get, for the $\langle \mathfrak{F}_i^{\,2}\rangle$ terms,

$$\begin{aligned}\langle \mathfrak{F}_i^{\,2}\rangle = (ig)^2 N 1 \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^1 ds_1 &\times \int_0^1 ds_2 \epsilon_{\mu\nu} \frac{\partial \xi_i^\mu}{\partial s} \frac{\partial \xi_i^\nu}{\partial \tau}(\tau_1, s_1) \\ &\times \epsilon_{\lambda\rho} \frac{\partial \xi_i^\lambda}{\partial s} \frac{\partial \xi_i^\rho}{\partial \tau}(\tau_2 s_2) \\ &\times \delta(\xi_i(\tau_1 s_1) - \xi_i(\tau_2 s_2)) ,\end{aligned}$$

where we used $\delta^{ab}T_aT_b = N1$. The integrals in this expression are invariant under arbitrary reparametrizations of ξ_i and therefore become an ordinary double surface integral with the τ ordering producing a factor of $\frac{1}{2}$. Since the surface of integration has area $(1/n)S_i$ we get

$$\langle \mathfrak{F}_i^{\,2}\rangle = -\frac{1}{n}\frac{g^2}{2} N S_i 1 . \quad (4.9a)$$

The contribution from the terms $\langle \mathfrak{F}_i \cdot \mathfrak{F}_j \rangle$ is found in a similar way. This time the $\delta^{(2)}$ function coming from the Wick contraction reduces the double surface integral to the region of overlap of the sheets ξ_i and ξ_j which has area $(1/n)S_{i,j}$. Since all these terms come from the products in Eq. (4.3), like $(U_{k+1,l}U_{P(1),l})$, where the two string variables correspond to paths traveling in opposite directions, the two sheets ξ_i and ξ_j have opposite orientations (cf. Fig. 5) and since this time the double surface integral is not ordered, we get

$$\langle \mathfrak{F}_i \cdot \mathfrak{F}_j \rangle = +\frac{1}{n} g^2 N S_{i,j} 1 . \tag{4.9b}$$

Finally, in the terms $\langle \mathfrak{F}_i \times \mathfrak{F}_j \rangle$ the two generators are not multiplied, and using

$$\delta^{ab}(T_a)^{\alpha_1}_{\beta_1}(T_b)^{\alpha_2}_{\beta_2} = \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}$$

we get

$$\langle (\mathfrak{F}_i)^{\alpha_i}_{\beta_i} \times (\mathfrak{F}_j)^{\alpha_j}_{\beta_j} \rangle = \mp \frac{1}{n} g^2 S_{i,j} \delta^{\alpha_i}_{\beta_j} \delta^{\alpha_j}_{\beta_i} , \tag{4.9c}$$

where the sign is minus (plus) if the two sheets ξ_i and ξ_j have the same (opposite) orientation. That is, the sign is minus if $i,j \leqslant k$ or $i,j \geqslant k$ and plus otherwise.

Putting all this together we get

$$\begin{aligned}\langle (U_{k+1,l}U_{P(1),l})^{\alpha_1}_{\beta_{P(1)}} \cdots (U_{2k,l}U_{P(k),l})^{\alpha_k}_{\beta_{P(k)}} \rangle = {} & \left[1 - \frac{1}{n}\frac{g^2N}{2}\sum_{i=1}^{k}(S_{P(i)} + S_{k+i} - 2S_{P(i),k+i}) + O(1/n^2)\right]\delta^{\alpha_1}_{\beta_{P(1)}} \cdots \delta^{\alpha_k}_{\beta_{P(k)}} \\ & + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\left[\frac{1}{n} g^2(-S_{k+i,k+j} + S_{k+i,P(j)} + S_{P(i),k+j} - S_{P(i),P(j)}) + O(1/n^2)\right] \\ & \times \delta^{\alpha_1}_{\beta_{P(1)}} \cdots \delta^{\alpha_i}_{\beta_{P(j)}} \cdots \delta^{\alpha_j}_{\beta_{P(i)}} \cdots \delta^{\alpha_k}_{\beta_{P(k)}} + O(1/n^2) .\end{aligned}$$

The coefficients $H_{P,Q}$, as defined in Eq. (4.5), can be read from the terms of order $1/n$ in this expression and we get

$$H_{P,P} = +\frac{g^2N}{2}\sum_{i=1}^{k} A_{P(i),k+i} , \tag{4.10a}$$

$$\begin{aligned} H_{P,t_{ij}\cdot P} = {} & -g^2(-S_{k+i,k+j} + S_{k+i,P(j)} \\ & + S_{P(i),k+j} - S_{P(i),P(j)}) \\ & (i,j = 1, \dots, k;\ i \neq j) ,\end{aligned} \tag{4.10b}$$

$$H_{P,Q} = 0 \quad \text{if } Q \neq t \cdot P , \tag{4.10c}$$

where t stands for a transposition, t_{ij} being the transposition of i and j, and in Eq. (4.10a) we have denoted by $A_{i,j}$ the combination

$$A_{i,j} = S_i + S_j - 2S_{i,j} \quad (i,j = 1, \dots, 2k;\ i \neq j) , \tag{4.11}$$

which is the area of the rectangle located between the paths γ_i and γ_j. The combination of $S_{i,j}$'s in Eq. (4.10b) can also be written always in terms of the areas $A_{i,j}$ but the explicit expression depends in general on the configuration of paths and the permutation P. Although these results were derived making reference to the line along which $A_2 = 0$ being located at the left of all the paths, it is easy to see that they remain the same regardless of the position of that line.

