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Equilibrium parity-violating current in a magnetic field

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It is argued that if the Hamiltonian of a system of charged fermions does not conserve parity, then an equilibrium electric current parallel to $\vec{B}$ can develop in such a system in an external magnetic field $\vec{B}$. The equilibrium current is calculated (i) for noninteracting left-handed massless fermions and (ii) for a system of massive particles with a Fermi-type parity-violating interaction. In the first case a nonzero current is found, while in the second case the current vanishes in the lowest order of perturbation theory. The physical reason for the cancellation of the current in the second case is not clear and one cannot rule out the possibility that a nonzero current appears in other models.

I. INTRODUCTION

Pseudoscalar quantities, such as the average projection of particle's spin on its momentum $\langle\vec{\sigma}\cdot\vec{p}\rangle$, can take nonzero values in a macroscopic system of particles with a parity-violating interaction. A well-known example is given by neutrinos, for which the intrinsic parity nonconservation makes $\langle\vec{\sigma}\cdot\vec{p}\rangle$ equal to $-p$ even without interaction. For particles other than neutrinos, the self-energy correction due to the weak interaction can have terms proportional to $\vec{\sigma}\cdot\vec{p}$, and it is possible that electrons and other particles have small nonzero values of $\langle\vec{\sigma}\cdot\vec{p}\rangle$ proportional to the weak-interaction constant. Imagine now a system of charged spin-$\frac{1}{2}$ particles with (say) $\langle\vec{\sigma}\cdot\vec{p}\rangle>0$. In such a system particles have a tendency to move in the direction parallel to their spin. If an external magnetic field $\vec{B}$ is applied to the system, then the particles are partially polarized in the direction of $\vec{B}$ and one can expect an equilibrium electric current to develop in the direction parallel to the magnetic field. The purpose of the present paper is to examine this possibility. We shall first consider a hypothetical case of massless left-handed charged fermions in a magnetic field and then turn to a more realistic situation of massive particles with a parity-violating Fermi-type interaction. It will be shown that in the first case there exists an equilibrium current given by Eq. (18), while in the second case the current is equal to zero[1] (at least in the lowest order of perturbation theory). The physical reason for the cancellation of the current in the second case is not clear: the existence of a nonzero current is not forbidden by CP and CPT symmetries and by gauge invariance. At this point one cannot exclude the possibility that a nonzero equilibrium current appears in different models of weak interactions (e.g., in the Weinberg-Salam model at $T\gtrsim m_w$).

II. MASSLESS LEFT-HANDED PARTICLES

Let us consider a system of charged, massless, left-handed spin-$\frac{1}{2}$ particles in a constant magnetic field $\vec{B}$. We shall first find the solutions of the Dirac equation

$$\gamma^{\mu}(i\partial_{\mu}-eA_{\mu})\psi=0 \tag{1}$$

with the subsidiary condition

$$(1+\gamma^5)\psi=0\ . \tag{2}$$

Here I use the system of units in which $\hbar=c=1$ and the γ matrices are taken in the representation of Bjorken and Drell.[2] Equation (2) ensures the left-handedness of the particles. The vector potential $A_{\mu}(\vec{x})$ can be chosen as

$$A_0=0,\quad \vec{A}=\tfrac{1}{2}\vec{B}\times\vec{x}\ . \tag{3}$$

The solutions of Eqs. (1)–(3) can be found directly or using the known solutions for an electron in a constant magnetic field.[3] In the latter case the electron wave functions have to be multiplied by $(1-\gamma^5)$ and by an appropriate normalization factor.

