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Cancellation of equilibrium parity-violating currents

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It has recently been shown that in a rotating equilibrium system there exist neutrino and antineutrino currents parallel to the rotation axis. This suggests that particles other than neutrinos can also develop equilibrium currents if parity-violating weak interactions are taken into account. This possibility is examined using quantum field theory at finite temperature, density, and angular velocity. Regularization and renormalization of such a theory are discussed. The result obtained is negative. It is found that in the model considered the equilibrium current vanishes in the lowest order of perturbation theory. This is a surprising result, since the existence of a nonzero current is not forbidden by the symmetries of the Lagrangian. The question naturally arising from this result is whether a nonzero current appears in higher orders of perturbation theory or in different models or there is some deep physical reason which makes the equilibrium current equal to zero for all particles except neutrinos.

I. INTRODUCTION

Reflectional symmetry imposes certain restrictions on the possible form of physical laws. It requires, for example, that in all vector relations like $\vec{A}=\vec{B}$, $\vec{A}$ and $\vec{B}$ should be both polar or both axial vectors. As we know, reflection symmetry is violated in weak interactions. (Another way to say it is that weak interactions do not conserve parity.) Consequences of this violation have been extensively studied in elementary particle processes, such as particle scattering and decays, splitting of nuclear energy levels, etc. One can expect, however, that parity nonconservation can manifest itself not only in the microworld, but on a macroscopic scale as well. For example, parity nonconservation removes a veto from processes like generation of electric currents by rotation ($\vec{j}\sim\vec{\Omega}$, where $\sim$ means proportionality) or by magnetic field ($\vec{j}\sim\vec{B}$). Macroscopic parity-violating effects received little attention, since one expects that in normal conditions they are vanishingly small, owing to the smallness of the weak-interaction constant. This, however, may not be the case in the early universe, in stellar collapse, or in neutron stars, where extremely high temperatures and densities and huge magnetic fields can be produced.

Two macroscopic effects of parity nonconservation have already been discussed: (i) asymmetric neutrino emission by rotating black holes and (ii) equilibrium neutrino current in a rotating system.[1] These effects are based on the two-component theory of neutrinos and can be easily understood if we recall that the left-handedness of neutrinos means that their momentum is always opposite to their spin. In a rotating equilibrium system, the average spin density of neutrinos is parallel to the angular velocity $\vec{\Omega}$, and thus neutrinos move, on average, in the direction opposite to $\vec{\Omega}$. Antineutrinos are right-handed and their current parallel to $\vec{\Omega}$. The magnitude of the neutrino current has been calculated in Refs. 2–4. If $\Omega \ll T$, χ then the current on the rotation axis is given by

$$\vec{J}_\nu = -\tfrac{1}{12}T^2\vec{\Omega} - \frac{1}{4\pi^2}\chi^2\vec{\Omega}\,, \tag{1}$$

where T is the temperature and χ is the chemical potential.[5] (I use the system of units in which $\hbar = c = k = 1$.) Equation (1) was derived assuming that the system is infinite in the direction of $\vec{\Omega}$ and that the size of the system in the plane perpendicular to $\vec{\Omega}$ is much greater than T^{-1} or χ^{-1} (see Ref. 3, Sec. IV). If the system is finite in all directions, we expect that neutrinos are emitted asymmetrically from its surface and that the neutrino current far from the the boundaries is still given by Eq. (1). The problem of determining the spatial distribution of the neutrino current for a finite system has not yet been solved.

It is clear that, for a finite system, the word "equilibrium" here does not mean complete thermal equilibrium. What is meant is that, far from the boundaries, neutrinos are in a local equilibrium with other particles. One can wonder what the state of the system would be if it were surrounded by walls impenetrable for neutrinos. However, this problem is difficult even to formulate, because of the well-known difficulty with confinement of massless fermions to a fixed volume (Klein paradox, see, e.g., Ref. 6). An additional difficulty arises[3] due to the left-handedness of neutrinos.

The asymmetry in the neutrino emission by black holes has been studied in Refs. 7, 8, and 3. It should be noted that there is a close relation between the two effects due to the theorem[9] that a black hole is in equilibrium with thermal radiation having the same temperature and angular velocity.

Particles other than neutrinos can be in both helicity states, and in the free-field approximation the currents of right- and left-handed particles exactly cancel. It is possible, however, that nonzero equilibrium currents can result if parity-violating weak interactions are taken into account. As we know, the particle energy is modified by interactions: $\epsilon(\vec{p})=\epsilon_0(\vec{p})+\delta\epsilon$, where $\vec{p}$ is momentum, $\epsilon_0(\vec{p})$ is the free-particle energy, and $\delta\epsilon$ depends on $\vec{p}, T$, chemical potentials, and the interaction constants. For a parity-violating interaction, $\delta\epsilon$ is different for left- and right-handed states. As a result, the equilibrium concentrations of left- and right-handed particles are different and their currents may not cancel completely. If this is the case, then the emission of the corresponding particle species by black hole has to be asymmetric. It has been argued[3,7-10] that macroscopic parity-violating effects can have important cosmological implications. In particular they can provide an explanation for the origin of cosmic magnetic fields and give rise to partial separation of matter and antimatter in the early universe.

The purpose of the present paper is to calculate the equilibrium current in a simple model with a parity-violating interaction. The result obtained is negative: it is shown that in the model considered, the equilibrium current vanishes in the lowest order of perturbation theory. Although it is said that all processes which are not forbidden by symmetries and conservation laws have to occur, this seems not to be the case for the parity-violating current. (The existence of such a current would not violate *CP* or *CPT* symmetries.) The crucial question now is whether a nonzero current appears in higher orders of perturbation theory or in different models or there is some deep physical reason which makes the equilibrium current equal to zero for all particles except neutrinos.

This paper has another, secondary purpose: to extend the finite-temperature -density, and -angular -velocity quantum field theory, which has been developed in Ref. 4, to the case of interacting fields. This includes the problem of divergences and renormalization of such a theory. Some subtle points of regularization and taking zero-temperature limit will be discussed. It will be seen that wrong results can be easily obtained unless these limiting procedures are performed very carefully. The method developed here may be useful in other problems, such as bag-model calculations for rotating nuclei.

The plan of the paper is the following. We shall first discuss the diagram technique and renormalization of quantum field theory at finite temperature, density, and angular velocity. Then the equilibrium current will be calculated in the simple case of a fermion field ψ with a parity-violating coupling to a classical external potential A_μ. Finally, the equilibrium current will be calculated for interacting quantum fields ψ and A_μ.

II. DIAGRAM TECHNIQUE AND RENORMALIZATION

The model to be discussed in this paper includes two massless spinor fields ψ and ψ' and one massless vector field A_μ with the interaction Lagrangian

$$L=-\tfrac{1}{2}eA_\mu\bar{\psi}\gamma^\mu(1-\gamma^5)\psi+\tfrac{1}{2}eA_\mu\bar{\psi}'\gamma^\mu(1-\gamma^5)\psi' . \quad (2)$$

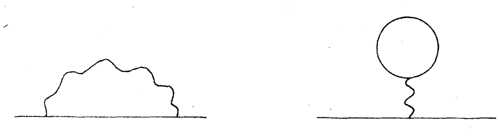
The second spinor field ψ' is needed for two reasons: (i) it cancels the γ^5 anomaly and makes the theory renormalizable,[11] and (ii) if the chemical potential χ of the field ψ is not equal to zero, the field ψ' is necessary to ensure the neutrality of the system. In the following I shall assume that $\chi=\chi'$, so that the "charge" density is equal to zero.

One could think that the equilibrium currents of the fields ψ and ψ' exactly cancel each other because the charges of these fields are equal and opposite. However, the currents still can have an observable effect, since some of the quantum numbers of ψ and ψ' can be different. I shall assume that the quantum number of interest is equal to 1 for ψ, and 0 for ψ', so that we have to calculate only the equilibrium current associated with ψ, $\vec{J}=\frac{1}{2}[\bar{\psi},\vec{\gamma}\psi]_-$. [Actually, we do not need the second field ψ' for the lowest-order calculation of the equilibrium current, since the problems associated with anomalies arise only in higher orders. The only role of ψ' in the following calculation is to cancel diagrams of the form shown in Fig. 1(b). The same result could be achieved by introducing an appropriate neutralizing background.]