Equations (4.10) and (4.6) provide the complete, although implicit solution to the problem. An explicit expression for the coefficients $L_{P,Q}$ requires the diagonalization of the matrix H, but in general it is only after the geometrical configuration of paths is specified that the off-diagonal elements $H_{P,t\cdot P}$ acquire special symmetries that allow their explicit diagonalization and the subsequent explicit solution for L.

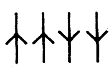
By the way of example consider the case with $k=2$ and the paths $\gamma_1, \dots, \gamma_4$ placed as shown in the first entry in Table I. The matrix H becomes

$$\begin{aligned} H &= \begin{bmatrix} \frac{g^2N}{2}(A_{13}+A_{24}) & g^2A_{23} \\ g^2A_{23} & \frac{g^2N}{2}(A_{23}+A_{14}) \end{bmatrix} \\ &= \begin{bmatrix} \frac{g^2N}{2}(A_{12}+2A_{23}+A_{34}) & g^2A_{23} \\ g^2A_{23} & \frac{g^2N}{2}(A_{12}+2A_{23}+A_{34}) \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} d_+ & 0 \\ 0 & d_- \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} ,\end{aligned}$$

where $d\pm = \frac{1}{2}g^2[N(A_{12}+2A_{23}+A_{34}) \pm 2A_{23}]$. Then, arranging the coefficients $L_{P,Q}$ in the same matrix form, we have

$$\begin{aligned} L &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-(d_+)} & 0 \\ 0 & e^{-(d_-)} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \exp\left[-\frac{g^2N}{2}(A_{12}+2A_{23}+A_{34})\right] \end{aligned}$$

TABLE I. The expectation values in Eq. (4.1) when $k=2$. A_{ij}, $i,j=1,\ldots,4$, is the area of the horizontal strip extending between the paths i and j.

Path configuration	$\langle (U_3U_1)^{\alpha_1}_{\beta_1}(U_4U_2)^{\alpha_2}_{\beta_2}\rangle$	$\langle (U_3U_2)^{\alpha_1}_{\beta_2}(U_4U_1)^{\alpha_2}_{\beta_1}\rangle$
 1 2 3 4	$\exp\left[-\frac{g^2N}{2}(A_{12}+2A_{23}+A_{34})\right] \times [\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\cosh g^2A_{23}-\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\sinh g^2A_{23}]$	$(\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\cosh^2 gA_{23}-\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\sinh g^2A_{23}) \times\exp\left[-\frac{g^2N}{2}(A_{12}+2A_{23}+A_{34})\right]$
 1 3 4 2	$\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\exp\left[-\frac{g^2N}{2}(A_{13}+A_{24})\right]$	$\left\{\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\exp(-g^2NA_{34})+\frac{1}{N}\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}[1-\exp(-g^2NA_{34})]\right\} \times\exp\left[-\frac{g^2N}{2}(A_{13}+A_{24})\right]$
 1 3 2 4	$\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\exp\left[-\frac{g^2N}{2}(A_{13}+A_{24})\right]$	$\left\{\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\exp(-g^2NA_{23})+\frac{1}{N}\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}[1-\exp(-g^2NA_{23})]\right\} \times\exp\left[-\frac{g^2N}{2}(A_{13}+A_{24})\right]$

$$\times\begin{bmatrix}\cosh(g^2A_{23}) & -\sinh(g^2A_{23})\\ -\sinh(g^2A_{23}) & \cosh(g^2A_{23})\end{bmatrix}$$

Thus, for this configuration of paths we obtain

$$\langle (U_3U_1)^{\alpha_1}_{\beta_1}(U_4U_2)^{\alpha_2}_{\beta_2}\rangle=\exp\left[-\frac{g^2N}{2}(A_{12}+2A_{23}+A_{34})\right]\times[\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\cosh(g^2A_{23})-\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\sinh(g^2A_{23})] \quad (4.12a)$$

and

$$\langle (U_3U_2)^{\alpha_1}_{\beta_2}({}_4U_1)^{\alpha_2}_{\beta_1}\rangle=\exp\left[-\frac{g^2N}{2}(A_{12}+2A_{23}+A_{34})\right]\times[\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\cosh(g^2A_{23})-\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\sinh(g^2A_{23})]\,. \quad (4.12b)$$

The explicit results for the other possible configurations with $k=2$ are listed in Table I. [The case of $k=1$ is included in the general result in Eq. (4.14).[3]]