The resulting solutions $\psi(\vec{x}\,|\,np_yp_z)$ are given by

$$\psi=\frac{1}{4\pi\sqrt{\epsilon}}\begin{pmatrix}(\epsilon-p_z)^{1/2}v_n(\xi)\\ i(\epsilon+p_z)^{1/2}v_{n-1}(\xi)\\ -(\epsilon-p_z)^{1/2}v_n(\xi)\\ -i(\epsilon+p_z)^{1/2}v_{n-1}(\xi)\end{pmatrix}\exp(ip_yy+ip_zz) \tag{4}$$

for $n\neq 0$ and

$$\psi=\frac{1}{2\pi\sqrt{2}}\begin{pmatrix}v_0(\xi)\\ 0\\ -v_0(\xi)\\ 0\end{pmatrix}\exp(ip_yy+ip_zz) \tag{5}$$

for $n=0$. Here

$$\epsilon_n^2 = p_z^2 + 2\,enB \tag{6}$$

are the energy eigenvalues,

$$v_n(\xi) = (eB)^{1/4}\pi^{-1/4}2^{-n/2}(n!)^{-1/2}e^{-\xi^2/2}H_n(\xi) , \tag{7}$$

$H_n(\xi)$ are Hermite polynomials, and

$$\xi = \sqrt{eB}\,(x - p_y/eB). \tag{8}$$

The functions $v_n(\xi)$ are defined so that

$$\int_{-\infty}^{\infty} v_n^2(\xi)dx = 1 . \tag{9}$$

The normalization condition is

$$\int \psi^\dagger(\vec{x}|np_yp_z)\psi(\vec{x}|n'p_y'p_z')d^3x = \delta_{nn'}\delta(p_y - p_y')\delta(p_z - p_z') . \tag{10}$$

The wave functions (4), (5) are spread in the directions of y and z and localized in the direction of x.

The equilibrium current density j_z can be written as

$$j_z(\vec{x}) = e\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z f(\epsilon_{np_z} - \chi) j_z(\vec{x}|np_yp_z) , \tag{11}$$

where

$$j_z(\vec{x}|np_yp_z) = \bar{\psi}(\vec{x}|np_yp_z)\gamma^3\psi(\vec{x}|np_yp_z) , \tag{12}$$

$$f(x) = (e^{\beta x}+1)^{-1} \tag{13}$$

is the Fermi distribution function, $\beta = T^{-1}$, T is the temperature, and χ is the chemical potential. At $T \neq 0$ antiparticles are also present in equilibrium with the chemical potential $-\chi$. The contribution of antiparticles to $j_z(\vec{x})$ can be easily found after calculating the integrals in Eq. (11) by changing χ to $-\chi$ and multiplying the whole expression by -1. [An alternative method is to find the antiparticle wave functions using the charge conjugation and to add the corresponding term to Eq. (11). Of course, both methods give the same result.] If $\chi \neq 0$, we shall assume the existence of a neutralizing background, so that the total charge density is zero everywhere.

Substituting the wave functions (4) and (5) in Eq. (12) we find

$$j_z(\vec{x}|np_yp_z) = -(8\pi^2\epsilon_n)^{-1}[(\epsilon_n - p_z)v_n^2(\xi) - (\epsilon_n + p_z)v_{n-1}^2(\xi)] \tag{14}$$

for $n \neq 0$ and

$$j_z(\vec{x}|0p_yp_z) = -(4\pi^2)^{-1}v_0^2(\xi) . \tag{15}$$

Now it is easily seen that after integration over p_y and p_z in Eq. (11) terms with $n\neq 0$ drop out and we obtain

$$j_z = -\frac{e^2B}{4\pi^2}\int_{-\infty}^{\infty} dp_z f(|p_z| - \chi) = -\frac{e^2B}{2\pi^2}\int_0^{\infty} dp f(p-\chi) . \tag{16}$$

Adding the contribution of antiparticles, we obtain finally

$$j_z = -\frac{e^2B}{2\pi^2}\int_0^{\infty} dp[f(p-\chi) - f(p+\chi)] = -\frac{e^2B}{2\pi^2}\chi \tag{17}$$

or, in the vector form,

$$\vec{j} = -(e^2\chi/2\pi^2)\vec{B} . \tag{18}$$

It is interesting that the equilibrium current (18) depends only on the chemical potential and not on the temperature.