It can be shown that the angular momentum operator $\vec{M}$ in our model coincides with its free-field form $\vec{M}_0$, and thus the effective interaction Hamiltonian[4]

$$H_i=H-H_0-\vec{\Omega}\cdot(\vec{M}-\vec{M}_0)$$

equals



a) b)

FIG. 1. Second-order corrections to the fermion Green's function.

$$H_i = \frac{e}{2}\int A_\mu \bar{\psi}\gamma^\mu(1-\gamma^5)\psi d^3$$

$$-\frac{e}{2}\int A_\mu \bar{\psi}'\gamma^\mu(1-\gamma^5)\psi' d^3x\,. \qquad (3)$$

The rules of diagram technique for the Green's functions can be easily derived from Eqs. (7) and (8) of Ref. 4. Each fermion line gives a factor $S(\vec{x}_1,\tau_1;\vec{x}_2,\tau_2)$, each boson line gives $D_{\mu\nu}(\vec{x}_1,\tau_1;\vec{x}_2,\tau_2)$, and each interaction vertex gives $-\frac{1}{2}e\gamma^\mu(1-\gamma^5)$. The resulting expression has to be integrated over $\vec{x}$ and τ at all vertices and multiplied by $(-1)^L$, where L is the number of fermion loops. For example, the lowest-order correction to the fermion Green's function is given by the diagram shown in Fig. 1(a). The corresponding analytic expression is

$$\delta S(\vec{x}_1,\tau_1;\vec{x}_2,\tau_2)=\tfrac{1}{4}e^2\int d^3x'd^3x''d\tau'd\tau''D_{\mu\nu}(\vec{x}',\tau';\vec{x}'',\tau'')$$
$$\times S(\vec{x}_1,\tau_1;\vec{x}',\tau')\gamma^\mu(1-\gamma^5)S(\vec{x}',\tau';\vec{x}'',\tau'')\gamma^\nu(1-\gamma^5)S(\vec{x}'',\tau'';\vec{x}_2,\tau_2)\,. \qquad (4)$$

The contribution of the diagram of Fig. 1(b) is equal to zero, since the loops of Ψ and Ψ' cancel each other.

In the momentum representation, each fermion line gives a factor $S(p_1,p_2,\zeta_l)$, each boson line gives $D_{\mu\nu}(p_1,p_2,\nu_n)$ and each vertex gives $-\frac{1}{2}e\gamma^\mu(1-\gamma^5)$. Energy and momentum have to be conserved at each vertex. The resulting expression has to be integrated over all internal momenta and summed over all internal energies. An overall factor $\beta^{-1}(2\pi)^{-3}$ has to be added for each energy summation, where $\beta=T^{-1}$, and a factor (-1) has to be added for each fermion loop. For example, the contribution of the diagram of Fig. 1(a) in the momentum representation has the form

$$\delta S(\vec{p}_1,\vec{p}_2,\zeta_l)=\tfrac{1}{4}e^2\beta^{-1}(2\pi)^{-3}\sum_n\int d^3p'd^3p''d^3q'd^3q''D_{\mu\nu}(\vec{p}'-\vec{q}',\vec{p}''-\vec{q}'',\zeta_l-\zeta_n)$$
$$\times S(\vec{p}_1,\vec{p}',\zeta_l)\gamma^\mu(1-\gamma^5)S(\vec{q}',\vec{q}'',\zeta_n)\gamma^\nu(1-\gamma^5)S(\vec{p}'',\vec{p}_2,\zeta_l)\,. \qquad (5)$$

The free-field Green's functions $\hat{D}$ and S have been found in Ref. 4:

$$\hat{D}(\vec{p}_1,\vec{p}_2,\nu_n)=\exp\left[i\vec{\Omega}\cdot(\vec{\nabla}_{p_1}\times\vec{p}_2)\frac{\partial}{\partial\nu_n}-i\vec{\Omega}\cdot\hat{M}\frac{\partial}{\partial\nu_n}\right]\times\delta(\vec{p}_1-\vec{p}_2)\hat{D}_0(\vec{p}_2,\nu_n)\,, \qquad (6)$$

$$S(\vec{p}_1,\vec{p}_2,\zeta_l)=\exp\left[i\vec{\Omega}\cdot(\vec{\nabla}_{p_1}\times\vec{p}_2)\frac{\partial}{\partial\zeta_l}+\tfrac{1}{2}\vec{\Omega}\cdot\vec{\Sigma}\frac{\partial}{\partial\zeta_l}\right]\times\delta(\vec{p}_1-\vec{p}_2)S_0(\vec{p}_2,\zeta_l)\,. \qquad (7)$$

An equivalent representation for $S(\vec{p}_1,\vec{p}_2,\zeta_l)$ is

$$S(\vec{p}_1,\vec{p}_2,\zeta_l)=\exp\left[-i\vec{\Omega}\cdot(\vec{\nabla}_{p2}\times\vec{p}_1)\frac{\partial}{\partial\zeta_l}\right]\delta(\vec{p}_1-\vec{p}_2)S_0(\vec{p}_1,\zeta_l)\times\exp\left(\tfrac{1}{2}\vec{\Omega}\cdot\vec{\Sigma}\frac{\overleftarrow{\partial}}{\partial\zeta_l}\right)\,. \qquad (8)$$

Here $\vec{\nabla}_{pi}\equiv\partial/\partial\vec{p}_i$, a caret indicates that the corresponding quantity is a matrix,

$$M_{\mu\nu}=\epsilon_{0\mu\nu3}\,, \qquad (9)$$

$$\vec{\Sigma}=\gamma^0\vec{\gamma}\gamma^5 \qquad (10)$$

is the fermion spin operator, $\hat{D}_0$ and S_0 are the free-field Green's functions for a nonrotating system,

$$D_{0\mu\nu}(\vec{p},\nu_n)=g_{\mu\nu}(\nu_n^2-\vec{p}^2)^{-1}\,, \qquad (11)$$

$$S_0(\vec{p},\zeta_l)=-\frac{\gamma^0(\zeta_l+\chi)-\vec{\gamma}\cdot\vec{p}+\mu}{(\zeta_l+\chi)^2-\vec{p}^2-\mu^2}\,. \qquad (12)$$

The matrices γ^μ and γ^5 are taken in the representation of Bjorken and Drell,[6] $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, and the arrow over $\partial/\partial\zeta_l$ in Eq. (8) indicates that this differential operator acts to the left. The quantities μ and χ in Eq. (12) are the fermion mass and chemical potential, respectively.[12] In our case $\mu=0$.

Expanding the exponents in Eqs. (6)–(8) one can calculate the contribution to δS proportional to an arbitrary power of Ω. This enables one to obtain an expansion of the equilibrium current $\vec{J}$ in powers of Ω/T or Ω/χ. Since in most physical situations $\Omega\ll T,\chi$, we shall keep only the linear terms in $\vec{\Omega}$.

The Green's functions (6)–(8) were obtained using the fermion and boson wave functions in infinite space. However, a rotating system cannot be infinite in the plane perpendicular to the rotation axis, its maximum size being $R_{\max}=c/\Omega$. In the following sections we shall calculate the equilibrium current on the rotation axis assuming that it is not affected by the conditions at the boundaries of the system. (An argument that this is the case for the neutrino current has been given in Ref. 4. It has been argued that the effect of boundary conditions can appear only in terms proportional to Ω^3 and to higher powers of Ω.)

The renormalization of quantum field theory at finite temperature and density has been discussed by a number of authors.[13-15] The conclusion is that the finite-temperature and -density theory can be renormalized by the same counterterms as the theory in a vacuum. In physical terms, this means that the ultraviolet behavior of the theory is not affected by the temperature and density effects. Mathematically, this result is easily understood on the one-loop level.[13] In the ultraviolet limit the temperature and chemical potentials effectively scale to zero (in other words, they can be neglected compared to the loop momentum), the sum over the loop energies can be replaced by an integral, and the divergent part of a diagram coincides with that of the theory with $T=\chi=0$. The situation is more complicated for multiloop diagrams, since they can give rise to T- and χ-dependent infinities. It has been shown in Ref. 15 that all such infinities cancel among themselves.

Physically, we expect that the same results hold for rotating systems as well. Like before, we can argue that on the one-loop level, the temperature and chemical potentials can be neglected in the ultraviolet limit and the divergences of the theory with nonzero T, χ, and Ω must coincide with those of the theory with $T=\chi=0$, $\Omega\neq 0$.

It is shown in the Appendix that the Green's functions at $\chi=T=0$. $\Omega\neq 0$ are related to those at $T=\chi=\Omega=0$ by a simple coordinate transformation $\varphi\to\varphi-i\Omega\tau$. For example, the full fermion Green's function in cylindrical coordinates is given by

$$S'_{\chi=T=0}(r_1,\varphi_1,z_1,\tau_1;r_2,\varphi_2,z_2,\tau_2) \tag{13}$$

$$=S'_{\chi=T=\Omega=0}(r_1,\varphi_1-i\Omega\tau_1,z_1,\tau_1;r_2,\varphi_2-i\Omega\tau_2,z_2,\tau_2)$$

(Here the primes indicate that the corresponding quantities are taken in cylindrical coordinates.) In momentum space, Eq. (13) takes the form

$$S_{\chi=T=0}(\vec{p}_1,\vec{p}_2,\zeta)=\exp\left[i\vec{\Omega}\cdot(\vec{\nabla}_1\times\vec{p}_2)\frac{\partial}{\partial\zeta}+\tfrac{1}{2}\vec{\Omega}\cdot\vec{\Sigma}\frac{\partial}{\partial\zeta}\right]\times\delta(\vec{p}_1-\vec{p}_2)S_{\chi=T=\Omega=0}(\vec{p}_2,\zeta)\,. \tag{14}$$

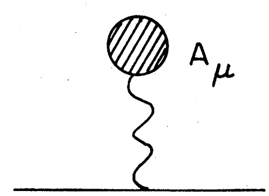


FIG. 2. First-order correction to the fermion Green's function in an external field.

