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Self-dual gauge fields

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Using ideas and techniques adopted from the theory of self-dual gravitational fields we investigate properties of self-dual gauge fields. A linear equation which generates these fields is the center for this investigation. Some of the main results are a natural choice of gauge which leads to (a) a Hertz potential, (b) the Lorentz gauge condition, (c) a *linear* relationship between the field and potential, and (d) the surprising solitonlike behavior of the solutions such that the future and past asymptotic fields are identical.

I. INTRODUCTION

It is the purpose of this paper to investigate the properties of, and to suggest an interpretation for, self-dual (or anti-self-dual) gauge fields. Although there is a considerable interest in self-dual fields, the interest has essentially centered on real fields on Euclidean four-space[1-3] and been associated with the instanton problem. We will be concerned only with (complex) self-dual fields on (real) Minkowski space. The basic point of view which we adopt is to view these self-dual (anti-self-dual) fields as representing right-helicity (left-helicity) "Yang-Mills particles," i.e., the analogs of right- and left-helicity photons, and that their field equations represent a (nonlinear) Schrödinger-type equation again analogous to the Schrödinger-equation interpretation of the Maxwell equations.[4,5]

In Sec. II we establish our basic notation and introduce some mathematical techniques (essential for the balance of the paper) that are used frequently in general relativity but are largely unknown in other fields. The main idea is to introduce four (null) basic vectors at each point in the Minkowski space which are all parametrized by two angles, i.e., by points on S^2; components of space-time fields, relative to this basis, become functions of x^a and S^2.

In Sec. III we exploit this idea by describing a *linear* differential equation on S^2 (with x^a appearing as parameters) for a function G which can be thought of as a generating function for the self-dual fields. By taking appropriate derivatives of G a self-dual field is obtained. We show that for essentially all self-dual fields the function G exists.

Using our linear equation we study in Sec. IV properties of the self-dual fields and show in particular that very natural gauge conditions can be introduced such that

$$\begin{aligned} \gamma^a &= \nabla_b H^{ab}, \\ \nabla_a \gamma^a &= 0, \qquad (1.1) \\ \tfrac{1}{2} F_{ab} &= \nabla_{[b}\gamma_{a]^+}, \end{aligned}$$

where H^{ab} is a Hertz-type potential, γ^a is the connection, F_{ab} is the gauge field, ∇_a is the Minkowski-space covariant derivative (unrelated to the gauge connection), and the + denotes the self-dual part.

Section V is devoted to a discussion of a general class of self-dual solutions which vanish in both the future and past null directions. We prove here one of our main results. This class of solutions has the rather surprising and attractive solitonlike property of having its asymptotic future behavior being identical to its asymptotic past behavior, resembling very much those Maxwell fields which are mixtures of half-advanced minus half-retarded fields.

Sections V and VI involve some conjectures about a Hilbert-space structure for the Yang-Mills fields and its relationship to a quantum theory of these fields.

II. NOTATION AND MATHEMATICAL PRELIMINARIES

On Minkowski space M we will consider the trivial vector bundle B (each fiber being an n-complex-dimensional vector space), i.e., $B = M \times C^n$. The (global) vector fields e_A ($A = 1, \ldots, n$) form a basis set as does

$$e'_A = G_A{}^B(x^a) e_B \qquad (2.1)$$

with $G_A{}^B(x^a)$ being GL(C,n) matrix-valued functions on M. The connection or parallel transfer of vectors in introduced by defining $\hat{\nabla}_a$ from

$$\hat{\nabla}_a e_A = \gamma_A{}^B{}_a e_B \qquad (2.2)$$

with

$$\gamma_A{}^B \equiv \gamma_A{}^B{}_a dx^a$$

being an arbitrary matrix-valued one-form. One defines the covariant derivative of an arbitrary vector $V = V^A e_A$ by

$$\hat{\nabla}_a V = (V^A{}_{,a} + V^B \gamma_B{}^A{}_a) e_A \qquad (2.3)$$

with a comma denoting ordinary derivatives with respect to the Minkowski coordinates x^a.

From (2.1) it follows that, under a change in basis

$$\gamma'^{B}_{A\ a} = G_A{}^C{}_{,a} G^{-1}{}_C{}^B + G_A{}^C \gamma_C{}^D{}_a G^{-1}{}_D{}^B , \tag{2.4a}$$

or using matrix notation,

$$\gamma'_a = G_{,a} G^{-1} + G\gamma_a G^{-1} . \tag{2.4b}$$

The curvature tensor or gauge field of this connection is defined by

$$F_{ab} = \gamma_{b,a} - \gamma_{a,b} - [\gamma_a, \gamma_b] \tag{2.5}$$

with $[\gamma_a, \gamma_b] = \gamma_a\gamma_b - \gamma_b\gamma_a$. From (2.4) one obtains

$$F'_{ab} = G F_{ab} G^{-1} . \tag{2.6}$$

The curvature tensor satisfies the generalized Bianchi identities

$$F_{[ab,c]} + F_{[ab}\gamma_{c]} - \gamma_{[c}F_{ab]} = 0 . \tag{2.7}$$

The dual field is defined by

$$F^*_{ab} = \tfrac{1}{2}\eta_{abcd}F^{cd}, \quad \eta_{abcd} = -\sqrt{-g}\,\epsilon_{abcd} \tag{2.8}$$

with ϵ_{abcd} the alternating symbol. We will now restrict ourselves to self-dual fields, $F^*_{ab} = iF_{ab}$. (With equal ease we could have made the restriction to anti-self-dual fields.)

Using (2.8) with (2.7) one easily shows that for self-dual fields

$$F^{ab}{}_{,b} + F^{ab}\gamma_b - \gamma_b F^{ab} = 0 , \tag{2.9}$$

i.e., that the (generalized) Yang-Mills field equations are automatically satisfied for self-dual fields.

Before we treat in detail the properties of self-dual fields we wish to review and summarize some technology which will be of great value to us. This technology, though used frequently in relativity, is not widely known.