An alternative derivation of Eq. (18) can be given using the finite-temperature and -density Green's-function formalism.[4] In the momentum representation, the equilibrium current can be written as

$$j^\mu(\vec{q}) = -\Pi^{\mu\nu}(\vec{q})A_\nu(\vec{q}) , \tag{19}$$

where

$$\Pi^{\mu\nu}(\vec{q}) = -e^2(2\pi)^{-3}\beta^{-1} \times \sum_{p_0}\int d^3p\, \mathrm{Tr}\{\gamma^\mu S(p)\gamma^\nu S(p-q)\} \tag{20}$$

is the polarization operator,

$$S(p) = -\frac{1}{2}\frac{\gamma^\mu p_\mu}{p^2}(1+\gamma^5) \tag{21}$$

is the left-handed fermion Green's function, $q_0 = 0$, $p^2 = p_0^2 - \vec{p}^2$, $p_0 = \zeta_l + \chi$, $\zeta_l = i\pi(2l+1)\beta^{-1}$, and the summation is taken over $l = 0, \pm 1, \pm 2, \ldots$.

We are interested only in the parity-violating contribution to $\Pi^{\mu\nu}(\vec{q})$ arising from the terms proportional to γ^5. Omitting all other terms and calculating the trace we find

$$\Pi^{\mu\nu}(\vec{q}) = -\frac{2ie^2}{\beta(2\pi)^3}\epsilon^{\mu\sigma\nu\tau} q_\tau \sum_{p_0} \int d^3p \frac{p_\sigma}{p^2(p-q)^2} . \quad (22)$$

We want to calculate the contribution to $\vec{j}$ which is proportional to the magnetic field $\vec{B}$. This corresponds to a linear in $\vec{q}$ term of $\Pi^{\mu\nu}(\vec{q})$. Therefore, we can neglect q in the denominator of Eq. (22). In other words, we shall keep only the first term of the expansion of $\Pi^{\mu\nu}(\vec{q})$ in powers of $\vec{q}$. Higher terms give contributions proportional to derivatives of $\vec{B}$.

Now the summation over p_0 can be performed using the standard formula[4]

$$\frac{1}{\beta}\sum_l F(\zeta_l) = -\frac{1}{2\pi i}\int_C d\zeta f(\zeta) F(\zeta) , \quad (23)$$

where $f(\zeta)$ is given by Eq. (13) and the contour C encircles, in the counterclockwise direction, all the poles of $f(\zeta)$ and none of $F(\zeta)$. Deforming the contour of integration and finding the contributions of the poles of $p^{-4} = [(\zeta+\chi)^2 - \vec{p}^2]^{-2}$ we obtain

$$\Pi^{ik} = \frac{ie^2}{(2\pi)^3}\epsilon^{ikn} q_n \times \int \frac{d^3p}{|\vec{p}|}[f'(|\vec{p}|-\chi) - f'(|\vec{p}|+\chi)], \quad (24)$$

$$\Pi^{00} = \Pi^{i0} = \Pi^{0i} = 0 .$$

Here, as usual, Latin indices take values from 1 to 3 (and Greek indices from 0 to 3). The integration over $\vec{p}$ is easily done and we find from Eq. (19)

$$\vec{j} = -(e^2\chi/2\pi^2) i\vec{q}\times\vec{A} = -(e^2\chi/2\pi^2)\vec{B} . \quad (25)$$

From this derivation it is clear that Eq. (25) is valid not only for a constant (in space) magnetic field, but also for a slowly varying field, when the characteristic variation length is much greater than the typical particle wavelength (T^{-1} or χ^{-1}).

Although no left-handed charged fermions are known to exist, I would like to discuss briefly the magnetic properties of a hypothetical medium consisting of such (and perhaps some other) particles. As was explained in the Introduction, we cannot rule out the possibility of the existence of equilibrium parity-violating currents in realistic systems, and this discussion will give us an idea of what kind of effects can be expected of such a possibility is materialized. The Maxwell equations for the magnetic field with the equilibrium current (25) can be written as[5]

$$\vec{\nabla}\times\vec{B} = 4\pi\vec{j}_0 - \lambda^{-1}\vec{B} , \quad \vec{\nabla}\cdot\vec{B} = 0 , \quad (26)$$

where $\vec{j}_0$ is the "regular" (dissipative) current and $\lambda = \pi/2e^2\chi$. In the regions where $\vec{j}_0 = 0$, Eq. (26) gives

$$(\nabla^2 + \lambda^{-2})\vec{B} = 0 . \quad (27)$$

To illustrate the nature of the solutions of this equation, let us imagine a semi-infinite medium occupying the half-space $z>0$. A magnetic field $\vec{B}_0 = \text{const}$ is applied parallel to x axis at $z<0$. Then the solution inside the medium is $B_x = B_0 \cos(z/\lambda)$, $B_y = B_0\sin(z/\lambda)$. The magnetic field oscillates along the z axis with a wavelength $l = 2\pi\lambda$. (The condition that Eq. (25) is valid in this case is $l \gg \chi^{-1}$, i.e., $e^2 \ll 1$.)