There is a particularly simple class of configurations of paths that deserve mentioning since the corresponding expectation value in Eq. (4.1) can be solved in closed form for arbitrary k, arbitrary gauge group, and with the string variables written in an arbitrary representation of the group. As can be expected, such path configurations are the ones involved, in the way discussed in Sec. III, in the evaluation of the averages of loops and products of loops that do not overlap with themselves or with each other (although they can intercept themselves by bending towards the outside of the loop). Consider the case in which for each i, $i=1,\ldots,k$ the paths γ_i and γ_{k+i} stand next to each other, either one to the left of the other but with no other paths in between (Fig. 8). We compute the expectation value in Eq. (4.1) with P being the identity permutation

$$\langle (U_{k+1}U_1)^{\alpha_1}_{\beta_1}\cdots(U_{2k}U_k)^{\alpha_k}_{\beta_k}\rangle=\sum_Q L_{1,Q}\delta^{\alpha_1}_{\beta_{Q(1)}}\cdots\delta^{\alpha_k}_{\beta_{Q(k)}}\,.$$

In the case of U(N) with the string variables in the fundamental representation we have, for Eq. (4.10b),

$$S_{k+i,k+j}-S_{k+i,j}-S_{i,k+j}+S_{i,j}=0 \quad (i\neq j)\,,$$

so $H_{1,t_{ij}}=0$ and we get

$$L_{1,1}=\exp(H_{1,1})=\exp\left(-\frac{g^2N}{2}\sum_{i=1}^{k}A_{i,k+i}\right),$$

$$L_{1,P}=0 \quad \text{if } P\neq 1\,.$$

For an arbitrary group, with the string variables in an arbitrary representation, the previous calculation can be repeated step by step. The results in Eq. (4.9a) and (4.9b) still hold except that the quadratic Casimir constant N of the fundamental representation of U(N) is now replaced by the

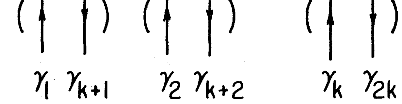


FIG. 8. Configuration of paths entering Eq. (4.14). The relative position of the two paths enclosed in each set of parentheses is arbitrary.

appropriate Casimir constant C:

$$\delta^{ab}T_aT_b=C1\,. \tag{4.13}$$

The contribution from the cross terms in Eq. (4.9c) need not be computed since, as in the case of U(N), with the paths in this configuration they all cancel against each other. Thus, for the paths in Fig. 8 we get

$$\langle (U_{k+1}U_1)^{\alpha_1}_{\beta_1}\cdots(U_{2k}U_k)^{\alpha_k}_{\beta_k}\rangle = \delta^{\alpha_1}_{\beta_1}\cdots\delta^{\alpha_k}_{\beta_k}\exp\left[-\frac{g^2C}{2}\sum_{i=1}^{k}A_{i,k+i}\right], \tag{4.14}$$

which holds for arbitrary representation.

In the large-N limit of the theory, where g^2N is kept fixed while $N\to\infty$, the matrix H in Eqs. (4.10) becomes diagonal and one could expect that, to leading order, only the diagonal coefficient $L_{P,P}$ would survive in Eq. (4.1). This would imply the complete factorizability of the expectation values of arbitrary loops and products of loops in this limit, including that of self-overlapping loops (i.e., a free theory of strings). However, as we will show by explicit examples in Sec. V, the latter does not hold and off-diagonal coefficients $L_{P,Q}$ can contribute to leading order in the $N\to\infty$ limit. This arises because the final gauge-invariant expressions for loop averages involve appropriate contractions of the group indices in Eq. (4.1) which can result in powers of N multiplying the off-diagonal coefficients $L_{P,Q}$. Nevertheless, provided one keeps terms of appropriate order in $1/N$ for each specific loop to be evaluated, in the large-N computations one can avoid the diagonalization of the matrix H by writing the coefficients $L_{P,Q}$ in an explicit power series in $1/N$. Separating H in Eqs. (4.10) into its diagonal and off-diagonal pieces we write

$$H=(g^2N)\left(H_0+\frac{1}{N}H'\right) \tag{4.15}$$

and Eq. (4.6) can be written as

$$L=\exp[-(g^2N)H_0]\,T\exp\left[-\frac{1}{N}\int_0^{g^2N}dt\,(e^{tH_0}H'e^{-tH_0})\right], \tag{4.16}$$

where T stands for t ordering. Since H_0 is diagonal, the coefficients in this expansion are easily evaluated to any desired order.

V. EXPLICIT RESULTS AND DISCUSSIONS

We now apply the general results derived in the previous sections to evaluate the expectation values of some specific loops which are of interest in the understanding of non-Abelian gauge theories.