The derivation is similar to that of Eq. (69) of Ref. 4. Relations like (13), (14) apply to all Feynman diagrams. This shows that vacuum counterterms renormalize the theory with $T=\chi=0$ and $\Omega\neq 0$, and thus provide a one-loop renormalization for the theory with nonzero T, χ, and Ω. We expect that this result can be extended to include multiloop diagrams.

III. CLASSICAL EXTERNAL FIELD

Before treating the complicated case of interacting fields, let us consider the equilibrium current in a classical time-independent external field $A_\mu(\vec{x})$. The corresponding physical situation can be pictured in the following way. As we know, the weak interactions at low energies can be described by a current-current Lagrangian of the form[11]

$$\mathcal{L}=-Gj_\mu\mathcal{T}^\mu\,. \tag{15}$$

If we take $j_\mu=\frac{1}{2}\bar{\psi}\gamma^\mu(1-\gamma^5)\psi$ and replace $\mathcal{T}^\mu$ by its average value (which is proportional to the density of the corresponding particles), then Eq. (15) reduces to Eq. (2) with $eA^\mu=G\mathcal{T}^\mu$ (and without ψ'). Thus the Lagrangian (2) with a classical field A_μ can be thought of as describing the effect of neutral-current interaction at low energies, when the temperature and chemical potential are much smaller than the vector-boson mass.

According to Eq. (6) of Ref. 4, the equilibrium current on the rotation axis is given by

$$\vec{J}=\langle\tfrac{1}{2}[\bar{\psi},\vec{\gamma}\psi]_-\rangle=-\mathrm{Tr}\{\vec{\gamma}\delta S(0,\tau;0,\tau+\epsilon)\}_{\epsilon\to 0}\,. \tag{16}$$

Here $\delta S=S-S$, $S(\vec{x}_1,\tau_1;\vec{x}_2,\tau_2)$ is the full fermion Green's function, angular brackets mean statistical averaging, and the limit $\epsilon\to 0$ is taken symmetrically for $\epsilon\to+0$ and $\epsilon\to-0$. In the first order of perturbation theory, δS is given by the diagram shown in Fig. 2 and we can write

$$\delta S(\vec{p}_1,\vec{p}_2,\zeta_l)=-2^{-1}(2\pi)^{-3}\int d^3p_1'\,d^3p_2'\,S(\vec{p}_1,\vec{p}_1',\zeta_l)\gamma^\mu(1-\gamma^5)S(\vec{p}_2',\vec{p}_2,\zeta_l)A_\mu(\vec{p}_1'-\vec{p}_2') \tag{17}$$

and

$$\vec{J}=(2\beta)^{-1}(2\pi)^{-6}\sum_l e^{\epsilon\zeta_l}\int d^3p_1\,d^3p_2\,d^3p_1'\,d^3p_2'\,\mathrm{Tr}\{\vec{\gamma}S(\vec{p}_1,\vec{p}_1',\zeta_l)\gamma^\mu(1-\gamma^5)S(\vec{p}_2',\vec{p}_2,\zeta_l)\}A_\mu(\vec{p}_1'-\vec{p}_2')\big|_{\epsilon\to 0}\,. \tag{18}$$

Let us first consider the case of a constant potential

$$A_\mu(\vec{x}) = (A_0, 0, 0, 0,), \tag{19}$$

where $A_0=$const, which corresponds to a static uniformly distributed source $\mathcal{T}_\mu$ in Eq. (15). In the momentum representation

$$A_\mu(\vec{q}) = (2\pi)^3\,\delta(\vec{q})(A_0, 0, 0, 0). \tag{20}$$

If we take $S(\vec{p}_1,\vec{p}_1', \zeta_l)$ in Eq. (17) in the representation of Eq. (7) and $S(\vec{p}_2',\vec{p}_2, \zeta_l)$ in the representation of Eq. (8), then the terms with gradients vanish upon integration by parts and we get

$$\vec{J} = \frac{eA_0}{2\beta(2\pi)^3}\sum_l e^{\epsilon\zeta_l}\int d^3p\,\mathrm{Tr}\{\vec{\gamma}S(\vec{p},\zeta_l)\gamma^0(1-\gamma^5)\tilde{S}(\vec{p},\zeta_l)\}_{\epsilon\to 0}. \tag{21}$$

Here

$$S(\vec{p},\zeta_l) = \exp\left(\tfrac{1}{2}\vec{\Omega}\cdot\vec{\Sigma}\,\frac{\partial}{\partial\zeta_l}\right)S_0(\vec{p},\zeta_l), \tag{22}$$

$$\tilde{S}(\vec{p},\zeta_l) = S_0(\vec{p},\zeta_l)\exp\left(\tfrac{1}{2}\vec{\Omega}\cdot\vec{\Sigma}\,\frac{\overleftarrow{\partial}}{\partial\zeta_l}\right). \tag{23}$$

From the symmetry of the problem it is clear that $\vec{J}$ is parallel to $\vec{\Omega}$, and thus it is sufficient to calculate $\vec{J}\cdot\vec{\Omega}$. Expanding S and $\tilde{S}$ in powers of Ω and keeping only the linear terms we obtain

$$\begin{aligned}\Omega\cdot\vec{J} &= \frac{eA_0}{4\beta(2\pi)^3}\sum_l e^{\epsilon\zeta_l}\int d^3p\,\mathrm{Tr}\left\{\vec{\Omega}\cdot\vec{\gamma}S_0\gamma^0(1-\gamma^5)\frac{\partial S_0}{\partial\zeta_l}\,\vec{\Omega}\cdot\vec{\Sigma} + (\vec{\Omega}\cdot\vec{\gamma})(\vec{\Omega}\cdot\vec{\Sigma})\frac{\partial S_0}{\partial\zeta_l}\,\gamma^0(1-\gamma^5)S_0\right\}\\ &= -\frac{eA_0\Omega^2}{4\beta(2\pi)^3}\sum_l e^{\epsilon\zeta_l}\int d^3p\,\frac{\partial^2}{\partial\chi^2}\,\mathrm{Tr}\{\gamma^0 S_0(\vec{p},\zeta_l)\}.\end{aligned} \tag{24}$$

Here I have used the cyclic property of the trace and the relations

$$(\vec{\gamma}\cdot\vec{\Omega})(\vec{\Sigma}\cdot\vec{\Omega}) = (\vec{\Sigma}\cdot\vec{\Omega})(\vec{\gamma}\cdot\vec{\Omega}) = \Omega^2\gamma^0\gamma^5, \tag{25}$$

$$S_0\gamma^0 S_0 = \frac{\partial S_0}{\partial\zeta_l} = \frac{\partial S_0}{\partial\chi}. \tag{26}$$

The summation over l can now be easily performed with the aid of the standard formula

$$\frac{1}{\beta}\sum_l F(\zeta_l) = -\frac{1}{2\pi i}\oint_c d\zeta f(\zeta)F(\zeta), \tag{27}$$

where $\zeta_l = i\pi\beta^{-1}(2l+1)$,

$$f(\zeta) = (e^{\beta\zeta}+1)^{-1}, \tag{28}$$

and the integration contour C encircles, in the counterclockwise direction, all the poles of $f(\zeta)$ but none of the function $F(\zeta)$. For further reference, I write a similar formula for summation over boson frequencies, $\nu_n = 2\pi i\beta^{-1}n$:

$$\frac{1}{\beta}\sum_n F(\nu_n) = \frac{1}{2\pi i}\oint_c d\nu n(\nu)F(\nu), \tag{29}$$

where

$$n(\nu) = (e^{\beta\nu}-1)^{-1}. \tag{30}$$

Deforming the contour of integration and finding the contribution of the poles of $S_0(\vec{p},\zeta)$ we obtain

$$\vec{\Omega}\cdot\vec{J} = \frac{eA_0\Omega^2}{2(2\pi)^3}\frac{\partial^2}{\partial\chi^2}\int d^3p[f(\epsilon_p-\chi) - f(\epsilon_p+\chi)], \tag{31}$$

where $\epsilon_p = |\vec{p}|$ and $f(x)$ is the Fermi function (28).

The integral over momenta can be evaluated[16] and we get finally

$$\vec{J} = (2\pi^2)^{-1}eA_0\chi\vec{\Omega}. \tag{32}$$

This result can be easily understood in the following way. The effect of the constant potential A_0 on a massless spinor field is to change the chemical potential of the left-handed component of the field by $-eA_0$, while the chemical potential of the right-handed component remains unchanged. Thus we can write

$$\vec{J} = -eA_0\frac{\partial}{\partial\chi}\vec{J}_\nu, \tag{33}$$

where $\vec{J}_\nu$ is the neutrino current. Substituting $\vec{J}_\nu$ from Eq. (1) we obtain Eq. (32).