III. INTERACTING MASSIVE PARTICLES

Let us consider a system consisting of two types of spin-$\frac{1}{2}$ particles described by the field operators ψ and ψ' with the interaction Lagrangian

$$\mathcal{L} = G(\bar{\psi}\gamma^\mu\gamma^5\psi)(\bar{\psi}'\gamma_\mu\psi') , \quad (28)$$

where $G = \text{const}$. It will be clear from the calculation that the same results hold for a more general model

$$\mathcal{L} = G[\bar{\psi}\gamma^\mu(a+b\gamma^5)\psi][\bar{\psi}'\gamma_\mu(a'+b'\gamma^5)\psi'] . \quad (29)$$

Lagrangians of the form (28), (29) are used to describe the neutral-current weak interactions at low energies.[6] We shall assume for simplicity that the field ψ' does not couple to the electromagnetic field and calculate the equilibrium current $\vec{j}$ of the field ψ. The model (28) is not renormalizable, but this does not cause any problems in the lowest order of perturbation theory.

The polarization operator in the first order of perturbation theory in G is represented by the diagrams shown in Fig. 1. Solid lines correspond to the Green's functions of particles ψ,

$$S(p) = -\frac{\gamma^\mu p_\mu + m}{p^2 - m^2} , \quad (30)$$

where $p_0 = \zeta_l + \chi$; double lines correspond to the

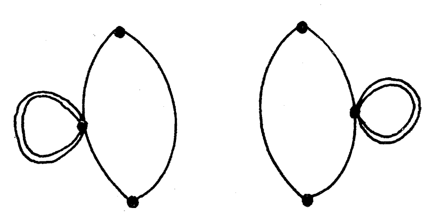

FIG. 1. First-order corrections to the polarization operator.

Green's functions of particles ψ', $S'(p)$, and are given by the same expression (30) with m and χ replaced by m' and χ', respectively. The contributions of the closed loops of ψ' particles are given by

$$-\beta^{-1}(2\pi)^{-3}\sum_{p_0}\int d^3p\,\mathrm{Tr}\{\gamma_\sigma S'(p)\}$$
$$=2\delta_{0\sigma}(2\pi)^{-3}\int d^3p[f(\epsilon'_p-\chi')-f(\epsilon'_p+\chi')]$$
$$=\delta_{0\sigma}n', \qquad (31)$$

where $\epsilon_{p'}=(\vec{p}^2+m'^2)^{1/2}$, n' is the density of ψ' particles (to be exact, the difference of particle and antiparticle densities), and I have dropped the infinite, T- and χ-independent vacuum density. The analytic expression corresponding to Fig. 1 can now be written as

$$\Pi^{\mu\nu}(\vec{q})=-Ge^2n'\beta^{-1}(2\pi)^{-3}$$
$$\times\sum_{p_0}\int d^3p\,\mathrm{Tr}\{\gamma^\mu S(p+q)\gamma^\nu S(p)\gamma^0\gamma^5 S(p)$$
$$+\gamma^\mu S(p+q)\gamma^0\gamma^5$$
$$\times S(p+q)\gamma^\nu S(p)\}, \qquad (32)$$

where $q=(0,\vec{q})$.

