Zero-point oscillations, local stability, and the effective action

Barry J. Harrington* and Charles H. Tabb
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106
(Received 19 June 1980)

We define a gauge-invariant action functional for non-Abelian gauge theories by deriving 't Hooft's generating functional from the vacuum-to-vacuum transition amplitude, and we explicitly demonstrate its gauge-invariance properties. We then examine the effective action in the loop expansion and show that in the presence of a nonvanishing background field the unrenormalized one-loop term can be written as the change in the zero-point energy of the theory. Finally, we consider the interpretation of an imaginary part of the effective action and its relation to local stability by examining the source of such an imaginary part within the Euclidean theory.

I. INTRODUCTION

The effective action functional $\Gamma(\phi_i)$ of a quantum field theory is commonly used to study symmetry properties. It is widely used in a semiclassical (loop) approximation to study the effect of quantum phenomena on certain classical symmetries and hence is extremely useful in the study of spontaneous symmetry breakdown.[1] The effective action, like the classical action, may also be used to study the dynamics of the theory. It is a generator of Green's functions and has been used in the study of vacuum stability. In this regard, it is hoped it will be useful in the study of confinement in quantum chromodynamics (QCD), where a nontrivial vacuum configuration may be responsible for color confinement.[2]

There are a number of definitions of Γ in the literature.[1,3–8] We will define Γ as the generator of one-particle-irreducible Green's functions.[1,5] It has been shown for nongauge theories that this definition is such that[3]

$$\left\langle 0\left|\frac{\delta S}{\delta \underline{\phi}_i}\right|0\right\rangle=\frac{\delta\Gamma}{\delta\phi_i}\,, \tag{1}$$

where i runs over all indices of the theory including those of space-time, $S(\underline{\phi}_i)$ is the classical action, $\underline{\phi}_i$ is the operator field, and

$$\phi_i=\langle 0|\underline{\phi}_i|0\rangle\,, \tag{2}$$

i.e., ϕ_i is the vacuum expectation value of the quantum field. With this definition, we can also show that

$$\frac{\delta\Gamma}{\delta\phi_i}=-J_i \tag{3}$$

in analogy with the classical equations of motion, i.e., $J_i(\phi_i)$ is an external source coupled linearly to the fields, and also

$$\frac{\delta^2\Gamma}{\delta\phi_i\delta\phi_j}\,G^{jk}=-\,\delta_i^k\,, \tag{4}$$

where G^{jk} is the Feynman propagator of the theory. The higher functional derivatives of Γ are the full irreducible vertex functions of the theory, whereas the functional derivatives of the classical action give the bare vertex functions.[3,4] In the tree approximation, the effective action $\Gamma(\phi_i)$ reduces to the classical action $S(\phi_i)$.

Γ is usually derived from the vacuum-to-vacuum transition amplitude[1,3–5]

$$Z(J)=\langle 0^+|0^-\rangle^J\,, \tag{5}$$

where

$$Z(J)=\exp[iW(J)] \tag{6}$$

and $W(J)$ is the generating functional for connected Green's functions. Now Γ is defined by the Legendre transformation

$$\Gamma(\phi_i)=W(J_i)-J_i\phi_i\,, \tag{7}$$

where

$$\phi_i=\frac{\delta W}{\delta J_i}\,. \tag{8}$$

If the above definitions are extended to gauge theories, $\Gamma(A_\mu)$ is, in general, not gauge invariant except perhaps when evaluated at its minimum ($J_i=0$).[9] Renormalization procedures needed to regularize Γ will also be gauge dependent, although physical processes should still maintain the gauge invariance of the theory. Recently, a number of alternative functionals have been proposed in order to define a gauge-invariant functional from which we could give a gauge-invariant renormalization scheme.[10–14] These employ the so-called background-field methods and gauges. 't Hooft[15] defines such a functional. We will show in Sec. II that this functional can be derived from the usual definition of the effective-action functional by way of a Legendre transformation. We will then explicitly demonstrate its gauge-invariance properties.

Our interest in the above study has been piqued by recent developments in the literature. In an attempt to discover a model for the nontrivial vacuum of QCD, several authors have investigated the one-loop approximation to the effective action in the presence of a constant color magnetic field B_a^i.[16-23] Such a configuration provides an effective infrared cutoff for the theory, although renormalization-group calculations cannot be readily extended to small values of B. For such small values of B, the one-loop term lowers the field energy and, although the range of applicability of this calculation is unclear, it has been surmised that a constant color magnetic field leads to a lower ground-state energy than the zero-field (perturbative) ground-state energy. Nielsen and Olesen[16] calculate this one-loop effective action by extending the result that the one-loop effective action for nongauge theories in the presence of a nonzero constant background field can be written as the change in the zero-point energy of the theory due to the background field.[24-26] In Sec. III, we verify that this result can be extended to gauge theories with Γ defined as in Sec. II and to theories with spatially dependent background fields and we demonstrate under what conditions it will presumably break down.

The calculation of the effective action in the presence of a constant color magnetic background field also leads to an imaginary part of the effective action[16,18,22] in analogy with the effective action for a constant electric field in QED.[6] This has been interpreted as an indication of an instability of the theory when expanded around a constant B field, and Nielsen and Olesen proceed to search for other field configurations which presumably lead to even lower energies and a stable vacuum.[27-31] In Sec. IV, we investigate the field modes responsible for this imaginary part by examining the Euclidean version of the theory. We point out some ambiguities in the definition of the imaginary part when looked at from this point of view. By going to Euclidean space we are also able to make analogies with the classical stability analysis and we conclude that this instability is actually of classical origin although it appears in the first quantum correction. (Indeed, the classical Yang-Mills theory in the presence of a constant magnetic field has been shown to be unstable.[32-35]) We compare this type of instability with those of quantum origin pointed out by Callan and Coleman[1,36-38] and with those due to pair production pointed out by Schwinger.[6] In analogy with the last example we show under what conditions the imaginary part can be used to calculate a decay rate and under what conditions the loop expansion is still well defined.

In Sec. V we discuss and summarize our main results.

II. GAUGE-INVARIANT ACTION FUNCTIONAL

A. The effective action

't Hooft's background-field method[15] reduces to an older formulation[10,12,14] at the one-loop level. This older formulation generates reducible diagrams when we go to higher order in the loop expansion, whereas with 't Hooft's method we generate only irreducible diagrams. 't Hooft's functional is

$$\exp\left[\frac{i}{\hbar}G(\overline{A},J)\right]=\int[dA(x)][d\phi(x)][d\phi^*(x)]\exp\left[\frac{i}{\hbar}\int d^4x\left(\mathcal{L}(\overline{A}+A)+J_\mu^a A^{\mu a}-\frac{1}{2\alpha}[D_\mu^{ab}(\overline{A})A^{\mu b}]^2 - \phi^{a*}[D_\mu^{ac}(\overline{A})D^{\mu cb}(\overline{A}+A)]\phi^b\right)\right]. \tag{9}$$

A few comments are in order: $A_\mu^a(x)$ is a c-number field, representing the quantum field over which we are evaluating the functional integral. $\overline{A}_\mu^a(x)$ is a c-number external field around which we are expanding. $J_\mu^a=J_\mu^a(\overline{A})$, where $J_\mu^a(\overline{A})$ is to be determined by the equations of motion

$$\frac{d}{d\overline{A}_\mu^a}G[\overline{A},J(\overline{A})]=-J^{\mu a}(\overline{A}), \tag{10}$$

where the functional derivative above is meant in the sense of a total derivative rather than a partial derivative. The Lagrangian density $\mathcal{L}$ may in general contain other sources composed of charged or Higgs particles coupled in a gauge-invariant way. We will not consider this possibility in what follows. $\phi^a(x)$ and $\phi^{a*}(x)$ are ghost fields and

$$D_\mu^{ab}(\overline{A})=\delta^{ab}\partial_\mu+c^{acb}\overline{A}_\mu^c.$$

Also, consistent with Eq. (10), $J^{\mu a}(\overline{A})$ can be constructed so as to satisfy

$$D_\mu^{ab}(\overline{A})J^{\mu b}(\overline{A})=0. \tag{11}$$

Now define

$$\Gamma(\overline{A})\equiv G[\overline{A},J(\overline{A})]. \tag{12}$$

't Hooft demonstrates that $\Gamma(\bar{A})$ is indeed the generating functional of the irreducible vertices. To define the physical theory we demand that

$$J_\mu^a(\bar{A})=0\ . \tag{13}$$

This determines $\bar{A}_\mu^a$ and is analogous to the classical equations of motion since

$$\frac{d\Gamma(\bar{A})}{d\bar{A}_\mu^a}=0\ . \tag{14}$$

In analogy with the scalar theories, the second functional derivative of Γ is the inverse propagator and the higher functional derivatives give the irreducible vertices.

We will now proceed to derive $\Gamma(\bar{A})$ from the vacuum-to-vacuum transition amplitude:

$$\langle 0^+|0^-\rangle^J \equiv Z[J,\bar{A}(J)]=\exp\left(\frac{i}{\hbar}W[J,\bar{A}(J)]\right). \tag{15}$$

$W[J,\bar{A}(J)]$ is the generating functional for the connected Green's functions. $Z[J,\bar{A}(J)]$ can be derived from the Hamiltonian formulation of the functional integral within the A gauges[39] to give

$$Z[J,\bar{A}(J)]=\int [dA]\delta(\chi^a[A,\bar{A}(J)])\det\left(\frac{d\chi^a}{d\theta^b}\right)\times\exp\left\{\frac{i}{\hbar}\int d^4x[\mathcal{L}(A)+J_\mu^a A^{\mu a}]\right\}, \tag{16}$$

where $\bar{A}_\mu^a(J)$ is an as-yet-unspecified but invertible function of J. $\chi^a[A,\bar{A}(J)]$ is the gauge constraint and $\det(d\chi^a/d\theta^b)$ is the appropriate Faddeev-Popov determinant. Since χ^a is a cyclic variable in the sourceless theory we may freely fix χ^a as long as the theory remains well defined. We choose the so-called covariant background-gauge condition

$$\chi^a=D_\mu^{ab}(\bar{A})(A^{\mu b}-\bar{A}^{\mu b})-f^a(x)\ . \tag{17}$$

The action is invariant under inhomogeneous gauge transformations of $A_\mu^a(x)$:

$$A_\mu^a(x)\to A_\mu^a+c^{abc}\theta^b A_\mu^c-\partial_\mu\theta^a=A_\mu^a-D_\mu^{ab}(A)\theta^b\ . \tag{18}$$

Therefore,

$$\frac{d\chi^a}{d\theta^b}=\frac{d\chi^a}{dA_\mu^c}\frac{dA_\mu^c}{d\theta^b}=-D_\mu^{ac}(\bar{A})D^{\mu cb}(A)\ . \tag{19}$$

We now make the change of variables $A_\mu^a\to A_\mu^a+\bar{A}_\mu^a(J)$ to get

$$\exp\left\{\frac{i}{\hbar}W[J,\bar{A}(J)]\right\}=\int[dA]\delta(D_\mu^{ab}(\bar{A})A^{\mu b}-f^a(x))\det[-D_\mu^{ac}(\bar{A})D^{\mu cb}(A+\bar{A})]\times\exp\left\{\frac{i}{\hbar}\int d^4x[\mathcal{L}(A+\bar{A})+J^{\mu a}(A_\mu^a+\bar{A}_\mu^a)]\right\}. \tag{20}$$