As has been mentioned above, a constant potential A_0 corresponds to a static uniformly distributed source $\mathcal{T}_\mu$ in Eq. (15). This means that particles contributing to $\mathcal{T}_\mu$ are out of equilibrium and do not rotate with the system. Let us now see what happens in the equilibrium situation with a rotating source $\mathcal{T}^\mu = \mathcal{T}^0(1, \vec{\Omega}\times\vec{x})$. Then

$$A_\mu(\vec{x}) = A_0(1, -\vec{\Omega}\times\vec{x}), \tag{34}$$

$$A_\mu(\vec{q}) = (2\pi)^3 A_0(1, -i\vec{\Omega}\times\vec{\nabla}_q)\delta(\vec{q}), \tag{35}$$

where $A_0 = \text{const}$. The equilibrium current is now equal to a sum of two terms, $\vec{J} = \vec{J}_1 + \vec{J}_2$, where $\vec{J}_1$ is the contribution of $A_0(\vec{q})$ and is given by Eq. (32) and $\vec{J}_2$ is the contribution of the vector potential $\vec{A}(\vec{q})$. Since $\vec{A}(\vec{q})$ is proportional to $\vec{\Omega}$, we can neglect the Ω-dependent terms in the fermion propagators:

$$\vec{\Omega}\cdot\vec{J}_2 = \frac{ieA_0}{2\beta(2\pi)^3}\sum_l e^{\epsilon\zeta_l}\int d^3p\, \mathrm{Tr}\{(\vec{\gamma}\cdot\vec{\Omega})[\vec{\Omega}\times\vec{\nabla}_p S_0]\cdot\vec{\gamma}(1-\gamma^5)S_0\}_{\epsilon\to 0}. \tag{36}$$

It is easily understood that in $\vec{\nabla}_p S_0$ only the numerator of Eq. (12) has to be differentiated,

$$\vec{\nabla}_p S_0(\vec{p},\zeta_l) \to \vec{\gamma}[(\zeta_l+\chi)^2-\epsilon_p^2]^{-1}, \tag{37}$$

since the other term vanishes upon taking the trace with γ^5. Using the relation

$$\vec{\Sigma} = \tfrac{1}{2} i\,\vec{\gamma}\times\vec{\gamma} \tag{38}$$

and Eq. (25) we obtain

$$\begin{aligned}\vec{\Omega}\cdot\vec{J}_2 &= \frac{4eA_0\Omega^2}{\beta(2\pi)^3}\sum_l e^{\epsilon\zeta_l}\int d^3p\,\frac{\zeta_l+\chi}{[(\zeta_l+\chi)^2-\epsilon_p^2]^2}\\ &= -(2\pi)^{-3}eA_0\Omega^2\frac{\partial}{\partial\chi}\\ &\quad\times\int\frac{d^3p}{\epsilon_p}[f(\epsilon_p-\chi)+f(\epsilon_p+\chi)].\end{aligned} \tag{39}$$

It can be easily shown, using integration by parts, that

$$\int\frac{d^3p}{\epsilon_p}[f(\epsilon_p-\chi)+f(\epsilon_p+\chi)] = \frac{1}{2}\frac{\partial}{\partial\chi}\int d^3p[f(\epsilon_p-\chi)-f(\epsilon_p+\chi)]. \tag{40}$$

Now we see from Eqs. (31), (39), and (40) that $\vec{J} = \vec{J}_1 + \vec{J}_2 = 0$. The equilibrium current vanishes if the source of the potential A_μ rotates with the system.

IV. INTERACTING FIELDS

We shall now turn to the calculation of the equilibrium current in the case of interacting quantum fields with the interaction Lagrangian (2). The calculations here are rather complicated and will be carried out only in the zero-temperature limit. We shall see, however, that the temperature cannot be set equal to zero at the beginning by replacing the sums over frequencies by integrals

$$\frac{1}{\beta}\sum_{q_0} \to \frac{1}{2\pi}\int_{-i\infty}^{i\infty} dq_0.$$

If this is done, then some of the terms are lost and the answer is incorrect. Examples of this sort are known in solid-state physics.[17] The source of the difficulty is the singularity of the Fermi function at $T=0$. The correct procedure is to take the limit $T\to 0$ after the summation over the frequencies.

In the second order of perturbation theory, the equilibrium current on the rotation axis can be written as

$$\begin{aligned}\vec{J} &= -\beta^{-1}(2\pi)^{-3}\sum_l e^{\epsilon\zeta_l}\int d^3p_1\, d^3p_2\,\mathrm{Tr}\{\vec{\gamma}\,\delta\mathcal{S}(\vec{p}_1,\vec{p}_2,\zeta_l)\}_{\epsilon\to 0}\\ &= -\beta^{-1}(2\pi)^{-3}\sum_l e^{\epsilon\zeta_l}\int d^3p_1\, d^3p_2\, d^3p'\, d^3p''\,\mathrm{Tr}\{\vec{\gamma}S(\vec{p}_1,\vec{p}',\zeta_l)\mathfrak{M}(\vec{p}',\vec{p}'',\zeta_l)S(\vec{p}'',\vec{p}_2,\zeta_l)\}_{\epsilon\to 0}.\end{aligned} \tag{41}$$

Here

$$\mathfrak{M}(\vec{p}_1,\vec{p}_2,\zeta_l) = \tfrac{1}{4}e^2\beta^{-1}(2\pi)^{-3}\sum_n\int d^3p'\, d^3p''\, D_{\mu\nu}(\vec{p}_1-\vec{p}',\vec{p}_2-\vec{p}'',\zeta_l-\zeta_n)\gamma^\mu(1-\gamma^5)S(\vec{p}',\vec{p}'',\zeta_n)\gamma^\nu(1-\gamma^5) \tag{42}$$

is the mass operator and I have used Eq. (5) for $\delta\mathcal{S}$.

The integrals (41) and (42) are divergent and have to be regularized. Dimensional regularization has proved to be a very powerful and elegant regularization method of field theories both in a vacuum and in a medium.[14] However, it appears not to be appropriate for our problem, since the matrix γ^5 and the vector product are not uniquely defined in n-dimensional space with $n\neq 4$. A natural choice in our case is the Pauli-Villars method. As we shall see, it is sufficient to introduce just one boson regulator field. This amounts to replacing the free boson propagator $D_{0\mu\nu}$ in Eq. (6) by

$$D_{0\mu\nu}(q)\to D^R_{0\mu\nu}(q)=-g_{\mu\nu}D^R_0(q),$$

where

$$D^R_0(q)=D_0(q|0)-D_0(q|M),\qquad D_0(q|M)=-(q^2-M^2)^{-1}, \tag{43}$$

and

$$q\equiv(\nu_m,\vec{q}). \tag{44}$$

The limit $M\to\infty$ has to be taken after all summations and integrations in Eq. (41). If some of the terms diverge in the limit $M\to\infty$, we expect that such divergences can be combined with the bare parameters of the theory to give the corresponding renormalized parameters. We shall see, however, that no renormalization is necessary in Eq. (41). The reason is that the mass renormalization is absent for a massless fermion field and the charge renormalization is not necessary in the second order of perturbation theory. Thus the equilibrium current (41) has only to be regularized.

A. The mass operator

In this section we shall find the linear in $\vec{\Omega}$ contribution to $\mathfrak{M}(\vec{p}_1,\vec{p}_2,\zeta_l)$.