A. Planar loops and confinement

We first consider the simplest, yet important case of loops that do not overlap. Since in this case we have the general result in Eq. (4.14) we need not specify the gauge group or the representation carried by the loop variable. Taking the trace of that equation with $k=1$ we get, for a simple rectangular loop,

$$\left\langle\frac{1}{N}\,\mathrm{tr}U[\gamma]\right\rangle=\exp\left(-\frac{g^2C}{2}A\right), \tag{5.1}$$

where A is the area enclosed by the loop γ and N is now the dimension of the representation of the group. In the case of U(N) with the loop in the fundamental representation, this result, with $C=N$, agrees with the original computation of 't Hooft[4] obtained in the large-N limit (which in this case is exact), and with weak-coupling lattice calculations.[5]

Thus, in the two-dimensional theory the expectation value of the Wilson loop obeys the area law regardless of the gauge group and the representation carried by the loop, with the representation entering only through the appropriate quadratic Casimir multiplying the coupling constant. This is of course not surprising since in two dimensions, that is, (1+1)-dimensional space-time, there is no room for the color flux to spread out, so a quark-antiquark pair in a physical gauge-invariant state will necessarily be joined by a string of color flux giving rise to a confining linear potential. The different representations that the quarks may carry amount only to different color charges given by $g\sqrt{C}$ which give rise to the result in Eq. (5.1). On the other hand, the usual argument for the absence of confinement of triality-zero quarks, that is that they can form bound states with gluons thus having no long-range interactions, does not hold here since, in two dimensions, there are no physical gluons.

Following the steps outlined in Sec. III, we can use the result in Eq. (4.14) to evaluate the expectation values of arbitrary nonoverlapping loops, even if self-intersecting, obtaining always an area-law result (cf. first entry in Table II).

B. Nonplanar loops and $N\to\infty$ limit

Let us consider now the expectation value of the self-overlapping loop in Fig. 10(b), now in the fundamental representation of U(N). The contribution from the region with no overlap is obtained by using Eq. (4.14) which leaves us with the expectation value of the loop shown in Fig. 9. This can be evaluated using Eq. (4.12a) with the trace involving in this case the contraction

$$\frac{1}{N}\,\delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_2}(\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\cosh g^2A' - \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\sinh g^2A') = \cosh g^2A' - N\sinh g^2A' ,$$

where A' is the area of overlap. Then, denoting by γ the exterior contour of the loop and by γ' the small loop in the inside, we obtain

$$\left\langle\frac{1}{N}\,\mathrm{tr}(U[\gamma]U[\gamma'])\right\rangle = (\cosh g^2A' - N\sinh g^2A') \times\exp\left[-\frac{g^2N}{2}\,(A+A')\right], \qquad (5.2)$$

where A is the total area enclosed by the loop γ (including the region of overlap).

In a similar way, using Eqs. (4.12) as well as the other results in Table I, one can evaluate the expectation values of the other overlapping loops shown in Fig. 10. The results are listed in Table II. (Notice that each of them reduces to the corresponding Abelian expression if one sets $N=1$.)

From the results in this table we see that in the $N\to\infty$ limit (with g^2N fixed) the expectation values of products of loops satisfy the factorization property

$$\left\langle\frac{1}{N}\,\mathrm{tr}U[\gamma_1]\,\frac{1}{N}\,\mathrm{tr}U[\gamma_2]\right\rangle = \left\langle\frac{1}{N}\,\mathrm{tr}U[\gamma_1]\right\rangle\left\langle\frac{1}{N}\,\mathrm{tr}U[\gamma_2]\right\rangle + O((1/N)^2) \qquad (5.3)$$

as required by general arguments in the $1/N$ expansion.[6] An analogous result does not hold however for loops that overlap with themselves. Indeed, corresponding to the loop in Fig. 10(b), for large N Eq. (5.2) reduces to

$$\left\langle\frac{1}{N}\,\mathrm{tr}(U[\gamma]U[\gamma'])\right\rangle = (1-g^2NA')\exp\left[-\frac{g^2N}{2}\,(A+A')\right] + O((1/N)^2) \neq \left\langle\frac{1}{N}\,\mathrm{tr}U[\gamma]\right\rangle\left\langle\frac{1}{N}\,\mathrm{tr}U[\gamma']\right\rangle. \qquad (5.4)$$

Thus, self-overlapping loops do not disentangle in the large-N limit. (A similar result has also been proved in the lattice case.[7])

One notices also that the correlation function in Eq. (5.2) is not a positive definite, vanishing, and changing sign when the area of overlap satisfies $\tanh g^2A' = 1/N$. A similar behavior is exhibited also by the average of the other self-overlapping loop given in Table II. It may seem disturbing that the sign of these loop averages depends only on the areas of overlap and therefore can be changed by rescaling the entire loop. However, this qualitative change under rescaling is a common feature of all correlation functions involving overlapping loops and, since in two dimensions the coupling has units of mass, is only when g^{-1} is changed in the same way as the length scales that the physical results remain invariant. Yet, the physical meaning of the indefinite sign of these loop averages, if any, is not clear at this point.

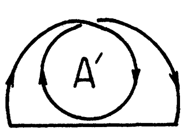


FIG. 9. Overlapping loop to be evaluated after removing the lower portion of the loop in Fig. 10(b).