At each point in M we introduce four independent null vectors l^a, n^a, m^a, and $\bar{m}^a$ (a null tetrad set) satisfying $l\cdot n = -m\cdot\bar{m} = 1$, all other products vanishing. If, for example, one has an orthonormal tetrad t^a, a^a, b^a, c^a with $t\cdot t = -a\cdot a = -b\cdot b = -c\cdot c = 1$ then we could choose

$$l^a = \frac{1}{\sqrt{2}}(t^a - a^a), \quad m^a = \frac{1}{\sqrt{2}}(b^a + ic^a),$$
$$\bar{m}_a = \frac{1}{\sqrt{2}}(b^a - ic^a), \quad n^a = \frac{1}{\sqrt{2}}(t^a + a^a).$$

These null vectors now serve as a basis set and one can find the components of a vector or tensor with respect to the tetrad. For the curvature tensor and connection form we define

$$\|\chi_{0A}{}^B\| \equiv \chi_0 = F_{ab}l^a m^b ,$$
$$\|\chi_{1A}{}^B\| \equiv \chi_1 = \tfrac{1}{2}F_{ab}(l^a n^b + \bar{m}^a m^b) , \tag{2.10a}$$
$$\|\chi_{2A}{}^B\| \equiv \chi_2 = F_{ab}\bar{m}^a n^b ,$$
$$\|\tilde{\chi}_{0A}{}^B\| \equiv \tilde{\chi}_0 = F_{ab}l^a \bar{m}^b ,$$
$$\|\tilde{\chi}_{1A}{}^B\| \equiv \tilde{\chi}_1 = \tfrac{1}{2}F_{ab}(l^a n^b - \bar{m}^a m^b) , \tag{2.10b}$$
$$\|\tilde{\chi}_{2A}{}^B\| \equiv \tilde{\chi}_2 = F_{ab}m^a n^b ,$$
$$\gamma_{00'} = \gamma_a l^a, \quad \gamma_{01'} = \gamma_a m^a ,$$
$$\gamma_{10'} = \gamma_a \bar{m}^a, \quad \gamma_{11'} = \gamma_a n^a . \tag{2.11}$$

If one knows the behavior of the tetrad vectors as functions of position in M then the Yang-Mills equations could be rewritten in terms of the χ's and γ's.[6] To study different questions one could choose the tetrad in different appropriate fashions. To study radiation the following choice is very convenient. Introduce null polar coordinates in M such that the metric has the form

$$ds^2 = \eta_{ab}dx^a dx^b = 2du^2 + 2du\,dr - \frac{r^2}{2P_0^2}d\eta\, d\bar{\eta} , \tag{2.12a}$$

with $P_0 = \frac{1}{2}(1 + \eta\bar{\eta})$ and where η and $\bar{\eta}$ are the complex stereographic coordinates ($\eta = e^{i\phi}\cot\frac{1}{2}\theta$). The tetrad is then choosen as

$$l = l^a\frac{\partial}{\partial x^a} = \frac{\partial}{\partial r}, \quad m = -\frac{P_0}{r}\frac{\partial}{\partial\eta},$$
$$\bar{m} = -\frac{P_0}{r}\frac{\partial}{\partial\bar{\eta}}, \quad n = \frac{\partial}{\partial u} - \frac{\partial}{\partial r} . \tag{2.12b}$$

Another useful choice is based on null-plane coordinates with

$$ds^2 = 2(du\,dv - dw\,d\bar{w}) , \tag{2.13a}$$

$$l = \frac{\partial}{\partial v}, \quad m = \frac{\partial}{\partial w},$$
$$\bar{m} = \frac{\partial}{\partial\bar{w}}, \quad n = \frac{\partial}{\partial u} . \tag{2.13b}$$

In Appendix A we give the field equations for self-dual Yang-Mills theory in both systems, (2.12) and (2.13).

Note that the tetrad (2.13) is covariantly constant throughout M. It is very convenient to study tetrad systems with this property but where the vectors l, n, m, $\bar{m}$ are *not* associated with coordinate directions. The following formalism (which is similar or even equivalent to the two-component spinor formalism of Penrose) accomplishes this.

Using coordinate components in some Minkowski coordinate system $x^a = (t, x, y, z)$, let

$$l^a = \frac{1}{2\sqrt{2}P_0}(1+\zeta\bar{\zeta}, \zeta+\bar{\zeta}, i(\bar{\zeta}-\zeta), -1+\zeta\bar{\zeta}), \tag{2.14a}$$

$$m^a = \frac{1}{2\sqrt{2}P_0}(0, 1-\bar{\zeta}^2, -i(1+\bar{\zeta}^2), 2\bar{\zeta}), \tag{2.14b}$$

$$\bar{m}^a = \frac{1}{2\sqrt{2}P_0}(0, 1-\zeta^2, i(1+\zeta^2), 2\zeta)), \tag{2.14c}$$

$$n^a = \frac{1}{2\sqrt{2}P_0}(1+\zeta\bar{\zeta}, -(\zeta+\bar{\zeta}), -i(\bar{\zeta}-\zeta), 1-\zeta\bar{\zeta}). \tag{2.14d}$$

The ζ (and $\bar{\zeta}$) are complex stereographic coordinates labeling points on a sphere or the extended complex plane. Note that for any values of ζ and $\bar{\zeta}$ we have the usual scalar-product conditions $l\cdot n = -m\cdot\bar{m} = 1$, etc. As ζ and $\bar{\zeta}$ move over the complex plane l^a and n^a range over all real null directions. *Note further that even when $\bar{\zeta}$ is treated as an independent complex variable, free from ζ,* the above remarks remain true except that l^a and n^a now range over all complex null directions. For this case we will denote the free variable $\bar{\zeta}$ by $\tilde{\zeta}$ and use $\bar{\zeta}$ only to mean the complex conjugate of ζ.

For fixed but arbitrary ζ and $\tilde{\zeta}$ (2.14) can be considered as a global tetrad and tetrad components of arbitrary tensors or tensor fields could be taken, e.g.,

$$\begin{aligned} A_{00'}(x^a, \zeta, \tilde{\zeta}) &= A_a(x^b)l^a(\zeta, \tilde{\zeta}), \\ A_{01'}(x^a, \zeta, \tilde{\zeta}) &= A_a m^a, \text{ etc.} \end{aligned} \tag{2.15}$$

The components are now functions not only of the coordinates but also of the orientation of the tetrad via the ζ and $\tilde{\zeta}$. We now study the differentiation of these functions. All functions which we use are assigned an integer s, known as the spin-weight. How the assignment is made will be described shortly. The differential operators $\eth$ and $\tilde{\eth}$ which act on spin-weight-s functions η defined by

$$\eth\eta \equiv 2P_0^{1-s}\frac{\partial}{\partial\zeta}(P_0^{s}\eta), \tag{2.16}$$

$$\tilde{\eth}\eta \equiv 2P_0^{1+s}\frac{\partial}{\partial\tilde{\zeta}}(P_0^{-s}\eta), \tag{2.17}$$

respectively, increase and decrease the spin-weight by one unit.