Let us take W to be formally only a function of J, although as yet we still have not specified $\bar{A}_\mu^a(J)$. Then we may define Γ by the Legendre transformation:

$$\Gamma(A_p)\equiv W(J)-\int d^4x\,J_\mu^a\frac{dW}{dJ_\mu^a}\ , \tag{21}$$

where we define

$$A_{a,p}^\mu\equiv\frac{dW}{dJ_\mu^a} \tag{22}$$

as the physical field. Therefore,

$$\exp\left[\frac{i}{\hbar}\Gamma(A_p)\right]=\int[dA]\delta(D_\mu^{ab}(\bar{A})A^{\mu b}-f^a(x))\det[-D_\mu^{ab}(\bar{A})D^{\mu bc}(A+\bar{A})]\times\exp\left\{\frac{i}{\hbar}\int d^4x[\mathcal{L}(A+\bar{A})+J^{\mu a}(A_p)A_\mu^a+J^{\mu a}(A_p)(\bar{A}_\mu^a-A_\mu^a)]\right\}. \tag{23}$$

Thus if we choose $\bar{A}_\mu^a(J)$ such that

$$\bar{A}_\mu^a(J)=A_{\mu,p}^a(J)=\frac{dW(J)}{dJ^{\mu a}}\ , \tag{24}$$

then all the J dependence at the tree level will cancel. We now use the usual exponentiation[1,40] of the gauge-fixing term and the Faddeev-Popov determinant to give

$$\exp\left[\frac{i}{\hbar}\Gamma(\bar{A})\right]=\int[dA(x)][d\phi(x)][d\phi^*(x)]\exp\left[\frac{i}{\hbar}\int d^4x\left(\mathcal{L}(\bar{A}+A)+J_\mu^a(\bar{A})A^{\mu a}-\frac{1}{2\alpha}[D^{ab}(\bar{A})A^{\mu b}]^2-\phi^{a*}[D_\mu^{ac}(\bar{A})D^{\mu cb}(\bar{A}+A)]\phi^b\right)\right], \tag{9'}$$

and we have reconstructed 't Hooft's generating functional. Also, Eq. (10) follows from the Legendre transformation, Eqs. (21) and (22). Again as in the scalar case Γ generates the one-particle-irreducible Green's functions. To the tree level Γ is just the classical action. $\overline{A}^a_\mu(x)$, however, is no longer the vacuum expectation value of the quantum field. To see this, note that

$$\overline{A}^a_\mu(J)=\frac{dW[J,\overline{A}(J)]}{dJ^{\mu a}} =\frac{\delta W}{\delta J^{\mu a}}+\frac{\delta W}{\delta \overline{A}^b_\nu}\frac{d\overline{A}^b_\nu}{dJ^{\mu a}}\ , \tag{25}$$

where by the partial functional derivative δ we mean the functional derivative of W with respect to only the explicit J or $\overline{A}$ dependence. From the definition of W, Eqs. (15) and (16), we can see that

$$\frac{\delta W}{\delta J^{\mu a}}=\frac{\langle 0^+|A^a_\mu|0^-\rangle}{\langle 0^+|0^-\rangle}\equiv\langle A^a_\mu\rangle\ . \tag{26}$$

Therefore, $\overline{A}^a_\mu\neq\langle A^a_\mu\rangle$. The distinction develops because of the dependence of the gauge condition on $\overline{A}^a_\mu$. The difference between this and the scalar theory is the implicit J dependence in χ^a through its dependence on $\overline{A}^a_\mu(J)$. To see the explicit distinction, we calculate $\langle A^a_\mu\rangle$. In Eq. (26), let us change variables $A^a_\mu\to A^a_\mu+\overline{A}^a_\mu(J)$ to give

$$\langle A^a_\mu\rangle=\overline{A}^a_\mu(J)+\frac{\int[dA]A^a_\mu\delta[\chi^a(\overline{A},A+\overline{A})]\det(d\chi^a/d\theta^b)\exp\left\{\frac{i}{\hbar}\int d^4x[\mathcal{L}(A+\overline{A})-\mathcal{L}(\overline{A})+J^a_\mu A^{\mu a}]\right\}}{\int[dA]\delta[\chi^a]\det(d\chi^a/d\theta^b)\exp\left\{\frac{i}{\hbar}\int d^4x[\mathcal{L}(A+\overline{A})-\mathcal{L}(\overline{A})+J^a_\mu A^{\mu a}]\right\}}. \tag{27}$$

The second term above is the expectation value of the quantum field calculated in a "shifted"[3] theory. Therefore let us write this as

$$\langle A^a_\mu\rangle=\overline{A}^a_\mu(J)+\langle 0'_+|A^a_\mu|0'_-\rangle/\langle 0'_+|0'_-\rangle\ . \tag{28}$$

The difference between $\langle A^a_\mu\rangle$ and $\overline{A}^a_\mu(J)$ is of order $\hbar$. In the usual formulation for scalar fields, the expectation value of the quantum field within the shifted vacuum vanishes. Here, however, in order to maintain gauge invariance (as we shall presently show) the shifted vacuum expectation value of A^a_μ does not vanish, or to put it another way, the physical field or order parameter of the theory differs from the vacuum expectation value of the quantum field. This result is similar to that of Fischler and Brout[41] where they demonstrate that, to preserve gauge invariance for an Abelian theory with scalar fields, the order parameter is not taken to be the naive vacuum expectation value of the scalar field.

B. Gauge invariance

We now proceed to demonstrate the gauge-invariant properties of the effective action. If

$$\overline{A}^a_\mu\to\overline{A}^a_\mu+c^{abc}\theta^b\overline{A}^c_\mu-\partial_\mu\theta^a\ , \tag{29}$$

then Eq. (11) is consistent with

$$J^a_\mu(\overline{A})\to J^a_\mu+c^{abc}\theta^bJ^c_\mu\ . \tag{30}$$

If we make these transformations within the functional integral, Eq. (9), and if we also make the change of variables

$$\begin{aligned}A^a_\mu(x)&\to A^a_\mu+c^{abc}\theta^bA^c_\mu\ ,\\ \phi^a(x)&\to\phi^a+c^{abc}\theta^b\phi^c\ ,\\ \phi^{a*}(x)&\to\phi^{a*}+c^{abc}\theta^b\phi^{c*}\ ,\end{aligned} \tag{31}$$

then the functional integral, and hence the effective action Γ, is shown to be invariant with respect to gauge transformations of its argument, $\overline{A}^a_\mu(x)$ (Refs. 12 and 15). $\Gamma(\overline{A})$ does, however, depend on the choice of the background gauge parameter α. We will show that for the physical theory defined by Eqs. (13) and (14), the value of $\Gamma(\overline{A})$ is left invariant by changing α. The value of $\overline{A}^a_\mu$ $(J=0)$, however, is not invariant.

If we keep $\overline{A}^a_\mu$ fixed but make the change of variables

$$\begin{aligned}A^a_\mu&\to A^a_\mu+c^{abc}\theta^b(A^c_\mu+\overline{A}^c_\mu)-\partial_\mu\theta^a\\ &=A^a_\mu-D^{ab}_\mu(A+\overline{A})\theta^b\ ,\end{aligned} \tag{32}$$

then the value of the functional integral is invariant. This change of variables is equivalent to a gauge transformation of the quantity $(A^a_\mu+\overline{A}^a_\mu)$. Therefore, the action $\int d^4x\ \mathcal{L}(A+\overline{A})$ is left invariant. The measure $[dA]$ is invariant with respect to such linear transformations. The Faddeev-Popov determinant is also invariant as demonstrated below. (This follows a similar proof due to Abers and Lee.[40]) Define $\Delta(A,\overline{A})$ by

$$\Delta(A,\overline{A})\int\delta[D^\mu(\overline{A})A^\Omega_\mu-f(x)]d\Omega=1\ , \tag{33}$$

where

$$A_\mu^\Omega = \Omega(A_\mu + \overline{A}_\mu)\Omega^{-1} + \Omega\partial_\mu\Omega^{-1} - \overline{A}_\mu \ . \tag{34}$$

This is just the finite version of the infinitesimal transformation, Eq. (32). Also, $A_\mu \equiv A_\mu^a T^a$, where T^a is a matrix representation of the group generators. Now, $A_\mu^{\Omega\Omega'}$ is defined by group multiplication, i.e., $A_\mu^{\Omega\Omega'} = A_\mu^{\Omega''}$, where $\Omega\Omega' = \Omega''$. Also $d\Omega'' = d(\Omega\Omega') = d\Omega$. Therefore, by changing variables $A_\mu \to A_\mu^{\Omega'}$, we have

$$\Delta(A^{\Omega'}, \overline{A}) \int \delta[D^\mu(\overline{A})A_\mu^{\Omega'\Omega} - f(x)]d(\Omega'\Omega) = 1 \ . \tag{35}$$

The integral above is identical to the integral in Eq. (33), since we are integrating over all Ω. Therefore we must have

$$\Delta(A^{\Omega'}, \overline{A}) = \Delta(A, \overline{A}). \tag{36}$$

Equation (33) can be solved for $\Delta(A, \overline{A})$ near the constraint surface defined by the δ function, to give

$$\Delta(A, \overline{A}) = \det[D_\mu^{ab}(\overline{A})D^{\mu bc}(\overline{A} + A)] \ . \tag{37}$$

This is just the Faddeev-Popov determinant. Equations (36) and (37), together, demonstrate the invariance of the Faddeev-Popov determinant under the change of variables, Eq. (32).

The only term that we have not discussed is the gauge-fixing term. In its exponentiated version, we have the exponential of the integral of $-(1/2\alpha) \times [D_\mu^{ab}(\overline{A})A^{\mu b}]^2$. If we make the change of variables, Eq. (32), then

$$D_\mu^{ab}(\overline{A})A^{\mu b} \to D_\mu^{ab}(\overline{A})A^{\mu b} - D_\mu^{ab}(\overline{A})D^{\mu bc}(\overline{A} + A)\theta^c \ . \tag{38}$$

If we now specify θ^c such that

$$D_\mu^{ab}(\overline{A})D^{\mu bc}(\overline{A} + A)\theta^c = \frac{\delta\alpha}{2\alpha} D_\mu^{ab}(\overline{A})A^{\mu b} \ , \tag{39}$$

where $\delta\alpha$ is an infinitesimal, then

$$-\frac{1}{2\alpha}[D_\mu^{ab}(\overline{A})A^{\mu b}]^2 \to -\frac{1}{2\alpha}\left(1 - \frac{\delta\alpha}{2\alpha}\right)^2 [D_\mu^{ab}(\overline{A})^{\mu b}]^2$$
$$\simeq -\frac{1}{2(\alpha + \delta\alpha)}[D_\mu^{ab}(\overline{A})A^{\mu b}]^2 \ . \tag{40}$$

Therefore, the only effect of changing variables with the transformation (32) and (39) is to change the value of α to $\delta\alpha$. Since the functional integral is invariant under changes of integration variables, it is therefore independent of the choice of α. (If $J_\mu^a \neq 0$, this is no longer true.) Furthermore, 't Hooft demonstrates that apart from renormalizations due to fermions or scalar fields in the theory, the infinities of Γ will be gauge independent. Thus we can contruct a gauge-invariant renormalization scheme for the vector part of the theory.