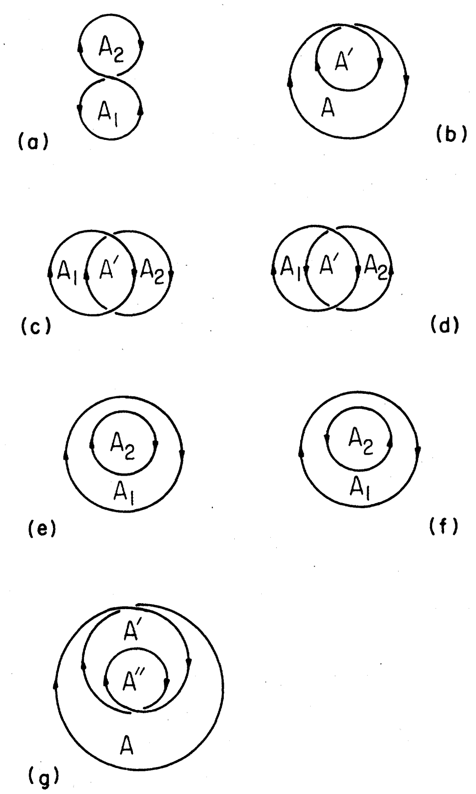


FIG. 10. Loops whose expectation values are listed in Table II. The areas of each region include always the area of the overlaps that occur within that region. So, for example, in (c) A_1 includes A' and in (g) A' includes A'' while A is the total area enclosed by the outer boundary.

TABLE II. Expectation values corresponding to the loops in Fig. 10, and their $N\to\infty$ limit.

Loop (figure)	$\langle\ \rangle$	$\langle\ \rangle_{N\to\infty}$
10(a)	$\exp\left[-\frac{g^2N}{2}(A_1+A_2)\right]$	
10(b)	$(\cosh g^2A' - N\sinh g^2A')\exp\left[-\frac{g^2N}{2}(A+A')\right]$	$(1-g^2NA')\exp\left[-\frac{g^2N}{2}(A+A')\right]+O(1/N^2)$
10(c)	$\left(\cosh g^2A' - \frac{1}{N}\sinh g^2A'\right)\exp\left[-\frac{g^2N}{2}(A_1+A_2)\right]$	$\exp\left[-\frac{g^2N}{2}(A_1+A_2)\right]+O(1/N^2)$
10(d)	$\left\{1-\frac{1}{N^2}[1-\exp(g^2NA')]\right\}\exp\left[-\frac{g^2N}{2}(A_1+A_2)\right]$	$\exp\left[-\frac{g^2N}{2}(A_1+A_2)\right]+O(1/N^2)$
10(e)	$\left(\cosh g^2A_2 - \frac{1}{N}\sinh g^2A_2\right)\exp\left[-\frac{g^2N}{2}(A_1+A_2)\right]$	$\exp\left[-\frac{g^2N}{2}(A_1+A_2)\right]+O(1/N^2)$
10(f)	$\left\{1-\frac{1}{N^2}[1-\exp(g^2NA_2)]\right\}\exp\left[-\frac{g^2N}{2}(A_1+A_2)\right]$	$\exp\left[-\frac{g^2N}{2}(A_1+A_2)\right]+O(1/N^2)$
10(g)	$\frac{1}{3}\{\cosh g^2(A'-A'')+2\cosh g^2(A'+2A'')$ $-3N\sinh g^2(A'+2A'')$ $+N^2[\cosh g^2(A'+2A'')-\cosh g^2(A'-A'')]\}$ $\times\exp\left[-\frac{g^2N}{2}(A+A'+A'')\right]$	$[1-g^2N(A'+2A'')+\frac{1}{2}(g^2N)^2A''(2A'+A'')]$ $\times\exp\left[-\frac{g^2N}{2}(A+A'+A'')\right]+O(1/N^2)$

C. Schwinger-Dyson equation for loops

Recently, there has been interest in studying the expectation value of non-Abelian loop operators by considering the variations induced by infinitesimal deformations of the contour. In the continuum formulation this leads to a second-order functional equation[8] which presents, however, technical difficulties regarding the limiting procedure involved in taking derivatives with respect to paths. A similar equation has been derived also in the lattice theory where it becomes a well-defined identity with the small deformations having the size of one fundamental plaquette.[9,10]

Borrowing the relevant diagrams from the lattice equation[11] and using the corresponding loop averages in Table II, one finds by direct substitution that

$$\lim_{A'\to 0}\frac{1}{2A'g^2N}(W_1+W_2-W_3-W_4)=W\,, \qquad (5.5)$$

where $W, W_1, \ldots, W_4$ are the expectation values corresponding to the loops $\gamma, \gamma_1, \ldots, \gamma_4$ in Fig. 11, and A' is the area of the small deformation in each of the loops $\gamma_1, \ldots, \gamma_4$. (This holds regardless of the shape of the loop γ away from the point where it is deformed since, as shown in Sec. III, the region of the deformation can always be isolated from the rest of the loop.) Thus, the continuum loop averages satisfy the naive continuum limit of the loop equation defined in the lattice, with the result being quite insensitive to the detailed shape of the small deformation.