We assign to the vector l^a in (2.14a) the spin-weight 0, from which it follows from (2.16), (2.17), and (2.14) that

$$l^a = \frac{1}{2\sqrt{2}P_0}(1+\zeta\tilde{\zeta}, \zeta+\tilde{\zeta}, i(\tilde{\zeta}-\zeta), -1+\zeta\tilde{\zeta}), \tag{2.18a}$$

$$m^a = \eth l^a, \tag{2.18b}$$

$$\tilde{m}^a = \tilde{\eth} l^a, \tag{2.18c}$$

$$n^a = l^a + \eth\tilde{\eth} l^a \tag{2.18d}$$

and hence m^a, $\tilde{m}^a$, and n^a have, respectively, spin-weight 1, -1, and 0. Furthermore, it follows that

$$\eth^2 l^a = \eth m^a = 0, \tag{2.19a}$$

$$\tilde{\eth}^2 l^a = \tilde{\eth}\tilde{m}^a = 0. \tag{2.19b}$$

If to ordinary (i.e., ζ and $\tilde{\zeta}$ independent) tensors we assign spin-weight 0 and if in addition spin-weights add when two spin-weighted functions are multiplied, we have a general method of constructing spin-weighted functions, e.g., $T_{ab}l^a l^b$, $T_{ab}l^a m^a$, $T_{ab}m^a m^b$, $T_{ab}m^a\tilde{m}^b$ have spin-weights, 0, 1, 2, 0.

Without entering into details[7] we mention that each regular spin-s function can be expanded in a basis set known as the spin-s spherical harmonics ${}_sY_{lm}$, the ordinary harmonics having $s=0$. Further, each regular spin-s function $\eta_{(s)}$ can be written for some regular $\eta_{(0)}$ as

$$\eta_{(s)} = \eth^s\eta_{(0)}.$$

The process of finding the regular $\eta_{(0)}(\zeta, \tilde{\zeta})$ might, however, be difficult. We will see later that the problem of finding self-dual Yang-Mills fields is closely related to the problem of finding $\eta_{(0)}$ when $s=1$. (We remark in passing that to find self-dual solutions of the Einstein equations one encounters a similar problem but for $s=2$.)

A simple but important example of a set of spin-weighted functions are those constructed from the position vector x^a, namely,

$$\begin{aligned} l &\equiv x^a l_a, \\ m &\equiv \eth l = x^a m_a, \\ \tilde{m} &\equiv \tilde{\eth} l = x^a\tilde{m}_a, \\ n &= l + \eth\tilde{\eth} l = x^a n_a, \end{aligned} \tag{2.20}$$

having spin-weights 0, 1, -1, and 0. These variables will play an important role.

From the vectors (2.18) one can construct the two sets of bivectors

$$\left.\begin{aligned} &l_{[a}m_{b]} \\ &\tfrac{1}{2}(l_{[a}n_{b]} + \tilde{m}_{[a}m_{b]}) \\ &n_{[a}m_{b]} \end{aligned}\right\} \text{anti-self-dual}, \tag{2.21}$$

$$\left.\begin{aligned} &l_{[a}\tilde{m}_{b]} \\ &\tfrac{1}{2}(l_{[a}n_{b]} - \tilde{m}_{[a}m_{b]}) \\ &n_{[a}\tilde{m}_{b]} \end{aligned}\right\} \text{self-dual}. \tag{2.22}$$

Expressions (2.21) are all anti-self-dual and form a basis set for anti-self-dual bivectors while (2.22) are self-dual and likewise form a basis for self-dual bivectors. The following relationships are easily checked:

$$\tilde{\eth} l_{[a}m_{b]} = l_{[a}n_{b]} + \tilde{m}_{[a}m_{b]}\,, \tag{2.23a}$$

$$\tilde{\eth}(l_{[a}n_{b]} + \tilde{m}_{[a}m_{b]}) = 2\tilde{m}_{[a}n_{b]}\,, \tag{2.23b}$$

$$\tilde{\eth}\tilde{m}_{[a}n_{b]} = 0, \quad \eth l_{[a}m_{b]} = 0 \tag{2.23c}$$

and

$$\eth l_{[a}\tilde{m}_{b]} = l_{[a}n_{b]} - \tilde{m}_{[a}m_{b]}\,, \tag{2.24a}$$

$$\eth(l_{[a}n_{b]} - \tilde{m}_{[a}m_{b]}) = 2m_{[a}n_{b]}\,, \tag{2.24b}$$

$$\eth m_{[a}n_{b]} = 0, \quad \tilde{\eth} l_{[a}\tilde{m}_{b]} = 0\,. \tag{2.24c}$$

From the self-dual and anti-self-dual character of (2.22) and (2.21), it is seen immediately that for self-dual fields

$$\chi_0 = \chi_1 = \chi_2 = 0\,. \tag{2.25}$$

Note that if we use the tetrad (2.18) then from (2.23)

$$2\chi_1 = \tilde{\eth}\chi_0\,, \quad 2\chi_2 = \tilde{\eth}^2\chi_0 \tag{2.26}$$

and hence if $\chi_0 = 0$, it follows that $\chi_1 = \chi_2 = 0$.

III. A LINEAR EQUATION FOR SELF-DUAL GAUGE FIELDS

In this section we summarize and review earlier work by Sparling,[8] Goldberg,[9] and Newman[6] on the Sparling equation

$$\eth G = -GA\,, \tag{3.1}$$

where G is a matrix-valued $s=0$ function of x^a, ζ, and $\tilde{\zeta}$, to be determined from (3.1) while A is a given but arbitrary matrix-valued $s=1$ function of l (from 2.20) and ζ and $\tilde{\zeta}$. Note that the x^a appears in (3.1) only as a set of parameters, entering via the l dependence on x^a.

We will now show that the linear equation (3.1) is equivalent to the self-dual Yang-Mills equations where the arbitrary A plays the role of (characteristic) initial data. First we will show that a regular solution of (3.1),

$$G = G(x^a, \zeta, \tilde{\zeta})\,, \tag{3.2}$$

does lead to a self-dual field simply by differentiation of (3.2). We then will show that (3.1) will exist for any self-dual field.

From a regular G we define

$$\gamma'_a(x^a, \zeta, \tilde{\zeta}) = G_{,a}G^{-1} + \eth h l_a - h m_a\,, \tag{3.3}$$

with

$$h = l^a(\zeta, \tilde{\zeta})\tilde{\eth}(G_{,a}G^{-1})\,. \tag{3.4}$$

We claim that γ'_a depends only on x^a, i.e., is independent of ζ and $\tilde{\zeta}$ and furthermore is automatically the connection for a self-dual field. To see this we first take the gradient of (3.1), i.e.,

$$\eth G_{,a} = -G_{,a}A - G\dot{A}l_a\,, \tag{3.5}$$

with $\dot{A} \equiv \partial A/\partial l$. Using (3.1) and

$$\eth G^{-1} = -G^{-1}\eth G G^{-1}\,, \tag{3.6}$$

we have

$$\eth(G_{,a}G^{-1}) = -G\dot{A}G^{-1}l_a \tag{3.7}$$

and

$$l^a\eth(G_{,a}G^{-1}) = m^a\eth(G_{,a}G^{-1}) = \tilde{m}^a\eth(G_{,a}G^{-1}) = 0\,, \tag{3.8}$$