When $J_\mu^a(\overline{A}) = 0$, the one-loop approximation to the above formalism is equivalent to an approximation in which $\overline{A}_\mu^a(x)$ satisfies the classical equations of motion. The one-loop diagrams satisfy the ordinary Ward identities rather than the Slavnov-Lee identities, and the only renormalization needed is a coupling-constant renormalization.[12,14,15]

III. THE CONSTANT MAGNETIC FIELD AND ZERO-POINT ENERGY

The calculation of the one-loop approximation to the effective action as a sum over the zero-point energies of the theory has been suggested in the work of Nielsen and Olesen,[16] where they consider the one-loop effective action in the presence of a constant color magnetic field. They advocate the study of such field configurations as a tool in understanding quark confinement. Although vacuum configurations in which $\langle \vec{H} \rangle \neq 0$ are not Lorentz invariant or even rotationally invariant, Nielsen and Olesen suggest that such configurations are important, for the apparent instabilities in these configurations may indicate something about the true nature of the Yang-Mills vacuum. Along this line, Ambjorn, Nielsen, and Olesen have suggested a theory based upon a condensate of color magnetic flux tubes.[7,27-31] Mandelstam[42] has suggested that the instabilities for constant H indicate an enhancement of the virtual, low-frequency modes in a Fourier expansion of the magnetic field. 't Hooft[43] has indicated how to study the local stability of theories with nonzero, topologically stable magnetic flux. A justification for these programs lies in the apparent result that the real part of the energy is lowered by the one-loop term in the presence of a constant magnetic field and in the interpretation of the calculated imaginary part of the effective action as an indication of an instability leading to a lower-energy, stable ground state.

The first indication that the real part of the energy is lowered for small external color magnetic fields was implicit in the work of Vanyashin and Terent'ev,[44] where they noted the "anomalous character of the charge renormalization" for a charged vector field in the presence of an external magnetic field. A decade later, Duff *et al.*[23] used Schwinger's elegant proper-time technique to write down the effective Lagrangian for a non-Abelian theory in the presence of general external fields. The first systematic calculations of the explict one-loop effective action for a constant external magnetic field using both proper time and renormalization-group techniques were by Savvidy,[19] Matinyan *et al.*,[20] and Batalin *et al.*[21] However, they failed to note the imaginary part. Nielsen and Olesen[16] redid the calculation by assuming that the one-loop term can be written as the change in the zero-point energy of the theory. Falomir and Schaposnik[18] extended the calculation to also include scalar particles in the adjoint representation, but their calculation appears to depend on their choice of gauge. Yildiz and Cox[22] also performed the calculation, in this case including scalars and spinors in an arbitrary representation, using the definition of the effective action due to Schwinger. The last three papers all use proper time techniques to evaluate their integrals, and all note the existence of an imaginary part.

The one-loop approximation to a pure vector-gluon theory gives the following for the real part of the effective energy density[17]:

$$\mathrm{Re}\,\mathcal{E}=\tfrac{1}{2}H^2+\frac{11N}{96\pi^2}g^2H^2\left(\ln\frac{gH}{\Lambda^2}-\tfrac{1}{2}\right), \tag{41}$$

where N depends on the gauge group SU(N). Renormalization-group arguments can be given to show that this is a good approximation for large H, i.e., for $gH \gg \Lambda^2$, where Λ^2 is the renormalization point.[16-22] If the above approximation is extended to smaller values of H, it is noted that there exists a minimum away from $H=0$. For the above approximation to hold in this region it is required that

$$\int_g^\infty \frac{dg}{\beta(g)}<\infty , \tag{42}$$

where $\beta(g)$ is the usual Callan-Symanzik β function. This behavior of $\beta(g)$ corresponds to the "ferromagnetic" classification of the theory due to Pagels and Tomboulis[45] and Gross and Wilczek.[46] For this to happen, the running coupling constant $\bar{g}^2(\Lambda)$ must become negative, a pathological result, unless a phase transition to some ordered system occurs. This is just what Nielsen and Olesen suggest, although the proof must somehow come from the theory itself.

As already stated, Nielsen and Olesen suggest that the one-loop approximation to the effective action can be written as the difference in the zero-point energy of the theory due to the presence of an external vector field. This result has previously been found for particle theories by Coleman[1] and for scalar field theories by Salam and Strathdee.[24] It was also pointed out by Dolan and Jackiw[26] and by Weinberg,[25] as the zero-temperature limit of scalar and fermion field theories formulated at finite temperature. The extension to vector gauge theories is complicated by the gauge problems associated with the construction of the effective action. The zero-point energy of the theory is a gauge-invariant quantity, and hence problems with the gauge noninvariance of the effective action would naturally lead to problems with this interpretation. With the formulation of the background-field method of Sec. II the effective action is gauge invariant and we are able to show the equivalence of the one-loop term to the change in the zero-point energy due to the presence of an external field. We will demonstrate this result for a limited class of external background fields $\bar{A}^a_\mu(x)$ and clarify to what extent this class may be enlarged without changing the interpretation. We will then verify the result explicitly for the case of a constant external magnetic field.

To calculate the one-loop approximation for Γ, we expand the exponential in Eq. (9) in powers of A_μ. Since we are interested in vacuum configurations, we take the case $J^a_\mu(\bar{A})=0$. This is therefore the gauge-invariant formulation of the theory. The one-loop approximation is found by keeping only quadratic terms in the fields A^a_μ, ϕ^a, and ϕ^{a*}. The effective action to this order is then

$$\begin{aligned}\Gamma(\bar{A})=\int d^4x\,\mathcal{L}(\bar{A})-i\hbar\ln\Big\{N\int[dA][d\phi][d\phi^*]\exp\Big[-\frac{i}{2\hbar}\int d^4x\Big([D_\mu(\bar{A})A_\nu]^a[D^\mu(\bar{A})A^\nu]^a-[D^{ab}_\mu(\bar{A})A^{\mu b}]^2\\ +2gF^{\mu\nu a}(\bar{A})c^{abc}A^b_\mu A^c_\nu+\frac{1}{\alpha}[D^{ab}_\mu(\bar{A})A^{\mu b}]^2\\ +2\phi^{a*}[D_\mu(\bar{A})D^\mu(\bar{A})]^{ab}\phi^b\Big)\Big]\Big\},\end{aligned} \tag{43}$$

where

$$\mathcal{L}(\bar{A})=-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a},\quad F^a_{\mu\nu}(\bar{A})=\partial_\mu\bar{A}^a_\nu-\partial_\nu\bar{A}^a_\mu+gc^{abc}\bar{A}^b_\mu\bar{A}^c_\nu , \tag{44}$$

and the normalization N is defined so that the one-loop term Γ_1 (the logarithm above) vanishes when $\bar{A}^a_\mu=0$. Note also that

$$B^{ia}=-\tfrac{1}{2}\epsilon^{ijk}F^a_{jk}$$

and (45)

$$E^{ia}=F^{i0a}\,.$$

As stated in Sec. II, to the one-loop order we may freely choose $\bar{A}^a_\mu$ to be solutions of the classical equations of motion. Integrating by parts, we find that the one-loop term alone becomes

$$\Gamma_1(\bar{A})=-i\hbar\ln\Big[N\int[dA][d\phi][d\phi^*]\times\exp\Big(\frac{i}{2\hbar}\int d^4x\{A^{\mu a}[g_{\mu\nu}(D_\alpha(\bar{A})D^\alpha(\bar{A}))^{ac}+2gc^{abc}F^b_{\mu\nu}(\bar{A})]A^{\nu c}-\phi^{a*}[D_\alpha(\bar{A})D^\alpha(\bar{A})]^{ac}\phi^c\}\Big)\Big]\,, \quad (46)$$

where we have used our gauge freedom to put $\alpha=1$. To evaluate the functional integral, we will define the theory in Euclidean space by first letting

$$t\to -i\tau\,,\quad A^0\to -iA^4\,. \quad (47)$$

The Euclidean theory is defined from this change of variables by letting the integral over τ go from $-\infty$ to $+\infty$. Then

$$Z_E(\bar{A}_E)\equiv\exp\Big[-\frac{1}{\hbar}\Gamma_E(\bar{A}_E)\Big]\,. \quad (48)$$

This implies that

$$\Gamma=i\Gamma_E\big|_{\tau=it,\,A^4=iA^0}\,. \quad (49)$$

Henceforth, all subscripts E will be dropped, and all four-vectors are Euclidean unless otherwise noted. The Euclidean one-loop term becomes

$$\Gamma_1(\bar{A})=-\hbar\ln\Big[N\int[dA][d\phi][d\phi^*]\exp\Big(-\frac{1}{2\hbar}\int d^4x\{A^{\mu b}[\delta_{\mu\nu}(D_\alpha(\bar{A})D^\alpha(\bar{A}))^{bc}-2gc^{abc}F^a_{\mu\nu}(\bar{A})]A^{\nu c}+2\phi^{a*}(D_\alpha(\bar{A})D^\alpha(\bar{A}))^{ab}\phi^b\}\Big)\Big], \quad (50)$$

where now the metric is $-\delta_{\mu\nu}$.

We shall label the square-bracketed differential operator in Eq. (50), $G^{bc}_{\mu\nu}(x)$. The functional integral over A^a_μ is then formally equal to

$$\det[G^{bc}_{\mu\nu}(x)\delta^4(x-y)]^{-1/2}\,, \quad (51)$$

where we have assumed that the eigenvalues of $G^{bc}_{\mu\nu}$ are positive definite in order for the Gaussian integral to be well defined. Rather than proceeding as above, we can examine the integral mode by mode expanding the fields $A^{\mu a}(x)$ in a basis which diagonalizes $G^{ab}_{\mu\nu}$. Since $G^{ab}_{\mu\nu}$ is a differential operator, it is not diagonal in coordinate space. As we shall see, in order to show the equivalence of Γ_1 to a sum over zero-point energies, we need $G^{ab}_{\mu\nu}$ to be diagonal in frequency space. To effect this we shall restrict $\bar{A}^a_\mu(x)$ and later we will examine to what extent this restriction can be relaxed without affecting the result. Since $\Gamma(\bar{A})$ is gauge invariant we will use this freedom to put $\bar{A}^a_\mu$ in the temporal gauge, $\bar{A}^a_4=0$. (This still leaves us the freedom to make an additional time-independent gauge transformation.) Now we will restrict $\bar{A}^a_i(x)$ to be static. This is a genuine constraint, but it is not very severe since our purpose is to find field configurations of lowest energy and in many cases this can be assumed to be static. (Jackiw and Rossi[47] have demonstrated the existence of time-dependent classical solutions which give rise to lower energies than certain "gyroscopically" stable static configurations in the classical theory with sources. It is still presumed, however, even in that theory, that the lowest energy field configurations would still be static. This has been shown in the work of Sikivie and Weiss[48] and Magg.[49]) With this restriction $G^{bc}_{\mu\nu}$ takes the form

$$G^{bc}_{\mu\nu}=-\delta_{\mu\nu}\delta^{bc}\partial_4{}^2+M^{bc}_{\mu\nu}(\vec{x})\,. \quad (52)$$