Notice, however, that due to the negative result in Eq. (5.4), this loop equation does not linearize in the large-N limit and therefore, in terms of the

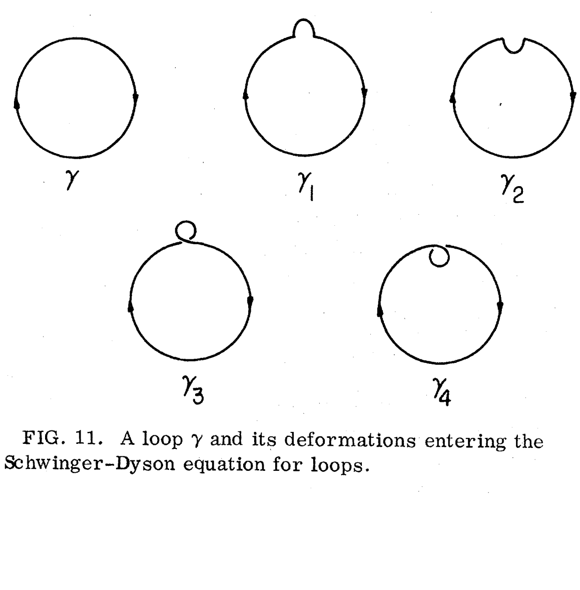


FIG. 11. A loop γ and its deformations entering the Schwinger-Dyson equation for loops.

propagation of strings, it includes nontrivial topological interactions.[10]

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APPENDIX: NON-ABELIAN STOKES'S THEOREM

We derive here a non-Abelian version of Stokes's theorem which, besides simplifying the computations in the two-dimensional theory, may be of interest in its own right. Throughout this appendix the gauge group, its representation, and the dimensionality of space-time are all arbitrary.

For our purposes it is convenient to define the non-Abelian group element $U[\gamma]$ by means of the differential equation describing parallel transport along the path γ. Using the notation introduced in Eqs. (3.2) this can be written as

$$\frac{d}{ds} U[{}_s\gamma_0]=igA_\mu(\gamma(s))\dot{\gamma}^\mu(s)U[{}_s\gamma_0] \tag{A1a}$$

with

$$U[{}_s\gamma_0]=1 \quad \text{at } s=0\,. \tag{A1b}$$

It is easy to show that all the usual properties of $U[\gamma]$ follow directly from these equations without need of using the explicit path-ordered expansion in Eq. (3.1). In particular, one has

$$U^{-1}[\gamma]=U[\gamma^{-1}] \text{ and } U[{}_{s_2}\gamma_s]U[{}_s\gamma_{s_1}]=U[{}_{s_2}\gamma_{s_1}] \tag{A2}$$

for arbitrary s_1, s_2, and s in the unit interval and, under a gauge transformation

$$A'_\mu=\Omega^{-1}A_\mu\Omega+\frac{i}{g}\Omega^{-1}\partial_\mu\Omega\,, \tag{A3a}$$

$U[\gamma]$ rotates to

$$U'[\gamma]=\Omega^{-1}(\gamma(1))U[\gamma]\Omega(\gamma(0))\,. \tag{A3b}$$

It also follows from Eqs. (A1) that

$$\frac{d}{ds} U[{}_1\gamma_s]=-igU[{}_1\gamma_s]A_\mu(\gamma(s))\dot{\gamma}^\mu(s) \tag{A4a}$$

with

$$U[{}_1\gamma_s]=1 \quad \text{at } s=1\,. \tag{A4b}$$

Let ξ now be a one-parameter family of paths, parametrized by τ, $0\le\tau\le1$, such that for all τ the paths $\xi(\tau)$ have fixed starting and ending points [Fig. 2(a)]:

$$\frac{\partial}{\partial\tau}\xi^\mu(\tau,s)=0 \quad \text{at } s=0,1\,. \tag{A5}$$

To each path $\xi(\tau)$ we assign the Lie algebra element

$$\mathcal{A}[\xi(\tau)]=iU^{-1}[\xi(\tau)]\frac{d}{d\tau}U[\xi(\tau)]\,, \tag{A6}$$

which is gauge covariant due to the fixed-end-point condition in (A5). Indeed, using Eq. (A3b) it follows that under the gauge transformation in (A3a) $\mathcal{A}[\xi(\tau)]$ rotates into

$$\mathcal{A}'[\xi(\tau)]=\Omega^{-1}(\xi(\tau,0))\mathcal{A}[\xi(\tau)]\Omega(\xi(\tau,0))\,. \tag{A7}$$

Given a loop γ we are interested in writing the group element $U[\gamma]$ in terms of a surface integral over an oriented two-dimensional sheet ξ whose boundary is given by γ. This amounts to associating a non-Abelian group element to a sheet, which we will do by means of an equation analogous to that in (A1). To that end let us carry over to the case of sheets the notation introduced in (3.2) for paths. Thus if $0\le\tau_1,\ \tau_2\le1$, we denote by ${}_{\tau_2}\xi_{\tau_1}$ that portion of the sheet ξ extending from the path $\xi(\tau_1)$ to the path $\xi(\tau_2)$ given by

$${}_{\tau_2}\xi_{\tau_1}(\tau)=\xi(\tau_1+(\tau_2-\tau_1)\tau)\,. \tag{A8a}$$

As before,

$$\begin{aligned}{}_1\xi_0&=\xi\,,\\ {}_0\xi_1&=\xi^{-1}\,.\end{aligned} \tag{A8b}$$