$$n^a\eth(G_{,a}G^{-1}) = -G\dot{A}G^{-1}\,. \tag{3.9}$$

Now by applying $\eth$ to (3.3) and $\eth^2$ to (3.4) we obtain, respectively,

$$\eth\gamma'_a = \eth(G_{,a}G^{-1}) + \eth^2 h l_a \tag{3.10}$$

and

$$\eth^2 h = -n^a\eth(G_{,a}G^{-1}) = G\dot{A}G^{-1}\,, \tag{3.11}$$

where extensive use was made of (3.8), (3.9), (2.18), and (2.19). By comparing (3.10) and (3.11) with (3.7) we see that

$$\eth\gamma'_a = 0\,. \tag{3.12}$$

From the assumed regularity of G and the fact that γ_a has $s=0$ it follows from (3.12) that

$$\tilde{\eth}\gamma'_a = 0 \tag{3.13}$$

and our first claim is proved. [Some useful identities are obtained by applying $\tilde{\eth}$ to (3.3):

$$\begin{aligned} &h = l^a\tilde{\eth}(G_{,a}G^{-1})\,,\\ &\eth h = m^a\tilde{\eth}(G_{,a}G^{-1})\,,\\ &\tilde{\eth} h = -\tilde{m}^a\tilde{\eth}(G_{,a}G^{-1})\,,\\ &h + \tilde{\eth}\eth h = -n^a\tilde{\eth}(G_{,a}G^{-1}).] \end{aligned} \tag{3.14}$$

In order to show that γ'_a is the connection all we must do is show that the field calculated from (3.3) is self-dual. From (2.5) and (3.3) after some simplification

$$F'_{ab} = 2\eth h_{,[b}l_{a]} - 2h_{,[b}m_{a]} - [G_{,a}G^{-1}, \eth h l_a - h m_b] + [G_{,b}G^{-1}, \eth h l_a - h m_a] + [\eth h, h]2l_{[a}m_{b]}\,.$$

It is seen immediately that

$$\chi_0 = F'_{ab}l^a m^b = 0 \tag{3.15}$$

and hence, from (2.26), that $\chi_1 = \chi_2 = 0$. Thus, the field is self-dual.

We now discuss the inverse problem, i.e., the problem of deriving (3.1) from a self-dual field.

The argument we give is a slight modification of the argument of Goldberg.[9] We are now *given* a $\gamma'_a(x^a)$ independent of ζ and $\tilde{\zeta}$ and wish to *determine* a $G(x^b, \zeta, \tilde{\zeta})$ by the differential equation

$$\gamma'_a = G_{,a}G^{-1} + \eth h l_a - h m_a , \quad h = l^a \tilde{\eth}(G_{,a}G^{-1}) . \tag{3.16}$$

By taking the curl (the skew derivative) of (3.16) and using the argument which led to (3.15) one is quickly led to the condition that γ'_a must be the connection for a self-dual field as the integrability conditions on (3.16).

Applying $\eth$ to (3.16) yields

$$\eth(G_{,a}G^{-1}) = -\eth^2 h l_a . \tag{3.17}$$

If we define $A(x^a, \zeta, \tilde{\zeta})$ by

$$A = -G^{-1}\eth G , \tag{3.18}$$

then

$$A_{,a} = G^{-1}G_{,a}G^{-1}\eth G - G^{-1}\eth G_{,a}$$

or

$$A_{,a} = -G^{-1}\eth(G_{,a}G^{-1})G . \tag{3.19}$$

From (3.17) we have

$$A_{,a} = G^{-1}\eth^2 h G l_a \tag{3.20}$$

and thus A is a function only of l, ζ, and $\tilde{\zeta}$, i.e., $\eth G = -GA(l, \zeta, \tilde{\zeta})$ which was to be proved.

This proof is based on the local existence of solutions to (3.16) for $G(x^a, \zeta, \tilde{\zeta})$ and thus the $A(l, \zeta, \tilde{\zeta})$ will in general be only a local function. We have also not been able to make an argument in general for the regularity of A as a function of ζ and $\tilde{\zeta}$. In a later section when we are discussing asymptotically "flat" Yang-Mills fields we will show that A is essentially the null or characteristic data for the field, appearing as the radiation field in the neighborhood of null infinity. This will provide an alternate proof of (3.1) albeit only in the asymptotically flat case.

IV. PROPERTIES OF SELF-DUAL FIELDS

We saw in Sec. III that regular solutions of (3.1), i.e., $G = G(x^a, \zeta, \tilde{\zeta})$, yield connections of the form

$$\gamma'_a(x^a) = G_{,a}G^{-1} + \eth h l_a - h m_a , \tag{4.1}$$

which can be rewritten as

$$\gamma'_a = G_{,a}G^{-1} + GG^{-1}(\eth h l_a - h m_a)GG^{-1}$$

or

$$\gamma'_a = G_{,a}G^{-1} + G\gamma_a G^{-1} , \tag{4.2}$$

with

$$\gamma_a(x^a, \zeta, \tilde{\zeta}) = G^{-1}(\eth h l_a - h m_a)G . \tag{4.3}$$

By comparing (4.2) with (2.4) we see that, for arbitrary but given $(\zeta, \tilde{\zeta})$, γ_a from (4.3) is an equivalent (gauge related) connection to γ'_a.

In solving (3.1) the freedom of solution is

$$G \to G^* = gG , \tag{4.4}$$

where $g(x^a)$ is an arbitrary $GL(n, C)$ matrix function of x^a alone. Using G^* in (4.1) yields a gauge-transformed connection

$$\gamma_a^{*\prime} = G^*_{,a}G^{*-1} + \eth h^* l_a - h^* m_a \tag{4.5}$$

with

$$h^* = ghg^{-1}, \quad \eth h^* = g\eth h g^{-1} .$$

Writing (4.5) as

$$\gamma_a^{*\prime} = G^*_{,a}G^{*-1} + G^*\gamma^*_a G^{*-1} , \quad \gamma^*_a = G^{*-1}(\eth h^* l_a - h^* m_a)G^* , \tag{4.6}$$

it is easy to check that

$$\gamma^*_a = \gamma_a , \tag{4.7}$$

i.e., the choice of solution G *does not affect* $\gamma_a(x^a, \zeta, \tilde{\zeta})$ in (4.3). What we have thus shown is that there is a canonical choice of gauge, up to a choice of ζ and $\tilde{\zeta}$, i.e., a complex null direction, or point on $S^2 \times S^2$. We will almost always adopt this choice of gauge.

From (4.3) we see that for each $(\zeta, \tilde{\zeta})$, γ_a lies in the anti-self-dual 2-blade or surface spanned by l_a and m_a. In fact, if we define the Hertz-type potential

$$H_{ab} = 2\Theta l_{[a}m_{b]}, \quad \Theta = G^{-1}\tilde{\eth} G , \tag{4.8}$$

which is an anti-self-dual bivector, then

$$\gamma^a = H^{ab}{}_{,b} = 2\Theta_{,b} l^{[a}m^{b]} . \tag{4.9}$$

To see this, note that

$$(G^{-1}\tilde{\eth}G)_{,a} = G^{-1}\tilde{\eth}(G_{,a}G^{-1})G , \tag{4.10}$$

then using (3.14) and (4.3), Eq. (4.9) follows.