Now by a similarity transformation let us diagonalize $G^{bc}_{\mu\nu}$ in Lorentz space and group space. This is a bit tricky (since G is also a differential operator) and has only been done for a countable set of external field configurations, among which are constant fields and plane waves.[50] Assuming that G is still in the form of Eq. (52), we now expand $A^a_\mu(x)$ is normalized eigenfunctions $\chi_n(\vec{x})$ of the differential operator G:

$$A^a_\mu=\sum_n\int\frac{dk_4}{\sqrt{2\pi}}C^a_{\mu,n}(k_4)\,e^{ik_4\tau}\chi_n(\vec{x})\,, \quad (53)$$

where n may be either a discrete or continuous variable (or both). Then

$$G^{bc}_{\mu\nu}A^{\nu c}=\sum_n \int \frac{dk_4}{\sqrt{2\pi}}(\delta_{\mu\nu}\delta^{bc}k_4{}^2+\lambda^{bc}_{n,\mu\nu})C^{\nu c}_n e^{ik_4\tau}\chi_n(\vec{x}), \tag{54}$$

where λ is diagonal in Lorentz and group indices. Then the integral over A^a_μ in Eq. (50) becomes

$$N'\int[dC]\exp\left[-\frac{1}{2\hbar}\sum_{n,m}C^{\mu b}_m(k_4)(\delta_{\mu\nu}\delta^{bc}k_4{}^2+\lambda^{bc}_{n,\mu\nu})\delta_{mn}\delta(k_4-k_4')C^{\nu c}_n(k_4')\right]=N'\det[(\delta_{\mu\nu}\delta^{bc}k_4{}^2+\lambda^{bc}_{n,\mu\nu})\delta_{mn}\delta(k_4-k_4')]^{-1/2}, \tag{55}$$

where N' has been defined to include any field-independent constant in the transformation from $A^a_\mu(x)\to C^b_{\mu,n}(k_4)$. Exponentiating the determinant, we find that Eq. (50) becomes

$$\Gamma_1(\bar{A})=N\hbar\,\mathrm{Tr}\,\delta_{mn}\delta(k_4-k_4')\times[\tfrac{1}{2}\ln(\delta_{\mu\nu}\delta^{bc}k_4{}^2+\lambda^{bc}_{n,\mu\nu})-\ln(\delta_{\mu\nu}\delta^{bc}k_4{}^2+\Lambda^{bc}_{n,\mu\nu})], \tag{56}$$

where the second term in the square brackets comes from the ghost part of the exponential and is of opposite sign (or power) because of the antisymmetry of the ϕ^a fields. The extra factor of 2 arises from an integration over both the real and imaginary parts of ϕ^a. The determinant in Eq. (55) has as its argument terms proportional to the unit operator $\delta_{mn}\delta(k_4-k_4')$, which have been slipped through the logarithms above. Now taking the trace, we set $m=n$, $\mu=\nu$, $b=c$, let $k_4'\to k_4$, and sum over these diagonal elements. Let us now evaluate the integral over k_4 for one of these terms. Following the procedure in Salam and Strathdee,[24] we let

$$N\int dk_4\ln(k_4{}^2+\lambda_n)=\int_{\lambda_n(0)}^{\lambda_n(\bar{A})}d\lambda_n'\int dk_4\frac{d}{d\lambda_n'}\ln(k_4{}^2+\lambda_n'), \tag{57}$$

where the limits of integration have been set to absorb the normalization N, i.e., N is defined so that $\lim_{\bar{A}\to 0}\Gamma_1(\bar{A})=0$. Performing the differentiation, we have

$$\int_{\lambda_n(0)}^{\lambda_n(\bar{A})}d\lambda_n'\int_{-\infty}^{+\infty}\frac{dk_4}{k_4{}^2+\lambda_n'}. \tag{58}$$

In assuming that the eigenvalues of G are positive definite, we note that λ_n' must also be positive definite. (When $k_4=0$, the eigenvalues of G are equal to λ.) In this case, the above integral reduces to

$$\int_{\lambda_n(0)}^{\lambda_n(\bar{A})}d\lambda_n'\frac{1}{(\lambda_n')^{1/2}}\tan^{-1}\left(\frac{k_4}{\lambda_n'}\right)\Bigg|_{-\infty}^{+\infty}=2\pi\{[\lambda_n(\bar{A})]^{1/2}-[\lambda_n(0)]^{1/2}\}. \tag{59}$$

Therefore, Eq. (56) reduces to

$$\Gamma_1(\bar{A})=\int d\tau\,\delta_{mm}\sum_{n,\mu,b}\frac{\hbar}{2}(\{[\lambda^b_{n,\mu}(\bar{A})]^{1/2}-[\lambda^b_{n,\mu}(0)]^{1/2}\}-2\{[\Lambda^b_{n,\mu}(\bar{A})]^{1/2}-[\Lambda^b_{n,\mu}(0)]^{1/2}\}), \tag{60}$$

where a single index on λ implies a diagonal element and where

$$\frac{1}{2\pi}\int d\tau=\lim_{k_4'\to k_4}\frac{1}{2\pi}\int d\tau\,e^{i(k_4'-k_4)\tau}=\lim_{k_4'\to k_4}\delta(k_4'-k_4). \tag{61}$$

Now writing $\Gamma_1(\bar{A})=\int d\tau\,L_1(\bar{A})$, we note that the first term in L_1 looks like the change in the zero-point energy of the theory due to the presence of the external field $\bar{A}$. To see that $\sqrt{\lambda}$ is an energy, continue the operator $G^{bc}_{\mu\nu}(\bar{A})$ back to Minkowski space where it is equivalent to the inverse propagator. If we examine its decomposition into normal modes, Eq. (56), we note that in Minkowski space the zeros of the energy transform (poles in the propagator) occur for $k_0{}^2=\lambda$. Therefore, the $[\lambda(\bar{A})]^{1/2}$ are just vector particle energy eigenvalues.

The second term in Eq. (60) is the contribution from the ghost determinant. It is of the same form as the first term and will be seen in specific examples to cancel the contribution from any unphysical degrees of freedom.

Continuing back to Minkowski space via Eq. (49), we now get

$$\Gamma_1(\bar{A})=-\int dt\,\delta_{mm}\sum_{n,\mu,b}\frac{\hbar}{2}(\{[\lambda^b_{n,\mu}(\bar{A})]^{1/2}-[\lambda^b_{n,\mu}(0)]^{1/2}\}-2\{[\Lambda^b_{n,\mu}(\bar{A})]^{1/2}-[\Lambda^b_{n,\mu}(0)]^{1/2}\}). \tag{62}$$

Therefore, the one-loop correction to the effective Lagrangian is the negative of the change in zero-point energy in the presence of an external field $\bar{A}^a_\mu$, where $A^a_0=0$ and $A^a_i(x)=A^a_i(\vec{x})$. For this field configuration the electric field $E^a_i(x)=0$, and thus the kinetic energy is zero. Therefore, the Lagrangian is minus the potential energy, so that the one-loop contribution to the energy is just the change in zero-point energy. This agrees with our intuition, since in the presence of an external field, the zero-point energy has changed. We have, naively, normalized the energy for $\bar{A}^a_\mu = 0$ by neglecting the zero-point energy (normal ordering). In the presence of an external field, the zero-point energy is now changed, and since zero-point energy is a quantum phenomenon, this change should appear in the first quantum correction to the energy. Another way of stating the same time thing is that due to the initial ($\bar{A}=0$) normal ordering of the theory, the change in zero-point energy shows up at the one-loop order as zero-point fluctuations around the original zero-point energy. This is analogous to the argument given by Coleman[1] of a particle sitting at the minimum of a potential. To get the first quantum correction to the energy, we approximate the potential by a harmonic oscillator and add the zero-point energy of the oscillator. This interpretation is also similar to one implicit in the work of Weinberg[25] in calculating higher-order corrections to theories at finite temperatures.

We may now ask what happens for $E^a_i \neq 0$. For time-dependent configurations which are not gauge equivalent to the previous case, our simple result clearly does not follow. We cannot separate out the k_4 dependence as needed. These configurations correspond to excited states of our system and the generalization of the discussion in the last paragraph would thus be more complicated. See Ref. 51 for an interpretation of Γ for time-dependent external field configurations.

For configurations in which $\bar{A}^a_4 \neq 0$, but $\bar{A}^a_\mu$ is still static, there would only be an addition to Eq. (54) linear in k_4. We could then complete the square and change variables to regain a formula similar to Eq. (54). It appears that the Euclidean one-loop effective action would resemble Eq. (60) with appropriate $\bar{A}^a_4$ dependence. This linear term in k_4 is similar in structure to a gyroscopic term in the sense of Ref. 47. For this situation the transition from Γ_E to Γ is not as straightforward since we must consider the $\bar{A}^a_4$ (or $\bar{A}^a_0$) dependence. Also since $E^a_i \neq 0$, the Lagrangian is not equal to minus the energy. It may still be argued, however, that the Euclidean Lagrangian is equal to an effective energy (see Sec. IV) and therefore, we should still be able to use our zero-point-energy interpretation for such "gyroscopic" configurations.

When $\bar{A}^a_\mu(x)$ represents a constant color magnetic field we may take

$$\bar{A}^a_\mu(x) = \delta^{a3}\bar{A}_\mu(x)\,, \tag{63}$$

i.e., $\bar{A}^a_\mu(x)$ points in the third direction in color space. [Here we are taking A^a_μ to be in the adjoint representation of the gauge group SU(2).] Also let

$$\begin{aligned}&\bar{A}^0 = \bar{A}^y = \bar{A}^z = 0\,,\\&\bar{A}^x = -By\,,\end{aligned} \tag{64}$$

so that from Eq. (45)

$$B^{ia} = \delta^{i3}\delta^{a3}B\,, \tag{65}$$

i.e., B^{ia} points in the third direction in both space and color space. Then define

$$T^{bc} \equiv \begin{bmatrix}1 & & \\ & 0 & \\ & & -1\end{bmatrix}, \qquad S_{\mu\nu} \equiv \begin{bmatrix}0 & & & \\ & 1 & & \\ & & 0 & \\ & & & -1\end{bmatrix}, \tag{66}$$

and Eq. (56) becomes (again in Euclidean space)

$$\begin{aligned}\Gamma_1(\bar{A}) = N\hbar\,\mathrm{Tr}\,\delta(k_4' - k_4)\delta(k_x' - k_x)\delta(k_z' - k_z)\delta_{mn}\\ \times \big(\tfrac{1}{2}\ln\{\delta_{\mu\nu}[\delta^{bc}(k_4{}^2 + k_z{}^2)\\ + (2n+1)g(T^2)^{bc}B] + 2gS_{\mu\nu}T^{bc}B\}\\ - \ln\delta_{\mu\nu}[\delta^{bc}(k_4{}^2 + k_z{}^2) + (2n+1)g(T^2)^{bc}B]\big)\,.\end{aligned} \tag{67}$$

Upon taking the trace, we find that the ghost term cancels with the two terms corresponding to the zero eigenvalues of $S_{\mu\nu}$, leaving

$$\Gamma_1(\bar{A}) = N\frac{\hbar}{2}\int d\tau\,dx\,dz\sum_n\int\frac{dk_4\,dk_z\,dk_x}{(2\pi)^3}\,2\{\ln[k_4{}^2 + k_z{}^2 + (2n+3)gB] + \ln[k_4{}^2 + k_z{}^2 + (2n-1)gB\}\,. \tag{68}$$

Notice the integral over dk_x. Since the integrand is independent of k_x, this is analogous to a zero-eigenvalue problem. This reflects a symmetry of the theory that we have overlooked. Instead of integrating over k_x we can put our problem in a box.[52] The sum over n above comes from the sum over the energy eigenvalues of a linear harmonic oscillator in the y direction of the form

$$P_y{}^2 + (gB)^2(y \mp y_0)^2\,, \tag{69}$$

where

$$y_0 = P_x/gB\,. \tag{70}$$

Therefore, in integrating over k_x, we are summing over all possible locations of the origin of the oscillator. If we put our system in a box of length L_y, then $0<y_0<L_y$ or $0<P_x<gBL_y$. Therefore, the integral over k_x should just give gBL_y. If we write L_y as $\int dy$, then

$$\Gamma_1(\bar{A})=N\frac{\hbar}{2}\frac{gB}{2\pi}\int d^4x \sum_n \frac{dk_4}{2\pi}\frac{dk_z}{2\pi} \times 2\{\ln[k_4{}^2+k_z{}^2+(2n+3)gB] + \ln[k_4{}^2+k_z{}^2+(2n-1)gB]\}. \quad (71)$$