It is clear that in a constant magnetic field the equilibrium current density $\vec{j}(\vec{x})$ is also a constant, and thus

$$j^i(\vec{x})=j^i(0)=-(2\pi)^{-3}\int d^3q\,\Pi^{ik}(\vec{q})A_k(\vec{q}), \qquad (33)$$

where I have used Eq. (19). The Fourier transform of the vector potential (3) corresponding to a constant magnetic field is given by

$$\vec{A}(\vec{q})=\tfrac{1}{2}(2\pi)^3 i\,\vec{B}\times\vec{\nabla}_q\,\delta(\vec{q}). \qquad (34)$$

Substituting Eqs. (32) and (34) in Eq. (33) we find

$$\vec{j}=\frac{iGe^2n'}{2\beta(2\pi)^3}\sum_{p_0}\int d^3p\,\mathrm{Tr}\{\vec{\gamma}[\vec{\nabla}_p S(p)\cdot(\vec{\gamma}\times\vec{B})]$$
$$\times S(p)\gamma^0\gamma^5 S(p)$$
$$-\vec{\gamma}S(p)\gamma^0\gamma^5 S(p)$$
$$\times[(\vec{\gamma}\times\vec{B})\cdot\vec{\nabla}_p S(p)]\}. \qquad (35)$$

From the symmetry of the problem it is obvious that $\vec{j}$ has to be parallel to $\vec{B}$. Therefore, it is sufficient to calculate the scalar product $\vec{j}\cdot\vec{B}$. It is clear also that $\vec{j}\cdot\vec{B}$ is independent of the direction of $\vec{B}$, and we can average over all directions. This amounts to replacing B^mB^n by $\frac{1}{3}B^2\delta^{mn}$ and Eq. (35) becomes

$$\vec{B}\cdot\vec{j}=\frac{iGe^2n'B^2}{3\beta(2\pi)^3}\epsilon_{ikl}\sum_{p_0}\int d^3p\,\mathrm{Tr}\{\gamma^i\nabla_k S(p)\gamma^l$$
$$\times S(p)\gamma^0\gamma^5 S(p)\}. \qquad (36)$$

Using the relation

$$\vec{\gamma}\times\vec{\gamma}=-2i\vec{\Sigma}, \qquad (37)$$

where $\vec{\Sigma}=\gamma^0\vec{\gamma}\gamma^5$, it can be shown that

$$\epsilon_{ikl}\gamma^i\nabla_k S(p)\gamma^l=-6i(p^2-m^2)^{-1}\gamma^0\gamma^5$$
$$+4i(p^2-m^2)^{-2}(\gamma^\mu p_\mu-m)\vec{\Sigma}\cdot\vec{p}. \qquad (38)$$

Now the trace in Eq. (36) is easily calculated and we obtain

$$\vec{B}\cdot\vec{j}=\frac{8Ge^2n'B^2}{3\beta(2\pi)^3}$$
$$\times\sum_{p_0}\int d^3p\left[\frac{3}{(p^2-m^2)^2}+\frac{4\vec{p}^2}{(p^2-m^2)^3}\right]. \qquad (39)$$

The integral in Eq. (39) is apparently logarithmically divergent and has to be regularized. We shall use the Pauli-Villars regularization, which amounts to subtraction from the integrand an identical expression with m replaced by M and taking the limit $M\to\infty$ after the p integration. Actually, we shall see that the integral in Eq. (39) is finite; however, the regulator term also gives a finite contribution. A similar situation occurs when Pauli-Villars regularization is used to calculate γ^5 and trace anomalies.[7]

The p_0 summation in Eq. (39) can be performed using Eq. (23):

$$\frac{1}{\beta}\sum_{p_0}\left(\frac{3}{(p^2-m^2)^2}+\frac{4\vec{p}^2}{(p^2-m^2)^3}\right)$$
$$=\left(3\frac{\partial}{\partial m^2}+2\vec{p}^2\frac{\partial^2}{(\partial m^2)^2}\right)\frac{1}{\beta}\sum_{p_0}\frac{1}{p^2-m^2}$$
$$=\frac{\vec{p}^2}{4\epsilon_p^3}\tilde{f}''(\epsilon_p)+\frac{3m^2}{4\epsilon_p^4}\tilde{f}'(\epsilon_p)$$
$$-\frac{3m^2}{4\epsilon_p^5}\tilde{f}(\epsilon_p)+\frac{3m^2}{4\epsilon_p^5}. \qquad (40)$$