The curvature tensor in our canonical gauge takes the form

$$F_{ab} = \gamma_{b,a} - \gamma_{a,b} + 2G^{-1}[\eth h, h]Gl_{[a}m_{b]} . \tag{4.11}$$

Since F_{ab} is self-dual and the last term is anti-self-dual, the skew-derivative terms must decompose into self-dual and anti-self-dual parts such that

$$F_{ab} = 2\gamma_{[b,a]+} , \tag{4.12a}$$

$$0 = 2\gamma_{[b,a]-} + 2G^{-1}[\eth h, h]Gl_{[a}m_{b]} , \tag{4.12b}$$

where the + and − mean, respectively, self-dual and anti-self-dual parts.

Summarizing, we stress the simplicity of the structure of self-dual Yang-Mills fields when (3.1) is solved for $G(x^a, \zeta, \tilde{\zeta})$. The function θ

$=G^{-1}\tilde{\eth}G$ determines the Hertz potential

$$H^{ab}=2\Theta l^{[a}m^{b]},$$

which leads to the connection

$$\gamma^a=H^{ab}{}_{,b}$$

and by linear relations to the field

$$F_{ab}=2\gamma_{[b,a]}.$$

We also have from (4.9) that

$$\gamma^a{}_{,a}=0\,, \tag{4.13}$$

i.e., γ^a satisfies the Lorentz gauge condition.

As a final comment to this section we point out that under the "gaugelike" transformation of $A \to A'$,

$$A'(l,\zeta,\bar{\zeta})=q^{-1}Aq-q^{-1}\eth q\,, \tag{4.14}$$

with $q=q(\zeta,\bar{\zeta})$ an arbitrary spin-weight zero matrix valued function of $\zeta,\bar{\zeta}$, the new equation

$$\eth G'=-G'A' \tag{4.15}$$

has as solution $G'=Gq$ where G is a solution of $\eth G=-GA$, and that the connection computed from the G' is identical to that computed from the G. The relevance of this remark is that when $A(u,\zeta,\bar{\zeta})$ has a limit in either the plus or minus $u=\infty$ direction, the A' can be made to vanish there. If both limits are equal the A' vanishes in both directions, if both limits exist but are different the A' can be made to vanish in either but not both directions.

V. STRUCTURE OF ASYMPTOTIC SOLUTIONS

Although Eq. (3.1) is a linear equation for G, the demand for regularity in the solutions makes finding solutions very difficult. To our knowledge only a handful of solutions are known.[8] We will, in this section, discuss how approximate or asymptotic solutions can be obtained. In particular, we will prove that for a wide class of solutions, the $A(l,\zeta,\bar{\zeta})$ constitutes the characteristic data (and is essentially the radiation field) and furthermore, if the data is given for incoming fields (on past null infinity) the propagation will yield data for outgoing fields (on future null infinity) which is (surprisingly) identical to the incoming data.

Equation (3.1) can be converted to the integral equation

$$G=-\eth^{-1}(GA)+I, \tag{5.1}$$

where $\eth^{-1}$ is the uniquely defined[10] integral operator inverse to $\eth$, i.e., $\eth\eth^{-1}\eta=\eta$ and I is the identity matrix.[11] By considering the iteration scheme

$$G_n=-\eth^{-1}(G_{n-1}A)+I,\quad G_1=I \tag{5.2}$$

we can form the formal solution as

$$G=I-\eth^{-1}A+\eth^{-1}(\eth^{-1}A)A \\ -\eth^{-1}(\eth^{-1}(\eth^{-1}A)A)A+\cdots\,. \tag{5.3}$$

We now assume that, for the solutions with which we are dealing, (5.3) converges. Our claim is that to study radiation in either the future or past null direction only the first nontrivial term is important, the further terms falling off faster than r^{-1}, Defining

$$F=-\eth^{-1}A\,, \tag{5.4}$$

we see that our problem is now to study the differential equation

$$\eth F=-A(l,\zeta,\bar{\zeta})\,. \tag{5.5}$$

[Note that each term in the series (5.3) involves an equation similar to (5.5), the right side always depending on information gained from the earlier terms.]

Since A has $s=1$, it can be expanded in the complete set ${}_1Y_{LM}$, i.e.,

$$A(l,\zeta,\bar{\zeta})=-\sum_{L,M}b_{LM}\,{}_1Y_{LM}\,. \tag{5.6}$$

We will consider only one arbitrary term of (5.6) in (5.5) since the general solution can be obtained by summing over L and M. We thus consider

$$\eth F=b(l)\,{}_1Y_{LM}=b(l)Y_L\,. \tag{5.7}$$

(Note that we have suppressed the M in the spherical function as it plays no role.)

Writing

$$b(l)=B^{(L)}(l)\equiv\frac{\partial^L B}{\partial l^L}\,, \tag{5.8}$$

the general solution to (5.7) is

$$F=F_P+H\,,$$

where

$$F_P=\frac{B^{(L-1)}}{\eth l}\,{}_1Y_L-\frac{B^{(L-2)}}{(\eth l)^2}\eth\,{}_1Y_L \\ +\cdots(-1)^{L+1}\frac{B}{(\eth l)^L}\eth^{L-1}\,{}_1Y_L \\ =\sum_{\mathcal{L}=1}^{L}\frac{B^{(L-\mathcal{L})}}{(\eth l)^{\mathcal{L}}}\eth^{\mathcal{L}-1}\,{}_1Y_L(-1)^{\mathcal{L}+1} \tag{5.9}$$

and where H satisfies

$$\eth H=0\,. \tag{5.10}$$

Using (2.14b) and (2.20), it is easily seen that $\eth l$ is of the form

$$\eth l=\frac{a(\bar{\zeta}-\bar{\zeta}_0)(\bar{\zeta}-\bar{\zeta}_1)}{1+\zeta\bar{\zeta}}\,, \tag{5.11}$$

where a, $\tilde{\zeta}_0$, and $\tilde{\zeta}_1$ are functions of x^a. Thus $\eth l$ has two first-order zeros in its $\tilde{\zeta}$ dependence and hence each term in (5.9) has two $\mathcal{L}$th-order poles. If we let

$$H=\sum_{\mathcal{L}=1}^{L}\frac{h_{\mathcal{L}}^{ab\cdots}\,m_a m_b\cdots}{(\eth l)^{\mathcal{L}}}\,, \tag{5.12}$$

where there are $\mathcal{L}$ indices on the $h_{\mathcal{L}}^{ab\cdots}$ and where $h_{\mathcal{L}}^{ab\cdots}$ is a trace-free symmetric three-dimensional tensor ($h_{\mathcal{L}}^{ab\cdots}V_a=0$, $V_a=l_a+n_a$) which is a function only of x^a, it will automatically satisfy (5.10) and be of the proper spin-weight $s=0$. Of the $2\mathcal{L}+1$ components of $h_{\mathcal{L}}^{ab\cdots}$, $2\mathcal{L}$ are determined by the demand that the poles of H cancel the poles in F_p, i.e., the F be regular on the real sphere $\tilde{\zeta}=\bar{\zeta}$. The remaining components (one for each $\mathcal{L}$) appear in the final F additively and constitute the gauge freedom.