Now note that, for $n=0$ and $k_4{}^2+k_z{}^2<gB$, the second logarithm has a negative argument. This traces back to the existence of negative eigenvalues for the operator $G^{bc}_{\mu\nu}$, and hence the Euclidean formulation of the theory is ill defined. We will, for the moment, naively proceed with our analysis, overlooking this problem, but we will return to it in the following section. If we do so, then in analogy to Eq. (62) we would get in Minkowski space

$$\Gamma_1(\bar{A})=-\int d^4x \sum_n \frac{gB}{2\pi}\int\frac{dk_z}{2\pi}\left(\frac{\hbar}{2}\{2[k_z{}^2+(2n+3)gB]^{1/2}+2[k_z{}^2+(2n-1)gB]^{1/2}\}-\frac{\hbar}{2}[2(k_z{}^2)^{1/2}+2(k_z{}^2)^{1/2}]\right). \quad (72)$$

Therefore, we have our result for constant magnetic field B. We can easily see the source of the imaginary part of Γ. For $n=0$ and $k_z{}^2<gB$, the second term above will be imaginary. Thus by writing the effective action as a sum over zero-point energies, we can readily distinguish the real and the imaginary parts. We should note, however, that some of the steps used in the above derivation are in fact ill defined for these imaginary energy field modes.

In the next section we will examine these field modes carefully and give an indication of when the one-loop approximation is still sensible. We will also investigate the interpretation of the imaginary part of the effective action and we will indicate the circumstances for which it may be useful in discussing local stability.

IV. NEGATIVE EIGENVALUES AND INSTABILITY

The imaginary part of the one-loop term of the effective action in the presence of a constant color magnetic field was first noted by Nielsen and Olesen[16] as due to the anomalous magnetic moment of the gluon [$2gc^{abc}F^b_{\mu\nu}(\bar{A})$ in Eq. (44)]. The result was implicit, however, in earlier works,[32] where the energy eigenvalues for vector fields with anomalous magnetic moments and electric quadrupole moments were determined. This work was then specialized to non-Abelian gauge theory and tachyonic states were noted.[32] An instability in the presence of a constant color magnetic field has also been pointed out in the classical theory by Chang, Weiss, and Sikivie,[33,34] and more recently by Cosenza and Neri.[35] The precise relationship between the classical instability and the imaginary part of the effective action at the one-loop (semiclassical) level has not been pointed out. In this section, we shall investigate the negative eigenvalue modes and in doing so will show that a relationship does indeed exist between the classical and semiclassical instabilities. We shall evaluate the validity of the one-loop approximation in the presence of negative eigenvalues and demonstrate how to "regularize" the functional integral in order to calculate the imaginary part. In this context, we can then give a physical interpretation of the imaginary part and connect this interpretation to that of imaginary effective potentials in the context of nucleation[38] or the "decay of the false vacuum"[36,37] and pair production in constant electric fields.[6]

As noted after Eq. (71), there exists negative eigenvalues for the operator $G^{bc}_{\mu\nu}$ in the presence of a constant color magnetic field. It can be shown that these are just due to the anomalous magnetic moment of the gluon, as noted above. The Euclidean functional integral, Eq. (55), for these modes will look, mode by mode, like

$$\int_{-\infty}^{+\infty} dC \exp[+\tfrac{1}{2}C(gB-k_4{}^2-k_z{}^2)C]\to\infty, \quad (73)$$

where we have suppressed indices. These are just the modes for which, in the second term of Eq. (68), $n=0$. For $k_4{}^2+k_z{}^2<gB$, the integral above is clearly infinite, and thus the Euclidean functional integral diverges. This does not mean that the theory is ill defined since we are only looking at the loop approximation. It does appear, however, that the loop approximation is invalid in the case of a constant color magnetic field. Let us be more careful and see if there is any information that we may salvage.

Imagine a theory such that

$$Z=\prod_k\int d\phi_k \exp[-\tfrac{1}{2}\Delta_k(k^2\phi_k{}^2-m^2\phi_k{}^2+\lambda\phi_k{}^4)]. \quad (74)$$

If this integral is expanded around $\phi=\phi_0=0$, then to quadratic order, when $m^2>k^2$, we will have the same form as Eq. (73), i.e., a tachyon [negative (mass)^2] is present. Here we are expanding around a maximum rather than a minimum of the potential (see Fig. 1). Clearly, a better approximation is to expand around one of the two symmetric minima. For the gauge-theory problem, we cannot solve for the minima, but the existence of negative eigenvalues indicates that we are expanding about a "bad" field configuration. Note that an expansion about $\phi=0$ in Fig. 1 is also classically unstable.

In a classical analysis, we demand for stability that we expand around a field configuration which minimizes the energy, unless, as shown by Jackiw and Rossi,[47] there exist gyroscopic terms in the Hamiltonian. Let

$$\mathcal{H}=\tfrac{1}{2}\pi^2+V(\phi)+G\pi\phi\ . \tag{75}$$

For the case $G=0$ (no gyroscopic terms), if we expand $\mathcal{H}$ around static solutions of Hamilton's equations:

$$\dot{\phi}_{\mathrm{cl}}=\frac{\delta\mathcal{H}}{\delta\pi}\bigg|_{\pi_{\mathrm{cl}}}=0,\quad \dot{\pi}_{\mathrm{cl}}=-\frac{\delta\mathcal{H}}{\delta\phi}\bigg|_{\phi_{\mathrm{cl}}}=-\frac{\delta V}{\delta\phi}\bigg|_{\phi_{\mathrm{cl}}}=0\ , \tag{76}$$

then

$$\frac{\delta^2\mathcal{H}}{\delta\pi^2}=1,\quad \frac{\delta^2\mathcal{H}}{\delta\phi^2}\bigg|_{\phi_{\mathrm{cl}}}=\frac{\delta^2 V}{\delta\phi^2}\bigg|_{\phi_{\mathrm{cl}}}\ . \tag{77}$$

For minimality we require that $(\delta^2V/\delta\phi^2)|_{\phi_{\mathrm{cl}}}>0$.

Now, in the Euclidean formulation of the functional integral, the argument of the exponential, modulo gauge-fixing terms and ghosts, is equal to minus the Euclidean action. The general structure of the Lagrangian density is

$$\mathcal{L}(\phi)=\tfrac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2+G\phi\frac{\partial\phi}{\partial t}-\tilde{V}(\phi)\ . \tag{78}$$

If we again specialize to the case $G=0$, then

$$\tilde{V}(\phi)=V(\phi)\ .$$

To go to the Euclidean formulation let

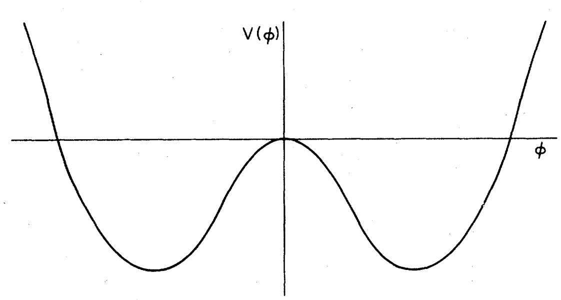

FIG. 1. Quartic potential $V(\phi)$ with a negative mass squared versus ϕ.

$$\mathcal{L}_E=-\mathcal{L}\,|_{t\to -i\tau}$$

or

$$\mathcal{L}_E(\phi)=\tfrac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2+V(\phi)\ . \tag{79}$$

To calculate the one-loop term we expand $\mathcal{L}_E$ around static solutions of the classical Euclidean equations of motion:

$$\frac{\partial^2\phi}{\partial\tau^2}\bigg|_{\bar{\phi}_E}=\frac{\delta V}{\delta\phi}\bigg|_{\bar{\phi}_E}=0\ , \tag{80}$$

and the quadratic term becomes

$$-\tfrac{1}{2}\,\phi\,\frac{\delta^2\mathcal{L}_E}{\delta\phi^2}\bigg|_{\bar{\phi}_E}\phi\ .$$

Therefore, for the theory to be well defined, we demand positivity of the eigenvalues of the above kernel, i.e.,

$$\frac{\delta^2\mathcal{L}_E}{\delta\phi^2}\bigg|_{\bar{\phi}_E}>0\ .$$

Explicitly

$$\frac{\delta^2\mathcal{L}_E}{\delta\phi^2}\bigg|_{\bar{\phi}_E}=-\frac{\partial^2}{\partial\tau^2}+\frac{\delta^2V}{\delta\phi^2}\bigg|_{\bar{\phi}_E}\ . \tag{81}$$

Now, under the assumption that we may separately diagonalize the two terms above, the first term will always be positive semidefinite. Now from Eqs. (80) and (76), $\phi_{\mathrm{cl}}=\bar{\phi}_E$. If our theory is classically stable, $(\delta^2V/\delta\phi^2)|_{\phi_{\mathrm{cl}}}>0$, then it will also give rise to positive eigenvalues at the one-loop level. Also, if we have negative eigenvalues, $(\delta^2V/\delta\phi^2)|_{\bar{\phi}_E}<0$, then the theory must be classically unstable. Note that it is possible to be classically unstable and still have positive eigenvalues.

There are two apparent exceptions to these conclusions. The first occurs in the theory of metastability due to Langer[38] and Callan and Coleman.[36] In this case we may expand the theory locally around a minimum of the potential, but quantum tunneling renders the theory unstable. However, the quantum-tunneling amplitude is a nonperturbative, Euclidean time-dependent effect while our analysis is for static, perturbative fields.

The second exception occurs in the work of Coleman and Weinberg[53] when considering theories with two coupling constants. In this case the one-loop effects due to one coupling may be of the same magnitude as the tree terms due to the other coupling. Thus, a minimum of the classical potential may turn into a local maximum when one-loop effects are included. In this case then, there is a gray area in the distinction between classical and semiclassical approximations, for which we may consider an effective classical theory which does not encounter this problem.

For gauge theories things are not quite so simple. To go to Euclidean space $t \to -i\tau$ and $A_0^a \to -iA_4^a$. If we expand around a configuration in which $\overline{A}_4^a \neq 0$, then there are gyroscopic terms present. It can be shown, however, (see the Appendix) that the above analysis does follow, at least in the case when we expand about static solutions for which $\overline{A}_4^a = 0$. With the presence of gyroscopic terms, the above statements do not hold. However, in this situation classical stability does not necessitate minimality.[47] It is therefore suggestive even in this case that there may be a relationship between classical stability and semiclassical stability.

Having made this connection, we find it clear that the existence of negative eigenvalues as in Eq. (73) leads immediately to the conclusion that the theory is also classically unstable as illustrated by Sikivie, Weiss, and Chang for the constant magnetic field.[33-35]

Let us return to the problem of dealing with the negative eigenvalues. Equation (74) has been written as a product of integrals. Each integral is equal to the area under one of the curves in Fig. 2. The quadratic approximation can be pictured as the product of the areas under the curves in Fig. 3. For values of $k^2 > m^2$, the curves are Gaussian and the area is finite. For $k^2 < m^2$ (the top set of curves in Fig. 3), the area under each curve diverges.