Then, using the Lie algebra element $\mathcal{A}[\xi(\tau)]$ in Eq. (A6) we define the group element $S[\xi]$ by means of the differential equation

$$\frac{d}{d\tau}S[{}_\tau\xi_0]=i\mathcal{A}[\xi(\tau)]S[{}_\tau\xi_0] \tag{A9a}$$

with

$$S[{}_\tau\xi_0]=1 \quad \text{at } \tau=0\,. \tag{A9b}$$

From here and (A6) it follows that

$$\frac{d}{d\tau}\left(U[\xi(\tau)]S[{}_\tau\xi_0]U^{-1}[\xi(0)]\right)=0$$

and using the boundary condition (A9b) we get

$$S[\xi]=U^{-1}[\xi(1)]U[\xi(0)]=U[\xi^{-1}(1)]U[\xi(0)]\,. \tag{A10a}$$

But, since the starting and ending points of the paths $\xi(0)$ and $\xi(1)$ coincide [cf. Eq. (3.7)] this result becomes

$$S[\xi]=U[\partial\xi]\,, \tag{A10b}$$

where $\partial\xi$ is the boundary of the sheet ξ, oriented according to the initial path $\xi(0)$ [Fig. 2(b)]. This equation has the desired form: it expresses the loop variable $U[\partial\xi]$ in terms of the sheet variable $S[\xi]$. It is a consequence of the pure gauge form of $\mathfrak{A}[\xi(\tau)]$ in Eq. (A6). The group element $S[\xi]$ can be interpreted as describing parallel transport along a path through "path space," that is, a sheet. But since the corresponding gauge connection $\mathfrak{A}[\xi(\tau)]$ has zero curvature, $S[\xi]$ depends only on the end points of that path, that is, the initial and final paths of the sheet ξ. From this point of view the "gauge potential" $\mathfrak{A}[\xi(\tau)]$ in Eq. (A6) is just a smooth, nonsingular version of Polyakov's chiral field in path space.[8]

To obtain an explicit expression for $S[\xi]$ we must write $\mathfrak{A}[\xi(\tau)]$ in terms of the original variables in the theory. Using the notation in (3.2) for the path $\xi(\tau)$ and replacing τ by $\tau+\delta\tau$ in $U[{}_s\xi(\tau)_0]$ we get

$$U[{}_s\xi(\tau+\delta\tau)_0]=U[{}_s\xi(\tau)_0]+\delta\tau\frac{d}{d\tau}U[{}_s\xi(\tau)_0]+\cdots .$$

Differentiating both sides with respect to s we get, using Eq. (A1a),

$$igA_\mu(\xi(\tau,s))\frac{\partial\xi^\mu}{\partial s}(\tau,s)U[{}_s\xi(\tau)_0]+\delta\tau\frac{d}{ds}\frac{d}{d\tau}U[{}_s\xi(\tau)_0]+O((\delta\tau)^2)=igA_\mu(\xi(\tau+\delta\tau,s))\frac{\partial\xi^\mu}{\partial s}(\tau+\delta\tau,s)U[{}_s\xi(\tau+\delta\tau)_0]$$

and expanding the right-hand side up to linear order in $\delta\tau$ we obtain

$$\frac{d}{ds}\frac{d}{d\tau}U[{}_s\xi(\tau)_0]=ig\partial_\nu A_\mu(\xi(\tau,s))\frac{\partial\xi^\mu}{\partial s}(\tau,s)\frac{\partial\xi^\nu}{\partial\tau}(\tau,s)U[{}_s\xi(\tau)_0]+igA_\mu(\xi(\tau,s))\frac{\partial^2\xi^\mu}{\partial s\partial\tau}(\tau,s)U[{}_s\xi(\tau)_0]+igA_\mu(\xi(\tau,s))\frac{\partial\xi^\mu}{\partial s}(\tau,s)\frac{d}{d\tau}U[{}_s\xi(\tau)_0]$$

or, using Eq. (A4),

$$\frac{d}{ds}\left(U[{}_1\xi(\tau)_s]\frac{d}{d\tau}U[{}_s\xi(\tau)_0]\right)=ig\frac{\partial\xi^\mu}{\partial s}(\tau,s)\frac{\partial\xi^\nu}{\partial\tau}(\tau,s)U[{}_1\xi(\tau)_s]\partial_\nu A_\mu(\xi(\tau,s))U[{}_s\xi(\tau)_0]+ig\frac{\partial^2\xi^\mu}{\partial s\partial\tau}(\tau,s)U[{}_1\xi(\tau)_s]A_\mu(\xi(\tau,s))U[{}_s\xi(\tau)_0].$$