We now illustrate these solutions with the cases $L=1$ and $L=2$. (There are no global $L=0$ solutions.)

It turns out to be much more convenient to use coordinates u, r, η, and $\tilde{\eta}$ of (2.12) instead of x^a. The transformation can be written simply as

$$x^a=uv^a+r\hat{l}^a(\eta,\tilde{\eta})\,, \tag{5.13}$$

where $\hat{l}^a$ is the same as l^a in (2.18a) but with $(\eta,\tilde{\eta})$ replacing the $(\zeta,\tilde{\zeta})$ and v^a is a constant vector with $v^a v_a=2$, $v^a l_a=1$. We will use, in addition to $\hat{l}_a$, the vectors $\hat{m}_a$, $\hat{\tilde{m}}_a$, and $\hat{n}_a$, also defined by (2.18) with the $(\eta,\tilde{\eta})$ substituted for $(\zeta,\tilde{\zeta})$. With (5.13), (2.20) becomes

$$l=x^a l_a=u+rl_a\hat{l}^a \tag{5.14}$$

and

$$\eth l=rm_a\hat{l}^a\,. \tag{5.15}$$

Note that $\eth$ does not act on the hatted vectors. Directly from their definitions we have

$$l_a\hat{l}^a=\frac{(\zeta-\eta)(\tilde{\zeta}-\tilde{\eta})}{(1+\zeta\tilde{\zeta})(1+\eta\tilde{\eta})}\,, \tag{5.16a}$$

$$m_a\hat{l}^a=\frac{(\tilde{\zeta}-\tilde{\eta})(1+\tilde{\zeta}\eta)}{(1+\zeta\tilde{\zeta})(1+\eta\tilde{\eta})} \tag{5.16b}$$

and thus the zeros of $\eth l$ occur at[12]

$$\tilde{\zeta}=\tilde{\eta},\quad \zeta=\eta\,, \tag{5.17a}$$

which we call the positive pole and at

$$\tilde{\zeta}=-1/\eta,\quad \zeta=-1/\tilde{\eta}\,, \tag{5.17b}$$

which is called the negative pole. We frequently must evaluate functions of ζ and $\tilde{\zeta}$ at the two poles. We use the notation for $T(\zeta,\tilde{\zeta})$

$$T^+=T(\eta,\tilde{\eta})\,,$$
$$T^-=T(-\tilde{\eta}^{-1},-\eta^{-1})\,.$$

From the definition[7] of the spin-s spherical harmonics it is easy to show that

$${}_sY^-_{LM}=(-1)^{L-s}\left(\frac{\tilde{\eta}}{\eta}\right)^s{}_{-s}\hat{Y}_{LM}\,, \tag{5.18a}$$

which leads easily to the special cases

$$\begin{aligned}
&l_a^+=\hat{l}_a,\quad n_a^+=\hat{n}_a,\quad {}_1Y_1^-=\frac{\tilde{\eta}}{\eta}\,{}_{-1}Y_1\,,\\
&l_a^-=\hat{n}_a,\quad n_a^-=\hat{l}_a,\quad {}_1Y_2^-=-\frac{\tilde{\eta}}{\eta}\,{}_{-1}Y_2\,,\\
&m_a^+=\hat{m}_a\,,\quad \tilde{m}_a^+=\hat{\tilde{m}}_a\,,\\
&m_a^-=-\frac{\tilde{\eta}}{\eta}\,\hat{\tilde{m}},\quad \tilde{m}_a^-=-\frac{\eta}{\tilde{\eta}}\,\hat{m}_a,\quad {}_2Y_2^-=\left(\frac{\tilde{\eta}}{\eta}\right)^2{}_{-2}Y_2\,.
\end{aligned} \tag{5.18b}$$

[Note that the mapping $\eta\to-\tilde{\eta}^{-1}$, $\tilde{\eta}\to-\eta^{-1}$ in the case of the real sphere, i.e., $\tilde{\eta}=\bar{\eta}$, maps points of the sphere into antipodal points and that we can think of P, the parity operator acting on spin-weight s function $T_{(s)}$ by

$$PT_{(s)}\equiv T^P_{(-s)}=\left(\frac{\eta}{\tilde{\eta}}\right)^s T_{(s)}(-\tilde{\eta}^{-1},-\eta^{-1})\,. \tag{5.18c}$$

Under P the spin-weight changes sign and in the expansion in spin-s harmonics the coefficients change sign by (5.18a).]

Using (5.18) with (5.14) we have

$$l^+=u,\quad \tilde{\eth}\eth l^+=r\,, \tag{5.19a}$$

$$l^-=u+r,\quad \tilde{\eth}\eth l^-=-r\,, \tag{5.19b}$$

so that from $B(l)$ we have

$$B^+=B(u)\,, \tag{5.20a}$$

$$B^-=B(u+r)\,. \tag{5.20b}$$

For $L=1$, (5.7) and (5.8) become

$$\eth F=\dot{B}\,{}_1Y_1$$

and

$$F=\frac{B}{\eth l}\,{}_1Y_1+\frac{m_a[\alpha\hat{m}^a+\beta\hat{\tilde{m}}^a+\gamma(\hat{n}^a-\hat{l}^a)]}{\eth l}\,, \tag{5.21}$$

where for $h_1{}^a(x^a)$ we have written

$$\alpha\hat{m}^a+\beta\hat{\tilde{m}}^a+\gamma(\hat{n}^a-\hat{l}^a)$$

with α, β, and γ to be determined.

In order to avoid the positive pole in F [using (5.18)] we must have

$$\beta=B^+\,{}_1\hat{Y}_1 \tag{5.22a}$$

and

$$\alpha=-B^-\,{}_{-1}\hat{Y}_1 \tag{5.22b}$$

to avoid the negative pole in F. The γ which is an arbitrary function of x^a (the gauge freedom) can be conveniently chosen as

$$\gamma = \tfrac{1}{2} B^{+} \hat{\bar{\eth}}_1 Y_1 \tag{5.22c}$$

so that

$$F^{+} = 0 . \tag{5.23}$$

This is easily checked by using L'Hopitals rule on (5.21), i.e., by applying $\bar{\eth}$ to the numerator and denominator. We thus have

$$F = \frac{1}{r}\left[\frac{B(l)_1 Y_1 + m_a(B(u)_1 \hat{Y}_1 \hat{\bar{m}}^a - B(u+r)_{-1}\hat{Y}_1 \hat{m}^a)}{m_a \hat{l}^a}\right] - \frac{B(u)\hat{\bar{\eth}}_1 \hat{Y}_1}{r} . \tag{5.24}$$

Note that the r behavior appears explicitly only in the r^{-1} factor and implicity in the $B(l)$ and $B(u+r)$.