Now, if we interpret the existence of the negative eigenvalues as indicating an instability of the theory when expanded around a bad field configuration, can the quadratic approximation tell us anything more about the nature of this instability? Imagine that we have two different theories, characterized by the solid curves in Fig. 4. If in each case we expand about a configuration corresponding to the local minimum of the integrand of Eq. (74) with $k^2 = 0$ (corresponding to minima in Fig. 4), then our theory is more unstable (we have farther to fall) for curve II than for curve I, where $m_{\rm II}{}^2 > m_{\rm I}{}^2$. We would like to somehow "regularize" the functional integral over the negative eigenvalue modes in order to gain some information about the degree of instability.

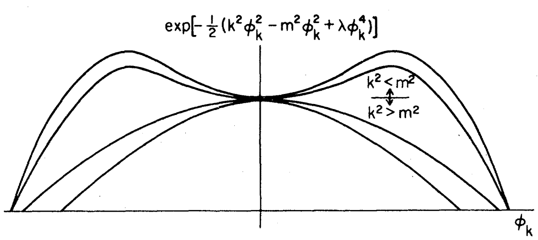

FIG. 2. Integrand of minus the Euclidean action versus the field modes ϕ_k for various values of k^2.

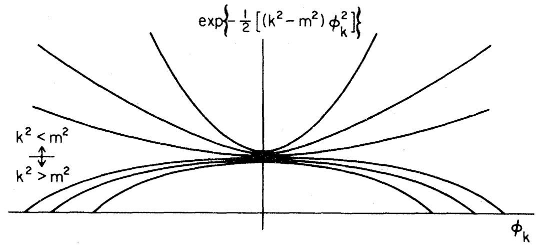

FIG. 3. Quadratic approximations to the integrands which appear in Fig. 2.

If we formulate the functional integral in Minkowski space, rather than in Euclidean space, then it is necessary to regularize the theory by taking the prescription of adding an infinitesimal term $i\epsilon\phi^2$ to the Lagrangian density in such a way as to ensure causal boundary conditions.[40] It turns out that this procedure also regularizes the infinities due to the presence of tachyons in the theory, and produces an imaginary contribution to Γ. If we continue a nontachyonic theory to Euclidean space, the $i\epsilon$ term is usually dropped, for the theory is now well defined mathematically. However, in the presence of tachyonic modes, the loop approximation breaks down. We would like to define some alternative "regularization" for the tachyonic modes which would reproduce the imaginary part of Γ. The advantage of doing so in Euclidean space will be to help us better understand the physical meaning of the imaginary contribution.

In the work due to Callan and Coleman on the decay of the false vacuum[1,36,37] and in the work due to Langer[38] on nucleation, a similar problem arises. There, the expansion around the metastable minimum gives a real contribution. They are also interested in calculating a nonperturbative quantum-mechanical correction due to the possibility of quantum tunneling to the lower vacuum configuration. To this effect they expand the integral around the Euclidean "bounce" finite-action

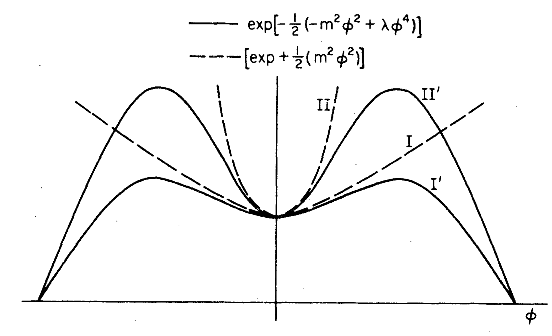

FIG. 4. Solid lines are the Euclidean functional integrand for $k^2 = 0$ versus the field modes ϕ for $m'_{\rm II} > m'_{\rm I}$. Dashed lines are the quadratic approximation to the solid lines where $m_{\rm II} = m'_{\rm II}$, $m_{\rm I} = m'_{\rm I}$.

solution to the classical equations. They also encounter negative eigenvalues due to an expansion around a maximum of the action, although not a maximum of the potential. In order to regularize these theories, they refer to another well-defined theory such that the analytic continuation of some parameter gives the physical theory. Thus to calculate a contribution due to the negative eigenvalue, they write down a steepest descent integral in the well-defined theory and, as they analytically continue to the physical theory, they also analytically continue both the path of integration and the boundary conditions into the complex field plane in such a way as to keep the theory well defined. This just corresponds to picking the "paths of steepest descent." The *ad hoc* feature of this procedure is that, to keep the theory finite, the boundary conditions are also continued, so that in effect we are computing a different functional integral and hence have regularized the theory. The justification in this case comes from the physical system. We are expanding around a metastable state and the imaginary part computed should represent the decay rate for this state. In fact, for a few examples taken from quantum mechanics, the decay rates calculated in this way agree with those calculated from more conventional methods. There is, however, no justification that this procedure and interpretation are correct in general. With the hope that this mathematical procedure does have some general physical significance, we will apply it to our problem.

First we shall discuss our sample theory, Eq. (74) with $k^2=0$. If we write down the Lagrangian for a ϕ^4 theory with $m^2>0$, then the tachyonic theory can be obtained by continuing $+m^2$ to $-m^2$. The theory with $+m^2$ has a saddle point at the origin in ϕ space and has two other saddle points along the imaginary axis in ϕ space [see Fig. 5(a)]. The saddle point at the origin is a maximum of the integrand along the real axis and hence the quadratic integral is well defined. Those saddle points on the imaginary axis are higher maxima. Thus if we rotate the complex plane, as in Fig. 5(b), we will reproduce the double humps in Figs. 2 and 4. Our procedure then is to rotate the contour as we rotate in the complex plane so as to always remain on the path of steepest descent from the saddle point. Now depending on how we rotate our contour we will either go up or down the imaginary axis. There are a number of ways in which to decide. If we keep the choice arbitrary, then at the end of the calculation, conservation of probability (unitarity) will demand a definite sign for the calculated imaginary part. Alternatively, we may keep the $i\epsilon\phi^2$ contribution to the Lagrangian in Minkowski space and, upon doing our calculations in Euclidean space, the pole structure of the negative eigenvalue contributions will be determined by the sign of ϵ. In regard to this question we have also found that instead of making the continuation noted above, we can rotate $\phi\to e^{i\pi/4}\phi$ so that, keeping the $i\epsilon\phi^2$ contribution, the integrand changes:

$$\exp(|\lambda|\phi^2+i\epsilon\phi^2)\to\exp(i|\lambda|\phi^2-\epsilon\phi^2).$$

This integrand is identical to the type we would have if we had formulated the theory in Minkowski space. If $\phi\to e^{-i\pi/4}\phi$, then the ϵ contribution would diverge. For the present we shall proceed as first mentioned and we will decide the direction of rotation by demanding unitarity.

Therefore, we shall write these negative eigenvalue contributions as

$$\pm\int_{-i\infty}^{+i\infty}d\phi_k\exp\left[-\tfrac{1}{2}\int_{-m}^{m}dk\,\phi_k(k^2-m^2)\phi_k\right]$$
$$\simeq\prod_{k=-m}^{m}\pm i\int_{-\infty}^{+\infty}d\phi_k\exp[-\tfrac{1}{2}\Delta_k(m^2-k^2)\phi_k{}^2]\quad(82)$$

which gives a finite imaginary result.

A few comments are in order. In the theory of nucleation there is an extra factor of $\frac{1}{2}$. In that problem we integrate over all of function space up to the negative eigenvalue mode and then deform the contour to go over only half of the Gaussian. In our case there is no metastable region. We expand around a local maximum of the potential. Hence, for the integral to remain well defined, we must integrate over the whole Gaussian.

Second, the contribution calculated by Callan and Coleman is of exponentially small order and, as already noted, has a nonperturbative origin, while our calculation is done in perturbation theory.

Third, in the metastable case some physical argument in terms of quantum tunneling can be given for performing such a continuation. In our case, expanding about a local maximum, no such argument exists. Our problem is more in analogy with Schwinger's calculation of an amplitude for

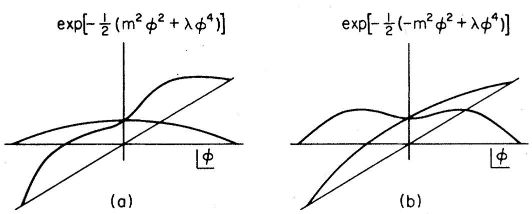

FIG. 5. (a) Euclidean functional integrand for $k^2=0$ and positive mass squared in the complex field-mode plane. (b) Same as in (a) except for negative mass squared.

pair production in the presence of a constant electric field, although his definition of the effective action does not involve path integrals.[6] We will therefore perform the analytic continuation in analogy with the theory of metastability, study our result, and finally compare it with that of Schwinger.

First let us examine the expression we get upon performing our analytic continuation to see if it contains the information we desire. For the case of our sample theory, the two inverted Gaussians (dashed lines) in Fig. 4 are replaced in the analytic continuation by the solid curves in Fig. 6, where the integrand (now imaginary) is plotted versus the continued ϕ axis. Since $\phi=0$ is a saddle point in the complex plane, and a minimum along the real axis, it is a maximum along the imaginary axis. If $m_{\rm II}^2>m_{\rm I}^2$, then the $k=0$ curve for II blows up faster along the real axis, but falls off faster along the imaginary axis than the $k=0$ curve for I. Equation (82) is equal to the product of the (imaginary) areas under the solid curves, above and including the solid line in Fig. 6. It appears then that the contribution from the theory with the larger tachyon mass $m_{\rm II}$ will give a larger contribution than the theory with the smaller mass $m_{\rm I}$. However, what is of primary interest is a calculation of Γ_E which is proportional to the logarithm of the functional integral. As mentioned in Sec. III, the logarithm of Eq. (82) is

$$\int_{-m}^{m} dk \ln[\pm i(m^2-k^2)^{-1/2}] = -\tfrac{1}{2}\int_{-m}^{m} dk \ln(k^2-m^2) \tag{83}$$

which is just the same form as we get for modes in which $k^2>m^2$. Therefore our analytic continuation for the negative eigenvalue modes will reproduce the second term in Eq. (71) for the case $n=0$, for all values of $k^2=k_4^2+k_z^2$. The logarithm in Eq. (83) can be rewritten as

$$\ln[\pm i(m^2-k^2)^{-1/2}] = \pm i\frac{\pi}{2} + \ln(m^2-k^2)^{-1/2}. \tag{84}$$

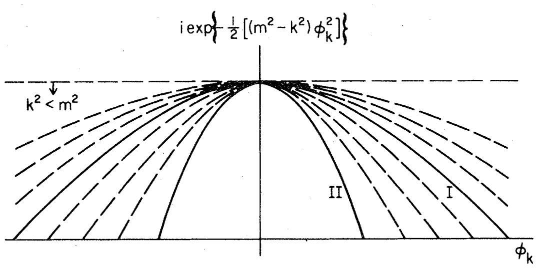

FIG. 6. Analytic continuation of the functional integrand represented in Fig. 3 for various values of $k^2<m^2$ where $m_{\rm II}>m_{\rm I}$.

Thus our analytic continuation has produced a real and an imaginary contribution to Γ_E, with the imaginary contribution proportional to $i\pi m$. Thus in Fig. 6, for $m_{\rm II}$ the contribution to the imaginary part is larger due to the larger range of integration, or the existence of a larger number of tachyonic curves above curve II. Thus the imaginary contribution to the effective action is clearly larger for larger tachyonic masses, or greater instabilities.