We can simplify this further by using again Eqs. (A1) and (A4) to obtain

$$\frac{d}{ds}\left\{U[{}_1\xi(\tau)_s]\left[\frac{d}{d\tau}U[{}_s\xi(\tau)_0]-igA_\mu(\xi(\tau,s))\frac{\partial\xi^\mu}{\partial\tau}(\tau,s)U[{}_s\xi(\tau)_0]\right]\right\}=-ig\frac{\partial\xi^\mu}{\partial s}(\tau,s)\frac{\partial\xi^\nu}{\partial\tau}(\tau,s)U[{}_1\xi(\tau)_s]F_{\mu\nu}(\xi(\tau,s))U[{}_s\xi(\tau)_0],$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ig[A_\mu,A_\nu]$ is the field strength. The boundary condition in Eq. (A1b) requires

$$\frac{d}{d\tau}U[{}_s\xi(\tau)_0]=0\quad\text{at } s=0$$

and using also the boundary condition in (A4b) and the fixed end-point condition in Eq. (A5) we get, after integrating over s,

$$\frac{d}{d\tau}U[\xi(\tau)]=-ig\int_0^1 ds\,\frac{\partial\xi^\mu}{\partial s}(\tau,s)\frac{\partial\xi^\nu}{\partial\tau}(\tau,s)U[{}_1\xi(\tau)_s]\times F_{\mu\nu}(\xi(\tau,s))U[{}_s\xi(\tau)_0],\qquad\text{(A11)}$$

which is again a smooth nonsingular version of the small deformation equation for the string variable $U[\gamma]$.[8] Using Eqs. (A2) we finally obtain

$$\mathfrak{A}[\xi(\tau)]=g\int_0^1 ds\,\frac{\partial\xi^\mu}{\partial s}(\tau,s)\frac{\partial\xi^\nu}{\partial\tau}(\tau,s)U[{}_0\xi(\tau)_s]\times F_{\mu\nu}(\xi(\tau,s))U[{}_s\xi(\tau)_0].\qquad\text{(A12)}$$

Equations (A9) can be integrated, as in the case of string variables, into a τ-ordered exponential

$$S[\xi]=\tau\exp\left\{i\int_0^1 d\tau\,\mathfrak{A}[\xi(\tau)]\right\},$$

which together with Eq. (A12) gives

$$S[\xi]=\tau\exp\left\{ig\int_0^1 d\tau\int_0^1 ds\,\frac{\partial\xi^\mu}{\partial s}\frac{\partial\xi^\nu}{\partial\tau}U[{}_0\xi(\tau)_s]\times F_{\mu\nu}(\xi(\tau,s))U[{}_s\xi(\tau)_0]\right\}.\qquad\text{(A13)}$$

This, together with the previous result

$$S[\xi]=U[\partial\xi]=U^{-1}[\xi(1)]U[\xi(0)]\,, \tag{A10}$$

constitutes the non-Abelian version of Stokes's theorem we were after.

Notice that, in the Abelian case, this result reduces to the usual expression for Stokes's theorem and, in the case of a loop in two dimensions that overlaps itself, it provides the correct winding number factor in Eq. (2.7). Indeed, since the one-parameter family of paths $\xi(\tau)$ evolves smoothly from one side of the loop to the other, the sheet ξ will fold on itself in the regions of overlap the appropriate number of times so as to provide the correct counting as given by the winding number.

In the non-Abelian case the string variables $U[{}_s\xi(\tau)_0]$ in Eq. (A13) are necessary in order to obtain a gauge-invariant result for the flux of the non-Abelian field strength through a surface. They introduce an explicit dependence on the gauge potential but, as shown in the main text for the two-dimensional case, this dependence can often be removed by exploiting the freedom of parametrization of the sheet ξ in order to match an appropriate choice of axial gauge.

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[1]To be precise, this holds when the sheet does not fold on itself and, for each τ, the path $\xi(\tau)$ is parametrized with constant speed so $(\partial/\partial s)[(\partial/\partial s)\xi^\mu_\tau(\tau,s)]^2=0$.

[2]The argument here and the resulting tensor structure in Eq. (3.9) is the same as in the case of the group integrals encountered in the strong-coupling expansion in the lattice theory. See, for example, S. Samuel, IAS report, 1980 (unpublished).

[3]The case $k=3$ involves 6×6 matrices but they are fairly simple and can be diagonalized explicitly. The results have not been included in Table I to save space, but one of them was used to evaluate the loop average in the last entry in Table II. I am grateful to M. Sweeny for his help with the diagonalization of some of these matrices on the computer. (This was done with MACSYMA, which is supported by DOE and other agencies.)

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[11]We use the notation of Ref. 10. Notice that to match the continuum limit of the lattice action in that reference with our notation we must identify

$$g^2=\frac{1}{2}\frac{\bar{g}^2}{a^2}$$

in the limit a, $\bar{g}\to 0$, with $\bar{g}$ being the coupling in the lattice and a the lattice constant.