The $L=2$ case, i.e., $\eth F = \ddot{B}_1 Y_2$, can be explicitly solved in a similar fashion obtaining

$$F = \frac{\gamma^1(u,\eta,\bar{\eta})}{r} + \frac{\gamma^2(u,\eta,\bar{\eta})}{r^2} + \frac{1}{r}\left[\frac{\dot{B}_1 Y_2 + \dot{B}^{+}{}_1 \hat{Y}_2 m_a \hat{\bar{m}}^a + \dot{B}^{-}{}_{-1}\hat{Y}_2 m_a \hat{m}^a}{m_a \hat{l}^a}\right]$$
$$+ \frac{(m_a \hat{l}^a)^{-2}}{r^2}\left[-B\eth_1 Y_2 + B^{+}\hat{\bar{\eth}}_1 \hat{Y}_2 (m_a \hat{\bar{m}}^a)^2 + B^{-}\hat{\bar{\eth}}_{-1} Y_2 (m_a \hat{m}^a)^2 + 4B^{+}{}_1 Y_2 m_a \hat{l}^a m_b \hat{\bar{m}}^b + 4B^{-}{}_{-1} Y_2 m_a \hat{l}^a m_b \hat{m}^b\right] . \tag{5.25}$$

The γ can be choosen so that $F^{+}=0$.

The general pattern of the solution for arbitrary L can now be seen, i.e.,

$$F = \frac{1}{r}\left[\frac{B^{(L-1)}{}_1 Y_L + B^{(L-1)+}{}_1 \hat{Y}_L m_a \hat{\bar{m}}^a + (-1)^L B^{(L-1)-}{}_{-1}\hat{Y}_L m_a \hat{m}^a}{m_a \hat{l}^a}\right] + \frac{1}{r^2}[\;] + \cdots + \frac{1}{r^L}[\;] + \frac{\gamma^1}{r} + \cdots + \frac{\gamma^L}{r^L} . \tag{5.26}$$

In order to keep F well behaved as $r \to \infty$ we restrict the u behavior in $A(u, \zeta, \bar{\zeta}) = b(u)_1 Y_L$. Specifically, we assume that in the neighborhood of $u = \pm\infty$,

$$b(u) = c + O(1/u^{L+1}) ,$$

where c is a constant which in general would be different in the two limits. This condition implies that

$$B(u) = \sum_0^L c_i u^i + O(u^{-1}), \quad c_L = c .$$

An important observation is to now notice that if (5.26) is used in order to generate higher terms in (5.3), *only* $F = -\eth^{-1}A$ will have an r^{-1} term; the others must begin with higher powers of r^{-1}. Thus if we concern ourselves only with asymptotic behavior ($u=$const, $r\to\infty$ for future infinity and $u+r=$const, $r\to\infty$ for past infinity) we need consider only the first term in (5.26) and an appropriate choice of γ^1.

We are now in a position to prove our original contention, namely, that $A(l, \zeta, \bar{\zeta})$ constitutes, for a wide class of solutions, the pure radiation data and furthermore this data could be given on either past or future null infinity.

It is most useful to calculate the gauge fields and potentials in the null polar coordinate and tetrad system of (2.12) (see Appendices A1, A2, and A3). In principle, it is a straightforward task to put $F(\zeta, \bar{\zeta}, n, \bar{n}, u, r)$ from (5.26), into (4.1) and find the γ_a and then construct the field. It is, however, a very nasty calculation and it turns out to be much simpler to do the calculation at the positive pole, i.e., when $\zeta = n$, $\bar{\zeta} = \bar{n}$. Great care must be exerted to be certain that all derivatives are taken before the pole values are inserted. As there are no new principles involved we will just state the results of the calculation. We have [remembering the tetrad comes from (2.12)]

$$\gamma_{00'} = \gamma_a \hat{l}^a = 0 , \tag{5.27a}$$

$$\gamma_{01'} = \gamma_a \hat{m}^a = -\frac{B^{(L)}(u)_1 \hat{Y}_L}{r} = \frac{A(u,\eta,\bar{\eta})}{r} , \tag{5.27b}$$

$$\gamma_{10'} = \gamma_a \hat{\bar{m}}^a = (-1)^L \frac{B(u+r)}{r}\, {}_{-1}\hat{Y}_L + O(r^{-2}) , \tag{5.27c}$$

$$\gamma_{11'} = \gamma_a \hat{n}^a = O(r^{-1}) . \tag{5.27d}$$

Equations (5.27a) and (5.27b) are exact and are consequences of the gauge conditions. (5.27d) is valid in the neighborhood of both past and future infinity. (5.27c) is valid in the neighborhood of past null infinity, i.e., $u+r=$const, $r\to\infty$; in the future direction it vanishes faster. Note further that (5.27c) can be written [from (5.18c)] as

$$\gamma_{10'} = -\frac{\eta}{\bar{\eta}}\frac{B_{(L)}(u+r)}{r}\,{}_1 Y_{\bar{L}} + O(r^{-2})$$
$$= \frac{A^P(u+r,\eta,\bar{\eta})}{r} + O(r^{-2}) . \tag{5.28}$$

It is now simple to calculate the radiation fields in the future and past null directions from (A3).

In the future null direction

$$\tilde{\chi}_0 = \tilde{\chi}_1 = O(r^{-2}) ,$$
$$\chi_2 = -\frac{\dot{A}(u,\eta,\bar{\eta})}{r} + O(r^{-2}) , \tag{5.29}$$

while in the past

$$\tilde{\chi}_2 = \tilde{\chi}_1 = O(r^{-2}),$$
$$\tilde{\chi}_0 = \frac{\dot{A}^P(u+r,\eta,\tilde{\eta})}{r} + O(r^{-2}). \tag{5.30}$$

We see that $A(u,\eta,\tilde{\eta})$ obviously constitutes the radiation data and that the data in the future and past are essentially the same (identifying antipodal points, $\eta \to -1/\tilde{\eta}$, $\tilde{\eta} \to -1/\eta$).