Let us first consider the contribution of terms with $\vec{\nabla}_1\times\vec{p}_2$ in S and $D_{\mu\nu}$. Omitting terms proportional to $\vec{\Sigma}$ and $\hat{M}$ we obtain

$$\mathfrak{M}(\vec{p}_1,\vec{p}_2,\zeta_l)\Rightarrow e^2\beta^{-1}(2\pi)^{-3}\sum_n\int d^3p'\,d^3p''\left\{1+i\vec{\Omega}\cdot[\vec{\nabla}_1\times(\vec{p}_2-\vec{p}'')]\frac{\partial}{\partial\zeta_l}\right\}$$
$$\times\delta(\vec{p}_1-\vec{p}'-\vec{p}_2+\vec{p}'')D^R_0(\vec{p}_2-\vec{p}'',\zeta_l-\zeta_n)\left[1+i\vec{\Omega}\cdot(\vec{\nabla}'\times\vec{p}'')\frac{\partial}{\partial\zeta_n}\right]$$
$$\times\delta(\vec{p}'-\vec{p}'')S_0(\vec{p}'',\zeta_n)(1-\gamma^5). \tag{45}$$

Here $\vec{\nabla}_1\equiv\vec{\nabla}_{p1}$, $\vec{\nabla}'\equiv\vec{\nabla}_{p'}$, and I used the identity

$$g_{\mu\nu}\gamma^\mu(1-\gamma^5)S_0\gamma^\nu(1-\gamma^5)=-4S_0(1-\gamma^5). \tag{46}$$

Integrating by parts in the term proportional to $\vec{\nabla}'\times\vec{p}''$, neglecting Ω^2 and using $\partial S_0(\vec{p},\zeta_n)/\partial\zeta_n=\partial S_0(\vec{p},\zeta_n)/\partial\chi$ we find

$$\mathfrak{M}(\vec{p}_1,\vec{p}_2,\zeta_l)\Rightarrow\left[1+i\vec{\Omega}\cdot(\vec{\nabla}_1\times\vec{p}_2)\frac{\partial}{\partial\chi}\right]\delta(\vec{p}_1-\vec{p}_2)\mathfrak{M}_0(\vec{p}_2,\zeta_l)$$
$$-\frac{e^2}{\beta(2\pi)^3}i\vec{\Omega}\cdot\left[[\vec{\nabla}_1\delta(\vec{p}_1-\vec{p}_2)]\sum_n\frac{\partial}{\partial\zeta_n}\int d^3p'(\vec{p}_2-\vec{p}')D^R_0(\vec{p}_2-\vec{p}',\zeta_l-\zeta_n)S_0(\vec{p}',\zeta_n)(1-\gamma^5)\right]. \tag{47}$$

Here

$$\mathfrak{M}_0(\vec{p},\zeta_l)=e^2\beta^{-1}(2\pi)^{-3}\sum_n\int d^3p'\,D^R_0(\vec{p}-\vec{p}',\zeta_l-\zeta_n)S_0(\vec{p}',\zeta_n)(1-\gamma^5) \tag{48}$$

is the mass operator for a nonrotating system.

Let us now consider the contribution of the terms proportional to $\vec{\Sigma}$ and $\hat{M}$. Neglecting all other terms we have

$$\mathfrak{M}(\vec{p}_1,\vec{p}_2,\zeta_l)\Rightarrow-\tfrac{1}{4}e^2\Omega\beta^{-1}(2\pi)^{-3}\delta(\vec{p}_1-\vec{p}_2)$$
$$\times\sum_n\int d^3p'\left[2i\epsilon_{0\mu\nu3}\gamma^\mu S_0(\vec{p}',\zeta_n)\gamma^\nu(1-\gamma^5)\frac{\partial}{\partial\zeta_l}D^R_0(\vec{p}_2-\vec{p}',\zeta_l-\zeta_n)\right.$$
$$\left.+\tfrac{1}{2}g_{\mu\nu}D^R_0(\vec{p}_2-\vec{p}',\zeta_l-\zeta_n)\gamma^\mu(1-\gamma^5)\Sigma_3\frac{\partial}{\partial\zeta_n}S_0(\vec{p}',\zeta_n)\gamma^\nu(1-\gamma^5)\right]. \tag{49}$$

Using the commutation relations for γ matrices it can be shown that

$$\epsilon_{0\mu\nu3}\gamma^\mu\gamma^\sigma\gamma^\nu=i(\Sigma_3\gamma^\sigma+\gamma^\sigma\Sigma_3), \tag{50}$$

$$g_{\mu\nu}\gamma^{\mu}(1-\gamma^5)\Sigma_3\gamma^{\sigma}\gamma^{\nu}(1-\gamma^5)=4\gamma^{\sigma}\Sigma_3(1-\gamma^5)\,, \tag{51}$$

and Eq. (49) takes the form

$$\begin{aligned}\mathfrak{M}(\vec{p}_1,\vec{p}_2,\zeta_l)\Rightarrow &\tfrac{1}{2}\vec{\Omega}\cdot\vec{\Sigma}\,\delta(\vec{p}_1-\vec{p}_2)\frac{\partial}{\partial\chi}\mathfrak{M}_0(\vec{p}_2,\zeta_l)\\ &-\tfrac{1}{2}\frac{e^2}{\beta(2\pi)^3}\delta(\vec{p}_1-\vec{p}_2)\sum_n\frac{\partial}{\partial\zeta_n}\int d^3p'D_0^R(\vec{p}_2-\vec{p}\,',\zeta_l-\zeta_n)\\ &\times[\vec{\Sigma}\cdot\vec{\Omega}S_0(\vec{p}\,',\zeta_n)+S_0(\vec{p}\,',\zeta_n)\vec{\Sigma}\cdot\vec{\Omega}](1-\gamma^5)\,.\end{aligned} \tag{52}$$

Combining Eqs. (47) and (52) we obtain finally

$$\begin{aligned}\mathfrak{M}(\vec{p}_1,\vec{p}_2,\zeta_l)=&\left[1+i\vec{\Omega}\cdot(\vec{\nabla}_1\times\vec{p}_2)\frac{\partial}{\partial\chi}+\tfrac{1}{2}\vec{\Omega}\cdot\vec{\Sigma}\frac{\partial}{\partial\chi}\right]\delta(\vec{p}_1-\vec{p}_2)\mathfrak{M}_0(p_2)\\ &-ie^2\vec{\Omega}\cdot\left\{[\vec{\nabla}_1\delta(\vec{p}_1-\vec{p}_2)]\int_{p'}\frac{\partial}{\partial p_0'}[(\vec{p}_2-\vec{p}\,')D_0^R(p_2-p')S_0(p')]\right\}(1-\gamma^5)\\ &-\tfrac{1}{2}e^2\delta(\vec{p}_1-\vec{p}_2)\int_{p'}\frac{\partial}{\partial p_0'}\{D_0^R(p_2-p')[\vec{\Sigma}\cdot\vec{\Omega}S_0(p')+S_0(p')\vec{\Sigma}\cdot\vec{\Omega}]\}(1-\gamma^5)\,.\end{aligned} \tag{53}$$

Here $p_2=(\zeta_l+\chi,\vec{p}_2)$, $p'=(p_0',\vec{p}\,')$, $p_0'=\zeta_n+\chi$, and I introduced a shorthand notation

$$\int_{p'}\equiv\frac{1}{\beta(2\pi)^3}\sum_{p_0'}\int d^3p'\,. \tag{54}$$

The last two terms in Eq. (53) are free of divergences and, as expected, all the divergences are confined to $\mathfrak{M}_0(p_2)$.

B. The equilibrium current

In Eq. (44) for the equilibrium current let us take $S(\vec{p}_1,\vec{p}\,',\zeta_l)$ in the representation of Eq. (7) and $S(\vec{p}\,',\vec{p}_2,\zeta_l)$ in the representation of Eq. (8). Then the terms with gradients vanish upon integration by parts and Eq. (44) takes the form

$$\vec{J}=-\beta^{-1}(2\pi)^{-3}\sum_l e^{\epsilon\zeta_l}d^3p_1\,d^3p_2\,\mathrm{Tr}\{\vec{\gamma}S(\vec{p}_1,\zeta_l)\mathfrak{M}(\vec{p}_1,\vec{p}_2,\zeta_l)\tilde{S}(\vec{p}_2,\zeta_l)\}_{\epsilon\to 0}\,, \tag{55}$$

where $S(\vec{p},\zeta_l)$ and $\tilde{S}(\vec{p},\zeta_l)$ are given by Eqs. (22) and (23), respectively.

Since $\vec{J}$ has to be parallel to $\vec{\Omega}$, it is sufficient to calculate $\vec{J}\cdot\vec{\Omega}$. Substituting Eq. (53) in Eq. (55) we obtain, after a simple transformation,

$$\begin{aligned}\vec{J}\cdot\vec{\Omega}=&-\tfrac{1}{2}e^2\int_p\int_{p'}\mathrm{Tr}\left\{(\vec{\gamma}\cdot\vec{\Omega})(\vec{\Sigma}\cdot\vec{\Omega})\frac{\partial S_0(p)}{\partial\chi}S_0(p')(1-\gamma^5)S_0(p)\right.\\ &\left.+\vec{\gamma}\cdot\vec{\Omega}S_0(p)S_0(p')(1-\gamma^5)\frac{\partial S_0(p)}{\partial\chi}\vec{\Sigma}\cdot\vec{\Omega}\right\}D_0^R(p-p')\\ &+ie^2\int_p\int_{p'}\mathrm{Tr}\left\{(\vec{\gamma}\cdot\vec{\Omega})[(\vec{p}\times\vec{\Omega})\cdot\vec{\nabla}_pS_0(p)]\frac{\partial S_0(p')}{\partial\chi}(1-\gamma^5)S_0(p)\right\}D_0^R(p-p')\\ &-\tfrac{1}{2}e^2\int_p\int_{p'}\mathrm{Tr}\left\{(\vec{\gamma}\cdot\vec{\Omega})S_0(p)(\vec{\Sigma}\cdot\vec{\Omega})\frac{\partial S_0(p')}{\partial\chi}(1-\gamma^5)S_0(p)\right\}D_0^R(p-p')\\ &-ie^2\int_p\int_{p'}\frac{\partial}{\partial p_0'}\mathrm{Tr}\{(\vec{\gamma}\cdot\vec{\Omega})[(\vec{p}-\vec{p}\,')\times\vec{\Omega}]\cdot\vec{\nabla}_pS_0(p)S_0(p')(1-\gamma^5)S_0(p)\}D_0^R(p-p')\\ &+\tfrac{1}{2}e^2\int_p\int_{p'}\frac{\partial}{\partial p_0'}\mathrm{Tr}\{(\vec{\gamma}\cdot\vec{\Omega})S_0(p)[\vec{\Sigma}\cdot\vec{\Omega}S_0(p')+S_0(p')\vec{\Sigma}\cdot\vec{\Omega}](1-\gamma^5)S_0(p)\}D_0^R(p-p')\\ \equiv&\,\vec{\Omega}\cdot(\vec{J}_1+\vec{J}_2+\vec{J}_3+\vec{J}_4+\vec{J}_5)\,.\end{aligned} \tag{56}$$

Here $p=(p_0,\vec{p})$, $p'=(p_0',\vec{p}\,')$, $p_0=\zeta_l+\chi$, $p_0'=\zeta_n+\chi$.