Let us now calculate the real and imaginary contributions of the regularized tachyonic modes in the constant color magnetic field example. The term we wish to calculate is

$$N\hbar\frac{gB}{2\pi}\int d^4x \int_{-\infty}^{+\infty}\frac{dk_4}{2\pi} \times \int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\, 2\ln(k_4^2+k_z^2-gB)^{-1/2},$$

where we have rotated τ back to Minkowski space. Now for $k_z^2>gB$, the k_4 integral is of the form $\int_{-\infty}^{+\infty} dk_4 \ln(k_4^2+x^2)^{-1/2}$ which has already been evaluated in Sec. III. This leads to a real contribution

$$-\frac{\hbar}{2}\frac{gB}{2\pi}\int d^4x \int_{\sqrt{gB}}^{\infty}\frac{dk_z}{2\pi}\, 4[(k_z^2-gB)^{1/2}-(k_z^2)^{1/2}].$$

For $k_z^2<gB$, the k_4 integral is of the form $\int_{-\infty}^{+\infty} dk_4 \times\ln(k_4^2-x^2)^{-1/2}$, where $x^2=gB-k_z^2>0$. For $k_4^2>x^2$, there are still positive eigenvalue contributions. For $k_4^2<x^2$,

$$\int_{-x}^{x} dk_4 \ln[\pm i(x^2-k_4^2)^{-1/2}] = \int_{-x}^{x} dk_4[\pm i\pi/2+\ln(x^2-k_4^2)^{-1/2}], \tag{85}$$

where the $\pm$ sign is the same as in Eq. (83) and depends upon which way we rotated the contour for the functional integral. Therefore, for $k_z^2<gB$ we have

$$\pm\pi i(gB-k_z^2)^{1/2} + \int_{-\infty}^{\infty} dk_4[\ln|k_4^2+k_z^2-gB|^{-1/2} - \ln(k_4^2+k_z^2)^{-1/2}].$$

The last term in brackets comes from the normalization N. The intergral above vanishes identically. That is, the real part of the negative eigenvalue contribution exactly cancels the real contribution from the positive eigenvalue modes for which $k_z^2<gB$. In analogy to the discussion by Langer,[38] the real part of the effective action in the tachyonic

sector of the theory as calculated above is the effective action due to the "metastable" phase. In Sec. III we interpreted the real part as a change in the zero-point energy. For $k_z^2 < gB$, the energy contribution $k_0 = (k_z^2 - gB)^{1/2}$ is imaginary and hence only contributes to the imaginary part of Γ. For $k_z^2 > B$, however, there is a nonvanishing contribution to the real part.

If we complete the evaluation of the imaginary part, we find

$$i\,\Gamma_{\rm imag} = \pm\, i\hbar\, \frac{g^2B^2}{8\pi} \int d^4x\,. \tag{86}$$

Note that the contribution to the imaginary part of the effective Lagrangian density is finite as it stands, i.e., before we renormalize the ultraviolet divergence of the theory. This is in keeping with the relationship of the instability to the classical analysis. To choose the sign of the imaginary part, we consider the vacuum generating functional $W(J)$. In the absence of a source, as shown in Eqs. (21) and (24), we can simply relate Γ and W:

$$\Gamma(\bar{A}) = W(0) \tag{87}$$

so that

$$|\langle 0^+ \,|\, 0^- \rangle|^2 = \left|\exp\left(\frac{i}{\hbar}\,\Gamma\right)\right| = \exp(-2\Gamma_{\rm imag}/\hbar)\,. \tag{88}$$

According to Schwinger's interpretation,[6] this is the probability that no pair creation occurs during the history of the field.

From Eq. (88), we see that, to conserve probability, $\Gamma_{\rm imag}$ must be greater than zero. This corresponds to letting $\phi \to e^{+i\pi/2}\,\phi$ in our analytic continuation or we have rotated our contour counterclockwise.

Note also that to make this connection to Schwinger's work, it is imperative to be able to relate the effective action Γ to the vacuum generating functional W, as was done in Sec II.

As to the imaginary part itself, Schwinger defines $2\,{\rm Im}\mathcal{L}$ as the probability, per unit time, per unit volume, of a pair being created by the external field. There are thus two interpretations of our calculation. The first is as above, that in the presence of an external color magnetic field, the vacuum is unstable and will pair produce. If we consider Eq. (88), then $g^2B^2/4\pi$ is the probability per unit four-volume of pair production. If this probability is of order one, the magnetic field cannot be maintained without radiating. For a four-volume larger than $(g^2B^2)^{-2}$, we may interpret our result as meaning that it would be impossible to maintain a constant magnetic field at all. In a smaller four-volume, the contribution of the negative eigenvalues decreases, in agreement with the conclusion of Chang and Weiss[34] that the classical unstable mode will not exist in a length L smaller than $(gB)^{-1/2}$. In a classical stability analysis, they find that the classical unstable modes fall off exponentially in space. They define a three-volume-per-unit unstable mode proportional to $(gB)^{-3/2}$. Chang and Weiss make the analogy between $\sqrt{gB}\,L$ and the Reynolds number in fluid mechanics. This is also reminiscent of the quantum fluid model of Nielsen, Olesen, and Ambjorn.[27-31] Perhaps the QCD vacuum is composed of "pockets" of constant magnetic field of size $L < (gB)^{-1/2}$.

The second interpretation of the calculation is that we are trying to approximate the effective action for the case of no external fields but that, under the assumption of a spontaneously generated constant magnetic field, our theory is unstable. In this regard note that for $E_i^a = 0$, the effective Hamiltonian density equals minus the effective Lagrangian density. The effective Hamiltonian thus has an imaginary part, indicating that the state we are studying is not to be found in our Hilbert space and is unstable.[36] From our discussion about the source of the instability as due to the expansion about a local maximum, it is interpreted that for small values of the imaginary part, the real part of Γ per unit time may still be a fair approximation to the energy functional. This is also in agreement with the discussion of Refs. 27–31, for the presumed minimum of the renormalization-group improved effective action occurs for small values of the magnetic field.

V. DISCUSSION AND SUMMARY

We have derived the gauge-invariant effective-action functional of 't Hooft[15] from a formulation of the vacuum-to-vacuum transition amplitude in the background-field gauge. Thus the definition of the gauge-invariant effective action for non-Abelian gauge theories is made consistent with the definition for nongauge theories. Also, this connection allows us to utilize Schwinger's interpretation of the imaginary part of the Lagrangian density in the absence of sources.[6] Further, we have explicitly demonstrated the gauge-invariance properties of the effective action and, noting this, have been able to justify the hypothesis of Nielsen and Olesen[16] that the unrenormalized one-loop approximation for the effective action, in the presence of an external field, can be written as minus the change in the zero-point energy of the theory due to the presence of an external field. We have justified this result for the case when the external field is static and $\bar{A}_0^a = 0$, although we suggest that it may be extended to $\bar{A}_0^a \neq 0$. In particular, we justity the use of this hypothesis for the case of a constant color magnetic field, at least for modes for which the functional integral is well defined (no negative eigenvalues in the Euclidean-space formulation). We have also noted that the physical field in 't Hooft's definition is not the customary vacuum expectation of the quantum field, but differs by the expectation value of the quantum field relative to a new "shifted" vacuum.

We have studied the "semiclassical" stability of sample theories by formulating the functional integral in Euclidean space and noting the existence of negative eigenvalues when negative mass squared terms are present in the Euclidean action. By examining the Euclidean formulation of the theory, we have been able to compare the "quantum instability" to instabilities arising in a classical analysis. In this regard, we have been able to explicitly demonstrate the equivalence of certain classical and quantum-mechanical stabilities and instabilities and, for example, have been able to equate the classical instability of the constant color magnetic field demonstrated by Sikivie[33] and Chang and Weiss,[34] to the quantum instability noted by Nielsen and Olesen,[16] Yildiz and Cox,[22] and Falomir and Schaposnik.[18]

As for the evaluation of the effective action when negative eigenvalues are present, we have referred to the prescription of Callan and Coleman in the decay of the false vacuum[36] and have regularized the negative eigenvalue modes by analytically continuing the functional integral into the complex field plane, where the direction of rotation is determined by demanding causal boundary conditions, as in the Minkowski-space formulation, or by demanding unitarity. This apparently *ad hoc* procedure reproduces the results of calculating the real and imaginary parts of the effective action in Minkowski space. This procedure resembles that of "Wick rotating" ordinary integrals and although the analogy is not complete, the equivalence of the Minkowski-space calculation encourages us to speculate that there is perhaps some deeper mathematical meaning to the analytic continuation and perhaps some kind of formal functional Wick rotation may indeed be formulated.

We have calculated the imaginary part of the effective action for the case of the constant magnetic field and have connected its interpretation to that of pair production in a constant electric field, thereby showing agreement with Chang and Weiss[34] on the existence of a minimal "length" for the unstable mode. We have also shown agreement for the constant magnetic field case with the interpretation of Nielsen and Olesen[16] that the imaginary part represents an imaginary energy density and hence the existence of lower-energy stable configurations.

By calculating the one-loop term as the change in the zero-point energy of the theory, we can easily separate the real and imaginary parts of the effective action. Our procedure of analytically rotating the field contour in Euclidean space reproduces the naive calculation obtained by ignoring the existence of negative eigenvalues and demonstrates that the imaginary part is a sum over zero-point energies which are totally imaginary. It should be noted that the imaginary part of the effective Lagrangian density is finite and does not suffer from the ultraviolet divergences which occur for all of the other momentum integrations.

In most of this work we have neglected the renormalization of physical parameters needed to regulate the above-mentioned divergences. Many renormalization schemes can be found in the literature and the renormalization for the case of the constant magnetic field is shown in Refs. 16, 18, and 22. Premature renormalization, though, can obscure the essential simplicity of the one-loop approximation as a change in the zero-point energy.

ACKNOWLEDGMENTS

We would like to thank Sudip Chakravarty, Michael Cornwall, Gary Horowitz, Roman Jackiw, John Richardson, Harvey Shepard, and Andy Strominger for many interesting and stimulating discussions. We would also like to thank everyone at the Institute for Theoretical Physics for their hospitality. The work of B. J. H. was supported in part by the National Science Foundation under Grant No. PHY79-08545. The work of C. H. T. was supported in part by a Dissertation Year Fellowship from the University of New Hampshire.