Here

$$\tilde{f}(\epsilon_p)=f(\epsilon_p-\chi)+f(\epsilon_p+\chi), \qquad (41)$$

$\epsilon_p=(\vec{p}^2+m^2)^{1/2}$ and prime means derivative with respect to ϵ_p. Using integration by parts, it can be shown that

$$\int d^3p(\vec{p}^2/\epsilon_p{}^3)\tilde{f}''(\epsilon_p) = \int d^3p[(3m^2/\epsilon_p{}^5)\tilde{f}(\epsilon_p) - (3m^2/\epsilon_p{}^4)\tilde{f}'(\epsilon_p)]\,. \tag{42}$$

From Eqs. (39), (40), and (42) we find

$$\vec{B}\cdot\vec{j} = \frac{2e^2Gn'B^2}{(2\pi)^3}\, m^2 \int d^3p\epsilon_p{}^{-5}\,,$$

$$= \pi^{-2}e^2Gn'B^2 \int_0^\infty dx x^2 (x^2+1)^{-5/2}\,. \tag{43}$$

This expression is mass independent and cancels with the infinite-mass regulator term. The final result is $\vec{j}=0$.

IV. CONCLUSION

The equilibrium current induced by an external magnetic field has been calculated for a system of massless left-handed charged particles and for a system of massive particles with a parity-violating Fermi-type interaction. A nonzero current is found in the first case, while in the second case the current is equal to zero in the lowest order of perturbation theory.

Similar results have been recently obtained for the equilibrium currents induced by rotation. It has been shown[8] that left-handed particles (neutrinos) develop a nonzero current parallel to the rotation axis. The equilibrium current has also been calculated[9] for a rotating system of interacting massless spinor and vector fields with the interaction Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}e\bar{\psi}\gamma^\mu(1+\gamma^5)\psi A_\mu\,.$$

The result is zero in the second order of perturbation theory.

The question naturally arising from these results is whether nonzero currents occur in different parity-violating models or in higher orders of perturbation theory, or is there some deep physical reason for the cancellation of the equilibrium currents, left-handed particles being the only exception.

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[1]Similar results for neutrinos and interacting particles were obtained in the case of rotating systems (see Sec. IV).

[2]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1969).

[3]See, e.g., A. I. Akhiezer and V. B. Berestetsky, *Quantum Electrodynamics* (GITTL, Moscow, 1951), AEC Translation No. 2876, 1957.

[4]For a review see, e.g., A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinsky, *Methods of Quantum Field Theory in Statistical Physics*, translated and edited by R. Silverman (Prentice-Hall, Englewood Cliffs, 1963). For application to relativistic systems see I. A. Akhiezer and S. V. Peletminsky, Zh. Eksp. Teor. Fiz. 38, 1829 (1960) [Sov. Phys. JETP 11, 1316 (1960)]; L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974); S. Weinberg, *ibid.* 9, 3357 (1974).

[5]The additional term in Eq. (26) is derivable from an addition

$$\delta\mathcal{L} = -(8\pi\lambda)^{-1}\vec{A}\cdot(\vec{\nabla}\times\vec{A})$$

to the effective electromagnetic Lagrangian. A gauge transformation $\vec{A}\to\vec{A}+\vec{\nabla}\psi$ changes $\delta\mathcal{L}$ by a divergence $-(8\pi\lambda)^{-1}\vec{\nabla}\cdot[(\vec{\nabla}\times\vec{A})\cdot\psi]$, and thus $\delta\mathcal{L}$ does not break the gauge invariance of the theory. [This is obvious also from the fact that Eqs. (26) are clearly gauge invariant.]

[6]See, e.g., J. C. Taylor, *Gauge Theories of Weak Interactions* (Cambridge University Press, Cambridge, 1976).

[7]S. L. Adler, Phys. Rev. 177, 2426 (1969); A. Vilenkin, Nuovo Cimento 44A, 441 (1978).

[8]A. Vilenkin, Phys. Lett. 80B, 150 (1978); Phys. Rev. D 20, 1807 (1979); 21, 2260 (1980).

[9]A. Vilenkin, preceding paper, Phys. Rev. D 22, 3067 (1980).