Using Eq. (25) and the identity

$$(\vec{p}\times\vec{\Omega})\cdot\vec{\nabla}_p S_0(p)=\frac{i}{2}[\vec{\Sigma}\cdot\vec{\Omega}S_0(p)-S_0(p)\vec{\Sigma}\cdot\vec{\Omega}] \tag{57}$$

we can rewrite the first three terms of Eq. (56) as

$$\vec{\Omega}\cdot(\vec{J}_1+\vec{J}_2+\vec{J}_3)=\tfrac{1}{2}e^2\Omega^2\frac{\partial}{\partial\chi}\int_p\int_q \mathrm{Tr}\{\gamma^0 S_0(p)S_0(p-q)S_0(p)\}D_0^R(q), \tag{58}$$

where I made a change of variables $q=p-p'$. Calculating the trace and using the identity

$$2p_0\vec{p}\cdot(\vec{p}-\vec{q})-(p_0-q_0)(p_0^{\,2}+\vec{p}^{\,2})=p_0[q^2-(p-q)^2]-q_0p^2 \tag{59}$$

we get

$$\vec{\Omega}\cdot(\vec{J}_1+\vec{J}_2+\vec{J}_3)=2e^2\Omega^2\lim_{M\to\infty}M^2\int_p\int_q\left\{\frac{p_0}{p^4(p-q)^2(q^2-M^2)}-\frac{p_0}{p^4q^2(q^2-M^2)}-\frac{q_0}{p^2(p-q)^2q^2(q^2-M^2)}\right\}. \tag{60}$$

Now let us consider $\vec{J}_4$. It is easily understood that in $\vec{\nabla}_p S_0(p)$ only the numerator has to be differentiated, since the other term vanishes on taking the trace with γ^5. Thus, we can replace $\vec{\nabla}_p S_0(p)$ by $\vec{\gamma}\,p^{-2}$. At this point it is convenient to average over directions of $\vec{\Omega}$. This amounts to replacing $\Omega^i\Omega^k$ by $\frac{1}{3}\Omega^2\delta^{ik}$. After such averaging

$$(\vec{\gamma}\cdot\vec{\Omega})[(\vec{p}-\vec{p}\,')\times\vec{\Omega}]\cdot\vec{\gamma}$$

becomes

$$-\tfrac{2}{3}i\Omega^2\vec{\Sigma}\cdot(\vec{p}-\vec{p}\,'),$$

where I have used $\vec{\Sigma}=\frac{i}{2}\vec{\gamma}\times\vec{\gamma}$. Now the trace is easily calculated and we obtain

$$\vec{\Omega}\cdot\vec{J}_4=-\tfrac{8}{3}e^2\Omega^2\lim_{M\to\infty}\int_p\int_{p'}\frac{\partial}{\partial p_0}\{p^{-4}p'^{-2}[p_0\vec{p}\,'\cdot(\vec{p}-\vec{p}\,')-p_0'\vec{p}\cdot(\vec{p}-\vec{p}\,')]D_0^R(p-p')\}. \tag{61}$$

Similarly, we find

$$\vec{\Omega}\cdot\vec{J}_5=4e^2\Omega^2\lim_{M\to\infty}\int_p\int_{p'}\frac{\partial}{\partial p_0}\{p_0'p^{-2}p'^{-2}D_0^R(p-p')\}+\tfrac{8}{3}e^2\Omega^2\int_p\int_{p'}\frac{\partial}{\partial p_0}\{p^{-4}p'^{-2}[p_0'\vec{p}^{\,2}-p_0\vec{p}\cdot\vec{p}\,']D_0^R(p-p')\}. \tag{62}$$

An explicit calculation shows (see the next subsection) that $\vec{\Omega}\cdot(\vec{J}_4+\vec{J}_5)$ is finite even if we use $D_0(p-p')$ instead of $D_0^R(p-p')$. Then it is clear from Eq. (43) that in the expression for $\vec{\Omega}\cdot(\vec{J}_4+\vec{J}_5)$ we can simultaneously replace $D_0^R(p-p')$ by $D_0(p-p'|M)$ and $\lim_{M\to\infty}$ by $(\lim_{M\to0}-\lim_{M\to\infty})$. This step simplifies the calculation, because it decreases the number of poles in the integrands of Eqs. (61) and (62). Introducing a new variable $q=p-p'$ and using the identity

$$3(p_0-q_0)p^2+2(p_0-q_0)\vec{p}\cdot(\vec{p}+\vec{q})-2p_0(\vec{p}^{\,2}-\vec{q}^{\,2})=(p_0-q_0)(p-q)^2+2p_0p^2-(3p_0-q_0)q^2 \tag{63}$$

we obtain

$$\vec{\Omega}\cdot(\vec{J}_4+\vec{J}_5)=\tfrac{4}{3}e^2\Omega^2(\lim_{M\to0}-\lim_{M\to\infty})\int_p\int_q\frac{\partial}{\partial q_0}\left\{\frac{p_0-q_0}{p^4(q^2-M^2)}-\frac{3p_0-q_0}{p^4(p-q)^2}-\frac{M^2(3p_0-q_0)}{p^4(p-q)^2(q^2-M^2)}+\frac{2p_0}{p^2(p-q)^2(q^2-M^2)}\right\}. \tag{64}$$

The integrals in Eqs. (60) and (64) will be calculated in the next subsection. Here I would like to indicate two traps one can easily get into. The first trap is the zero-temperature limit. If we set $T=0$ at the beginning and replace the sum over q_0 by an integral, then Eq. (64) would immediately give zero. However, as we shall see in the next subsection, the correct procedure (taking the limit $T\to0$ after calculating the sums over p_0 and q_0) gives a nonzero result. The second danger has to do with regularization. Since the term with $\lim_{M\to0}$ in Eq. (64) is finite, one can be tempted to drop the term with $\lim_{M\to\infty}$. However, it will be shown that the regulator term gives a finite contribution. A similar situation occurs when Pauli-Villars method is used to calculate axial[18] or trace[19] anomalies.

C. Calculation of the integrals

We shall first consider $\vec{J}_4+\vec{J}_5$. Using Eq. (29), let us perform the q_0 summation in the last term in the curly brackets of Eq. (64):

$$\frac{1}{\beta}\sum_n \frac{\partial}{\partial \nu_n}[(p-q)^2(q^2-M^2)]^{-1} = -\frac{1}{2\pi i}\int_c d\nu n'(\nu)[(p-q)^2(q^2-M^2)]^{-1}$$
$$=\frac{f'(\epsilon_{p-q}-\chi)}{2\epsilon_{p-q}[(p_0-\epsilon_{p-q})^2-E_q^2]}-\frac{f'(\epsilon_{p-q}+\chi)}{2\epsilon_{p-q}[(p_0+\epsilon_{p-q})^2-E_q^2]}$$
$$+\frac{n'(E_q)}{2E_q[(p_0-E_q)^2-\epsilon_{p-q}^2]}-\frac{n'(E_q)}{2E_q[(p_0+E_q)^2-\epsilon_{p-q}^2]},$$

where $E_q=(\vec{q}^{\,2}+M^2)^{1/2}$ and I have used for real ϵ, $n'(-\epsilon)=n'(\epsilon)$, $n'(\xi_l+\epsilon)=-f'(\epsilon)$. In the zero-temperature limit $n(\epsilon)=0$, $f(\epsilon-\chi)=\theta(\chi-\epsilon)$, $f'(\epsilon-\chi)=-\delta(\epsilon-\chi)$, and thus

$$\frac{1}{\beta}\sum_n \frac{\partial}{\partial \nu_n}[(p-q)^2(q^2-M^2)]^{-1}=-\frac{\delta(\epsilon_{p-q}-\chi)}{2\chi(\xi_l^2-E_q^2)}.$$