APPENDIX: COMPARISON OF CLASSICAL MINIMALITY AND POSITIVE DEFINITENESS OF THE ONE-LOOP EIGENVALUES FOR YANG-MILLS FIELDS

The classical Hamiltonian is

$$\mathcal{H}=\frac{1}{2}\int d^3x[(E_i^a)^2+(B_i^a)^2]. \tag{A1}$$

We wish to study the sufficient stability criterion of minimizing the energy H, subject to the constraint of Gauss's law:

$$\vec{\nabla}\cdot\vec{\mathrm{E}}^a-gc^{abc}\vec{\mathrm{A}}^b\cdot\vec{\mathrm{E}}^c=0. \tag{A2}$$

Following a derivation by Jackiw and Rossi,[47] we therefore extremize Eq. (A1) subject to Eq. (A2) by introducing the Lagrange multiplier A_0^a. Making the ansatz that the minimal configuration is static, we have

$$\frac{\partial E^{ia}}{\partial t}=0=\frac{\delta\mathcal{E}}{\delta A_i^a}=\epsilon^{ijk}\partial_j B_k^a-gc^{abc}\epsilon^{ijk}A_j^b B_k^c,$$

$$-\frac{\partial A^{ia}}{\partial t}=0=\frac{\delta\mathcal{E}}{\delta E_i^a}=E^{ia}+\nabla^i A_0^a-gc^{abc}A^{ib}A_0^c, \tag{A3}$$

where $\mathcal{E}$ is the constrained-energy density. The second variation of the constrained energy gives

$$\frac{\delta^2\mathcal{E}}{\delta E_i^c\delta A_j^a}=-\frac{\delta^2\mathcal{E}}{\delta A_i^c\delta E_j^a}=-\delta^{ij}gc^{abc}A_0^b,$$

$$\frac{\delta^2 E}{\delta E_i^a\delta E_j^b}=\delta^{ij}\delta^{ab}, \tag{A4}$$

$$\frac{\delta^2\mathcal{E}}{\delta A_i^c\delta A_j^a}=\frac{1}{2}\frac{\delta^2(B_k^b)^2}{\delta A_i^c\delta A_j^a}=\epsilon^{ikm}\epsilon^{nkj}D_m^{cb}D_n^{ba}-\epsilon^{ikj}gc^{cba}B_k^b.$$

The first term above is of a gyroscopic nature and leads to the complications noted in Ref. 47.

Now we also expand the action to second order around a static solution of the classical equations of motion in Euclidean space. The action is

$$S=\tfrac{1}{2}\int d^4x[(E_i^a)^2-(B_i^a)^2], \tag{A5}$$

with E_i^a and B_i^a as defined in Eq. (45). In Euclidean space

$$S_E=\tfrac{1}{2}\int d^4x_E[(E_i^a)^2+(B_i^a)^2], \tag{A6}$$

where now $E_i^a=F_{i4}^a$. If we expand around static solutions of $(\delta S_E/\delta A_\mu^a)_{\overline{A}}=0$, then the quadratic approximation to be functional integral has as its integrand the exponential of

$$-\tfrac{1}{2}A^{\mu c}\left[\frac{\delta^2 S_E}{\delta A^{\mu c}\delta A^{\nu a}}-\frac{1}{\alpha}(D_\mu D_\nu)^{ca}\right]_{\overline{A}}A^{\nu a}+\text{ghost terms}.$$

The ghost term always has positive-definite eigenvalues. The second term is the gauge-fixing term in the covariant background-field gauge, and all terms in the brackets are evaluated at the static classical field $\overline{A}_\mu^a$. Therefore, we desire that the bracketed operator is positive definite. Call this operator $G_{\mu\nu}^{ca}$. We have already shown [see Eq. (50)] that in the gauge $\alpha=1$ we have

$$G_{\mu\nu}^{ca}=\delta_{\mu\nu}(D_\alpha D^\alpha)^{ca}-2gc^{cba}F_{\mu\nu}^b. \tag{A7}$$

Thus

$$G_{00}^{ca}=(D_\alpha D^\alpha)^{ca}=-(D^\alpha D^\alpha)^{ca}, \tag{A8}$$

which is positive semidefinite while

$$G_{4i}^{ca}=-G_{i4}^{ca}=-2gc^{cba}F_{4i}^b=2gc^{cba}E^{ib} \tag{A9}$$

and

$$\begin{aligned}G_{ij}^{ca}&=\delta_{ij}(-D^\alpha D^\alpha)^{ac}-2gc^{cba}F_{ij}^b\\&=\frac{1}{2}\frac{\delta^2(E_k^b)^2}{\delta A^{ic}\delta A^{ja}}+\frac{1}{2}\frac{\delta^2(B_k^b)^2}{\delta A^{ic}\delta A^{ja}}-(D_iD_j)^{ca}\\&=-\delta_{ij}(D^4D^4)^{ca}+\frac{1}{2}\frac{\delta^2(B_k^b)^2}{\delta A^{ic}\delta A^{ja}}-(D_iD_j)^{ca}.\end{aligned} \tag{A10}$$

In general, the comparison with classical minimality is complicated. Classically we have a gyroscopic term, and quantum mechanically there are cross terms between A_4^a and A_i^a. If we restrict ourselves to the gauge choice $A_0^a=0$, however, things simplify. The classical analysis reduces to

$$\frac{1}{2}\frac{\delta^2(B_k^b)^2}{\delta A_i^c\delta A_j^a}>0,$$

and positive definiteness of the one-loop eigenvalues requires only that

$$-\delta_{ij}(D^4D^4)^{ca}+\frac{1}{2}\frac{\delta^2(B_k^b)^2}{\delta A^{ic}\delta A^{ja}}-(D_iD_j)^{ca}>0.$$

The first term in the latter expression is positive semidefinite. The second term is the same as in the classical expression, while the third is the gauge-fixing term. To compare with the classical case, let us use the remaining time-independent gauge freedom to fix the classical fluctuations with $D_i^{ab}(\overline{A})\delta A^{ib}=0$. Now in the expansion of the constrained Hamiltonian there are terms such as $\frac{1}{2}(\delta A^{ic})(\delta^2\mathcal{E}/\delta A^{ic}\delta A^{ja})_{\overline{A}}\,\delta A^{ja}$. If we use the above gauge choice, we may add a term $\delta A^{ic}[D_i(\overline{A})D_j(\overline{A})]^{ca}\times\delta A^{ja}$ to the above expression so that

$$(\delta^2\mathcal{E}/\delta A^{ic}\delta A^{ja})\to\tfrac{1}{2}[\delta^2(B_k^b)^2/\delta A^{ic}\delta A^{ja}]-(D_iD_j)^{ca}>0.$$

Comparing this to the one-loop expression, we conclude that classical stability again demands quantum stability and quantum instability demands classical instability (modulo exceptions mentioned in Sec. IV).

---

*Permanent address: Department of Physics, University of New Hampshire, Durham, New Hampshire 03824.

[1]S. Coleman, in *Laws of Hadronic Matter*, edited by A. Zichichi (Academic, New York, 1975) and G. Jona-Lasino, Nuovo Cimento 34, 1790 (1964).

[2]S. Mandelstam, in Proceedings of Les Houches Winter meeting, 1980 (unpublished).

[3]B. DeWitt, in *Relativity, Groups, and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).
[4]B. Zumino, in *Lectures on Elementary Particles and Quantum Field Theory*, 1970 Brandeis Summer Institute in Theoretical Physics, edited by S. Deser, M. Grisaru, and H. Pendleton (M.I.T. Press, Cambridge, Mass., 1970).
[5]R. Jackiw, Phys. Rev. D 10, 2491 (1974).
[6]J. Schwinger, Phys. Rev. 82, 664 (1951).
[7]S. Weinberg, Phys. Rev. D 7, 2887 (1973).
[8]W. Dittrich, Fortschr. Phys. 26, 289 (1978).
[9]R. Fukuda and T. Kugo, Phys. Rev. D 13, 3469 (1976).
[10]B. S. DeWitt, Phys. Rev. 162, 1195 (1967); 162, 1239 (1967).
[11]J. Honerkamp, Nucl. Phys. B48, 269 (1972).
[12]I. Ya. Arefieva, L. D. Faddeev, and A. A. Slavnov, Theor. Math. Phys. 21, 1165 (1974).
[13]R. Kallosh, Nucl. Phys. B78, 293 (1974).
[14]G. 't Hooft, Nucl. Phys. B62, 444 (1973).
[15]G. 't Hooft, in *Functional and Probabilistic Methods in Quantum Field Theory* (Wroclaw University, Wroclaw, Poland, 1976).
[16]N. K. Nielsen and P. Olesen, Nucl. Phys. B144, 376 (1978).
[17]J. Ambjorn and P. Olesen, Niels Bohr Institute Report No. NBI-HE-80-14, 1980 (unpublished).
[18]H. Falomir and F. Schaposnik, Phys. Rev. D 21, 2859 (1980).
[19]G. K. Savvidy, Phys. Lett. 71B, 133 (1977).
[20]S. G. Matinyan and G. K. Savvidy, Nucl. Phys. B134, 539 (1978).
[21]I. A. Batalin, S. G. Matinyan, and G. K. Savvidy, Yad. Fiz. 26, 407 (1977) [Sov. J. Nucl. Phys. 26, 214 (1977)].
[22]A. Yildiz and P. H. Cox, Phys. Rev. D 21, 1095 (1980); Harvard University Report No. HUTP-79/A032 (unpublished).
[23]M. J. Duff and M. Ramón-Medrano, Phys. Rev. D 12, 3357 (1975).
[24]A. Salam and J. Strathdee, Nucl. Phys. B90, 203 (1975).
[25]S. Weinberg, Phys. Rev. D 9, 3357 (1974).
[26]L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974).
[27]N. K. Nielsen and P. Olesen, Phys. Lett. 79B, 304 (1978).
[28]J. Ambjorn, N. K. Nielsen, and P. Olesen, Nucl. Phys. B152, 75 (1979).
[29]H. B. Nielsen and P. Olesen, Nucl. Phys. B160, 380 (1979).
[30]H. B. Nielsen and M. Ninomiya, Nucl. Phys. B156, 1 (1979).
[31]J. Ambjorn and P. Olesen, Nucl. Phys. B170, 60 (1980); H. B. Nielsen and P. Olesen, Niels Bohr Institute Report No. NBI-HE-79-45 (unpublished); H. B. Nielsen and M. Ninomiya, Nucl. Phys. B169, 309 (1980).
[32]W.-Y. Tsai and A. Yildiz, Phys. Rev. D 4, 3643 (1971); T. Goldman, W.-Y. Tsai, and A. Yildiz, *ibid.* 5, 1926 (1972); T. Goldman, Report No. SLAC-PUB-1363, 1973 (unpublished).
[33]P. Sikivie, Phys. Rev. D 20, 877 (1979).
[34]S. J. Chang and N. Weiss, Phys. Rev. D 20, 869 (1979).
[35]G. Cosenza and F. Neri, New York University Report No. NYU/TR3/80 (unpublished).
[36]S. Coleman, Phys. Rev. D 15, 2929 (1977); C. Callan and S. Coleman, *ibid.* 16, 1762 (1977).
[37]S. Coleman and F. DeLuccia, Phys. Rev. D 21, 3305 (1980).
[38]J. S. Langer, Ann. Phys. (N.Y.) 41, 108 (1967).
[39]R. Jackiw, M.I.T. Report No. C.T.P.-816, 1979 (unpublished).
[40]E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).
[41]W. Fischler and R. Brout, Phys. Rev. D 11, 905 (1975).
[42]S. Mandelstam, Phys. Rev. D 20, 3223 (1979).
[43]G. 't Hooft, Nucl. Phys. B153, 141 (1979).
[44]V. Vanyashin and M. Terent'ev, Zh. Eksp. Teor. Fiz. 48, 565 (1965) [Sov. Phys.—JETP 21, 375 (1965)].
[45]H. Pagels and E. Tomboulis, Nucl. Phys. B143, 485 (1978).
[46]D. G. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973); 9, 980 (1974).
[47]R. Jackiw and P. Rossi, Phys. Rev. D 21, 426 (1980).
[48]P. Sikivie and N. Weiss, Phys. Rev. D 20, 487 (1979).
[49]M. Magg, Nucl. Phys. B158, 154 (1979).
[50]H. Arodz, Phys. Lett. 84B, 524 (1979).
[51]R. Jackiw and A. Kerman, M.I.T. Report No. C.T.P.-763, 1979 (unpublished).
[52]L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, New York, 1977), pp. 456–458.
[53]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).