Carrying out the p_0 summation in a similar manner, we obtain

$$\frac{1}{\beta^2}\sum_{p_0,q_0}\frac{\partial}{\partial q_0}\frac{p_0}{p^2(p-q)^2(q^2-M^2)}=-\frac{1}{4\chi}\delta(\epsilon_{p-q}-\chi)\left\{\frac{\theta(\chi-\epsilon_p)}{(\epsilon_p-\chi)^2-E_q^2}+\frac{\chi}{E_q[(\epsilon_p+E_q)^2-\chi^2]}\right\}.$$

In the first and second terms of Eq. (64) the q_0 and p_0 summation can be done in a similar way (the first term gives no contribution at zero temperature). In the third term it is convenient to write

$$p^{-4}=\frac{\partial}{\partial m^2}(p^2-m^2)^{-1}\bigg|_{m=0}.$$

The resulting expression is

$$\vec{\Omega}\cdot(\vec{J}_4+\vec{J}_5)=-\frac{e^2\Omega^2}{3(2\pi)^6}\left(\lim_{M\to 0}-\lim_{M\to\infty}\right)$$
$$\times\int d^3p\,d^3q\,\delta(\epsilon_{p-q}-\chi)\left\{\frac{2\theta(\chi-\epsilon_p)}{\chi[(\epsilon_p-\chi)^2-E_q^2]}+\frac{3}{2\chi^2}\delta(\epsilon_p-\chi)-\frac{\theta(\epsilon_p-\chi)}{2\epsilon_p^3}+\frac{2}{E_q[(\epsilon_p+E_q)^2-\chi^2]}\right.$$
$$\left.-M^2\frac{\partial}{\partial m^2}\left[\frac{\theta(\chi-\tilde{\epsilon}_p)(2\tilde{\epsilon}_p+\chi)}{\chi\tilde{\epsilon}_p[(\tilde{\epsilon}_p-\chi)^2-E_q^2]}+\frac{E_q+3\tilde{\epsilon}_p}{E_q\tilde{\epsilon}_p[(E_q+\tilde{\epsilon}_p)^2-\chi^2]}\right]_{m=0}\right\}, \qquad (65)$$

where $\tilde{\epsilon}_p=(\vec{p}^{\,2}+m^2)^{1/2}$ and I have used the identity

$$\frac{3\chi-2E_q}{E_q[(E_q-\chi)^2-\tilde{\epsilon}_p^2]}+\frac{\chi-2\tilde{\epsilon}_p}{\tilde{\epsilon}_p[(\tilde{\epsilon}_p+\chi)^2-E_q^2]}=-\frac{\chi(E_q+3\tilde{\epsilon}_p)}{E_q\tilde{\epsilon}_p[(E_q+\tilde{\epsilon}_p)^2-\chi^2]}.$$

Now we have to calculate the integrals in Eq. (65) in the limits $M\to 0$ and $M\to\infty$. Let us consider the zero-mass limit first; then the term proportional to M^2 does not contribute. To avoid spurious infrared divergences, we shall not set $M=0$ in the remaining terms until after the $\vec{q}$ integration.

The $\vec{q}$ integration can be performed using the relation

$$\int d^3q\,\delta(\epsilon_{p-q}-\chi)F(\vec{q}^{\,2})=\frac{2\pi\chi}{\epsilon_p}\int_{|\epsilon_p-\chi|}^{\epsilon_p+\chi}F(q^2)q\,dq \qquad (66)$$

and we obtain

$$\vec{\Omega}\cdot(\vec{J}_4+\vec{J}_5)|_{M\to 0}=-\frac{e^2\Omega^2}{3(2\pi)^5}\int d^3p\left\{3\delta(\epsilon_p-\chi)-\frac{\chi^2}{\epsilon_p^3}\theta(\epsilon_p-\chi)-\epsilon_p^{-1}\theta(\chi-\epsilon_p)\ln\frac{4\chi\epsilon_p+M^2}{M^2}\right.$$
$$\left.+\epsilon_p^{-1}\ln\left|\frac{[(\epsilon_p+\chi)^2+M^2]^{1/2}+\epsilon_p-\chi}{[(\epsilon_p+\chi)^2+M^2]^{1/2}+\epsilon_p+\chi}\,\frac{[(\epsilon_p-\chi)^2+M^2]^{1/2}+\epsilon_p+\chi}{[(\epsilon_p-\chi)^2+M^2]+\epsilon_p-\chi}\right|\right\}_{M\to 0}. \qquad (67)$$

At this point it is convenient to go to the limit $M=0$. As $M\to 0$, the last logarithm in Eq. (61) becomes

$$\ln\frac{\epsilon_p^2}{\epsilon_p^2-\chi^2}$$

for $\epsilon_p>\chi$ and

$$\ln\frac{4\epsilon_p\chi(\chi-\epsilon_p)}{M^2(\chi+\epsilon_p)}$$

for $\epsilon_p<\chi$. Note that dangerous terms proportional to $\ln M^2$ cancel out. After a simple transformation we obtain

$$\vec{\Omega}\cdot(\vec{J}_4+\vec{J}_5)|_{M\to 0}=-\frac{2e^2\Omega^2\chi^2}{3(2\pi)^4}\left\{3+\int_0^1\ln\left(\frac{1-x}{1+x}\right)x\,dx+\int_1^\infty\left(\ln\frac{x^2}{x^2-1}-\frac{1}{x^2}\right)x\,dx\right\}.$$

To calculate the second integral in the curly brackets, let us introduce $z=x^2$ and define

$$I(\epsilon)=\tfrac{1}{2}\int_1^\infty\left(\ln\frac{z+\epsilon}{z+\epsilon-1}-\frac{1}{z}\right)dz$$

with $\epsilon>0$. $I(\epsilon)$ is easily calculated using integration by parts and we find $I(0)=\frac{1}{2}$. Similarly,

$$\int_0^1\ln\left(\frac{1-x}{1+x}\right)x\,dx=1$$

and thus

$$(\vec{J}_4+\vec{J}_5)_{M\to 0}=-\frac{5\alpha}{12\pi^3}\chi^2\vec{\Omega}, \qquad (68)$$

where $\alpha=e^2/4\pi$.

The infinite mass limit can be found in a similar manner, but we shall take a shorter route. Let us first examine the two terms proportional to M^2 in Eq. (65). In the first of these terms the integration is limited to the region $|\vec{p}|,|\vec{q}|\lesssim\chi$ and we can neglect $(\tilde{\epsilon}_p-\chi)^2$ and $\vec{q}^{\,2}$ compared to M^2 in the denominator. In the second term the integration region is infinite and we can neglect χ^2, but not $|\vec{p}|$ or $|\vec{q}|$. Note, however, that when $|\vec{p}|\sim M$ or $|\vec{q}|\sim M$, then the difference $|\vec{p}-\vec{q}|$ can be neglected (for $M\gg\chi$) because of the factor $\delta(\epsilon_{p-q}-\chi)$. Therefore, we can get $\vec{p}=\vec{q}$ everywhere except $\delta(\epsilon_{p-q}-\chi)$. Treating the remaining terms similarly, we obtain

$$\vec{\Omega}\cdot(\vec{J}_4+\vec{J}_5)|_{M\to\infty}=-\frac{e^2\Omega^2}{3(2\pi)^6}\int d^3p\,d^3q\,\delta(\epsilon_{p-q}-\chi)\left\{\frac{3}{2\chi^2}\delta(\epsilon_p-\chi)-\frac{1}{2\epsilon_p^3}\theta(\epsilon_p-\chi)+\frac{2}{E_p(\epsilon_p+E_p)^2}-M^2\frac{\partial}{\partial m^2}\left[\frac{\theta(\chi-\tilde{\epsilon}_p)(2\tilde{\epsilon}_p+\chi)}{\chi\tilde{\epsilon}_pM^2}-\frac{E_p+3\tilde{\epsilon}_p}{E_p\tilde{\epsilon}_p(E_p+\tilde{\epsilon}_p)^2}\right]_{m=0}\right\}_{M\to\infty}. \qquad (69)$$

Differentiating with respect to m^2 and setting $m=0$, we find after some algebra,

$$\begin{aligned}\vec{\Omega}\cdot(\vec{J}_4+\vec{J}_5)|_{M\to\infty}&=-\tfrac{1}{3}(2\pi)^{-6}e^2\Omega^2\lim_{M\to\infty}M^2\int d^3p\,d^3q\,\delta(\epsilon_{p-q}-\chi)\epsilon_p^{-1}E_p^{-1}(\epsilon_p+E_p)^{-3}\\&=-\frac{\chi^2e^2\Omega^2}{12\pi^4}\lim_{M\to\infty}M^2\int_0^\infty\frac{p\,dp}{E_p(p+E_p)^3}\\&=-\frac{\alpha\chi^2\Omega^2}{24\pi^3}.\end{aligned} \qquad (70)$$

Combining Eqs. (68) and (70) we finally obtain

$$\vec{J}_4+\vec{J}_5=-\frac{3\alpha}{8\pi^3}\chi^2\vec{\Omega}. \qquad (71)$$

A similar technique can be used to calculate $\vec{J}_1+\vec{J}_2+\vec{J}_3$ from Eq. (60). The result is

$$\vec{J}_1+\vec{J}_2+\vec{J}_3=\frac{3\alpha}{8\pi^3}\chi^2\vec{\Omega}. \qquad (72)$$

An independent test of this result can be obtained if we note that

$$n^{(2)}=-2e^2\int_p\int_q\mathrm{Tr}\{\gamma^0S_0(p)S_0(p-q)S_0(p)\}D_0^R(q) \qquad (73)$$

is the second-order correction to the electron density in (massless) QED. Then we can rewrite Eq. (58) as

$$\vec{\Omega}\cdot(\vec{J}_1+\vec{J}_2+\vec{J}_3)=-\tfrac{1}{4}\Omega^2\partial n^{(2)}/\partial\chi. \qquad (74)$$

The quantity $n^{(2)}$ has been calculated by a number of authors (see, e.g., Ref. 14) as

$$n^{(2)}=-\frac{\alpha}{2\pi^3}\chi^3. \qquad (75)$$

Substituting Eq. (75) in Eq. (74) we obtain Eq. (72).

Finally, combining Eqs. (71) and (72) we obtain

$$\vec{J}=\vec{J}_1+\vec{J}_2+\vec{J}_3+\vec{J}_4+\vec{J}_5=0.$$

V. CONCLUSION

Equilibrium currents induced by rotation have been calculated for a system described by the Lagrangian (2) in the following two cases:

(i) Classical external field $A^\mu = A^0(1, \vec{\Omega}\times\vec{x})$. Physically, this corresponds to a neutral-current interaction at low energies (see Sec. III).

(ii) Interacting quantum fields at finite density and zero temperature. It has been found that in both cases the equilibrium current on the rotation axis vanishes in the lowest order of perturbation theory. The question naturally arising from this result is whether a nonzero current appears in higher orders of perturbation theory or in different models or is there some deep physical reason which makes the equilibrium current equal to zero for all particles except neutrinos?

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APPENDIX

To discuss the relation between the theory with $\chi = T = 0$, $\Omega \neq 0$ and that with $\chi = T = \Omega = 0$, it is convenient to use the cylindrical coordinates. The fermion Green's function in cylindrical coordinates is given by[4]

$$S'(\vec{x}_1, \tau_1; \vec{x}_2, \tau_2) = \beta^{-1}\sum_l e^{-\zeta_l \tau} S'(\vec{x}_1, \vec{x}_2, \zeta_l), \quad \text{(A1)}$$

where $\zeta_l = (2l+1)\pi i \beta^{-1}$, $\tau = \tau_1 - \tau_2$,

$$S'(\vec{x}_1, \vec{x}_2, \zeta_l) = \exp\left(-i\Omega\frac{\partial}{\partial\zeta_l}\frac{\partial}{\partial\phi_1}\right) S_0'(\vec{x}_1, \vec{x}_2, \zeta_l), \quad \text{(A2)}$$

and S_0' is the Green's function for a nonrotating system.

At $\chi = T = 0$, ζ_l becomes a continuous variable and Eq. (A1) takes the form

$$S'(\vec{x}_1, \tau_1; \vec{x}_2, \tau_2) = (2\pi i)^{-1}\int_{-i\infty}^{i\infty} d\zeta\, e^{-\zeta\tau} S'(\vec{x}_1, \vec{x}_2, \zeta). \quad \text{(A3)}$$

From Eqs. (A2) and (A3) we find

$$S'(\vec{x}_1, \tau_1; \vec{x}_2, \tau_2) = \exp\left(-i\Omega\tau\frac{\partial}{\partial\phi_1}\right) S_0'(\vec{x}_1, \tau_1; \vec{x}_2, \tau_2), \quad \text{(A4)}$$

or

$$S'(r_1, \phi_1, z_1, \tau_1; r_2, \phi_2, z_2, \tau_2) = S_0'(r_1, \phi_1 - i\Omega\tau_1, z_1, \tau_1; r_2, \phi_2 - i\Omega\tau_2, z_2, \tau_2). \quad \text{(A5)}$$

Here I have used

$$S'(r_1, \phi_1, z_1, \tau_1; r_2, \phi_2, z_2, \tau_2) = S'(r_1, \phi_1 - \phi_2, z_1 - z_2, \tau_1 - \tau_2; r_2, 0, 0, 0). \quad \text{(A6)}$$

Quite similarly, it can be shown that the free-boson zero-temperature Green's function can be obtained from D_0' by the same coordinate transformation $\phi \to \phi - i\Omega\tau$.

The function S_0' can be written as

$$S_0'(r_1, \phi_1 z_1, \tau_1; r_2, \phi_2, z_2, \tau_2) = \sum_m e^{im(\phi_1 - \phi_2)} a_m(r_1, z_1, \tau_1; r_2, z_2, \tau_2), \quad \text{(A7)}$$

where the summation is taken over all half-integer values of m. A similar expression with an integer m can be written for the boson Green's function. After the ϕ integrations, only the terms with $\Sigma m_i = 0$ remain at each vertex in all Feynman diagrams, and it is easily understood from Eqs. (A5) and (A7) that the Ω dependence drops out everywhere except the external lines. This shows that Eq. (A5) holds for the Green's functions of interacting fields as well. This completes the proof of Eq. (13).

In conclusion of this Appendix, I want to give a direct proof of Eq. (13), without using the perturbation expansion. By definition, the fermion Green's function in cylindrical coordinates is given by (see Ref. 4)

$$\mathcal{S}'_{\alpha\beta}(\vec{x}_1, \tau_1; \vec{x}_2, \tau_2) = \mathrm{Tr}\{\rho T_\tau \psi_\alpha(\vec{x}_1, \tau_1)\bar\psi_\beta(\vec{x}_2, \tau_2)\}, \quad \text{(A8)}$$

where $\vec{x}$ stands for (r, ϕ, z), T_τ is the τ-ordering operator,

$$\psi(\vec{x}, \tau) = \exp[\tau(H - \vec{M}\cdot\vec{\Omega})]\psi(\vec{x}, 0) \times \exp[-\tau(H - \vec{M}\cdot\vec{\Omega})] \quad \text{(A9)}$$

is the Matsubara field operator in cylindrical coordinates, and ρ is the statistical operator. Assuming for definiteness that $\tau \equiv \tau_1 - \tau_2 > 0$ we can rewrite Eq. (A9) as

$$\mathcal{S}'_{\alpha\beta}(\vec{x}_1, \tau; \vec{x}_2, 0) = \mathrm{Tr}\{\rho e^{\tau(H - \vec{M}\cdot\vec{\Omega})}\psi_\alpha(\vec{x}_1, 0) e^{-\tau(H - \vec{M}\cdot\vec{\Omega})}\bar\psi_\beta(\vec{x}_2, 0)\} = \sum_{n,k}\rho_n \exp[\tau(\omega_{nk} - \Omega m_{nk})]\psi_{\alpha nk}(\vec{x}_1)\bar\psi_{\beta kn}(\vec{x}_2). \quad \text{(A10)}$$

Here

$$\psi_{\alpha nk}(\vec{x}) = \langle n|\psi_\alpha(\vec{x}, 0)|k\rangle, \quad \text{(A11)}$$

$|n\rangle$ and $|k\rangle$ are eigenstates of the Hamiltonian H

and z component of angular momentum M_z, $\omega_{nk}=E_n-E_k$, $m_{nk}=M_n-M_k$, M_n and E_n are eigenvalues of M_z and H, respectively. At

$$\chi=T=0\ ,\qquad \rho_n=\delta_{on}\ , \tag{A12}$$

where $|0\rangle$ is the vacuum state with $E_0=M_0=0$, and Eq. (A10) takes the form

$$\mathcal{S}'_{\alpha\beta}(\vec{x}_1,\tau;\vec{x}_2,0)_{\chi=T=0} = \sum_K \exp[-\tau(E_k-\Omega M_k)]\psi_{\alpha 0k}(\vec{x}_1)\bar{\psi}_{\beta k0}(\vec{x}_2)\ . \tag{A13}$$

Noticing that in cylindrical coordinates the ϕ dependence of $\psi_{\alpha 0k}(\vec{x})$ is given by the factor $\exp(iM_k'\phi)$, we can rewrite Eq. (A13) as

$$\mathcal{S}'_{\alpha\beta}(\vec{x}_1,\tau;\vec{x}_2,0)_{\chi=T=0} =\exp\left(-i\Omega\tau\frac{\partial}{\partial\phi_1}\right)\mathcal{S}'_{\alpha\beta}(\vec{x}_1,\tau;\vec{x}_2,0)_{\chi=T=\Omega=0}\ . \tag{A14}$$

Equation (13) immediately follows from Eq. (A14).

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