We emphasize that this result is true only for A's of the form

$$A(u,\eta,\tilde{\eta}) = \sum b_L(u)\,{}_1Y_L(\eta,\tilde{\eta}), \tag{5.31}$$

with the asymptotic behavior at $u \to \pm\infty$

$$b_L = O(u^{-L-1}) + \text{const}. \tag{5.32}$$

Although we have not verified it it seems overwhelmingly likely that from (5.31) and (5.32), (5.29) and (5.30) can be strengthened to the peeling theorem for gauge fields, namely,

$$\tilde{\chi}_0 = O(r^{-3}),$$
$$\tilde{\chi}_1 = O(r^{-2}),$$
$$\tilde{\chi}_2 = -\dot{A}/r + O(r^{-2}),$$

for the future and for the past

$$\tilde{\chi}_2 = O(r^{-3}),$$
$$\tilde{\chi}_1 = O(r^{-2}),$$
$$\tilde{\chi}_0 = \dot{A}^P/r + O(r^{-1}).$$

If we had restricted ourselves in this section to Abelian gauge theories (self-dual Maxwell fields), then Eq. (3.1) could be written as

$$\eth F = -A, \quad F = \ln G$$

and (5.4) would yield the complete solution. The fields obtained in this case would have been the familiar mixture of half-retarded minus half-advanced Maxwell fields.

The results of this section lead to the following interpreation of the self- (and anti-self-) dual solutions of the Yang-Mills equations.

Consider a *real*, "asymptotically-flat" Yang-Mills field with sources of compact support. In the neighborhood of future null infinity the fields are to have the form

$$\chi_0 = O(r^{-3}), \quad \tilde{\chi}_0 = O(r^{-3}),$$
$$\chi_1 = O(r^{-2}), \quad \tilde{\chi}_1 = O(r^{-2}), \tag{5.33}$$
$$\chi_2 = \frac{\chi_2^0(u,\eta,\tilde{\eta})}{r} + O(r^{-2}), \quad \tilde{\chi}_2 = \frac{\tilde{\chi}_2^0}{r} + O(r^{-2}).$$

In the Abelian (linear) case, one could construct new solutions from (5.33) by taking both the self-dual and anti-self-dual parts of the mixture of (5.33) minus the advanced version of (5.33). Although one clearly cannot do the same for the non-Abelian case, we can nevertheless do something quite similar.

If we write

$$\tilde{\chi}_2^0 = \dot{A}(u,\eta,\tilde{\eta})$$

and

$$\chi_2^0 = \dot{\bar{A}}(u,\eta,\tilde{\eta}),$$

then via (3.1) and its conjugate

$$\bar{\eth}\tilde{G} = -\tilde{G}\tilde{A}$$

we can produce self-dual and anti-self-dual solutions whose sum has exactly the same radiation field as the real field we began with. In this manner we can decompose the original field into two "pure radiation solutions" (the self-dual and anti-self-dual solutions) and a "longitudinal field" defined as the difference of the real field and the sum of the pure radiation solutions.

The basic idea now is to try to interpret the self-dual and anti-self-dual solutions as the quanta of the Yang-Mills fields.

VI. DISCUSSION

We have shown that Yang-Mills fields, obtained from $\eth G = -GA$ with A satisfying appropriate asymptotic conditions, have some remarkably attractive porperties: (a) the existence of a Hertz potential and a linear relationship between field and potential, (b) the solitonlike behavior with the future and past asymptotic fields being identical, and (c) a linear structure on the space $\mathfrak{F}$ of these fields. If in addition one could define a scalar product on $\mathfrak{F}$ and hence give $\mathfrak{F}$ a Hilbert-space structure one could construct a Fock space by taking formal sums and (symmetrized) products of elements of $\mathfrak{F}$; a one "Yang-Mills" particle state consisting of an element of $\mathfrak{F}$, a two "Yang-Mills" particle state consisting of a symmetrized pair from $\mathfrak{F}$, etc. Creation operators are essentially defined by multiplying elements of the Fock space by elements of $\mathfrak{F}$ and annihilation operators by taking the (Hilbert-space) scalar product of elements of the Fock space with elements of $\mathfrak{F}$. An enlarged Fock space could then be constructed as the sum of self-dual and anti-self-dual Fock spaces, the two parts being orthogonal. It seems reasonable to conjecture on the basis of the non-scattering property of the classical solutions that the quantum-mechanical S matrix would be trivial for the self-dual (or anti-self-dual) fields though certainly not for the real fields.

The point of view we have in mind is to consider this Fock space as the analog of the one constructed from the free Maxwell field. The idea would then be to learn how, from the real Yang-Mills theory, to construct an interaction between the self-dual and anti-self-dual solutions so that an S-matrix theory could be formulated. In this context we point out that any asymptotically flat real Yang-Mills field with no sources can be decomposed uniquely in one of two ways:

$$F^{ab} = F^{ab}_{(+)p} + F^{ab}_{(-)p} + F^{ab}_{(I)p} \qquad (6.1a)$$

and

$$F^{ab} = F^{ab}_{(+)f} + F^{ab}_{(-)f} + F^{ab}_{(I)f}, \qquad (6.1b)$$

where the $F^{ab}_{(+)p}$, $(F^{ab}_{(-)p})$ and $F^{ab}_{(+)f}$, $(F^{ab}_{(-)f})$ denote the self-dual (anti-self-dual) *solution* which agrees (to r^{-1}) with the self-dual (anti-self-dual) part of the real solution at past (p) null infinity and future (f) null infinity. The interaction $F^{ab}_{(I)}$ denotes the difference between the real solution and the sum of $F^{ab}_{(+)}$ and $F^{ab}_{(-)}$ and is not a solution. It can, however, be thought of in some sense as being the longitudinal part of F^{ab} while $F^{ab}_{(+)}$ and $F^{ab}_{(-)}$ can be thought of as the "pure" radiation parts of F^{ab}.

Equations (6.1a) and (6.1b) describe the classical transition from the pure radiation "in" state to pure radiation "out" state. We intend in the future to study the details of this transition.

As a final comment we remark that many of the ideas discussed here have analogs in gravitational theory. There is a gravitational equation corresponding to (3.1) which produces self-dual (complex) solutions to the Einstein equations. It has recently been proved[13] that their in and out states are the same, i.e., they have a solitonlike behavior similar to our self-dual Yang-Mills fields. The idea would be to find a Hilbert-space structure for these fields and hence construct a nonlinear graviton Fock space. This idea originated with R. Penrose.[14]

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APPENDIX

For any Yang-Mills field it is always possible to chose a gauge such that one component of the connection vanishes.[6] We will always choose

$$\gamma_{00'} = \gamma_a l^a = 0\,. \qquad (A1)$$

(a) With this condition we write out the self-dual Yang-Mills equations using the null polar coordinates and tetrad (2.12),

$$\frac{\partial}{\partial r} + \frac{1}{r}\,\gamma_{01'} = 0\,, \qquad (A2a)$$

$$\frac{\partial}{\partial r}\gamma_{11'} - \frac{1}{r}(\bar{\eth}\gamma_{01'} - \eth\gamma_{10'}) - [\gamma_{10'}, \gamma_{01'}] = 0\,, \qquad (A2b)$$

$$\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r} - \frac{1}{r}\right)\gamma_{10'} + \frac{1}{r}\bar{\eth}\gamma_{11'} - [\gamma_{11'}, \gamma_{10'}] = 0\,, \qquad (A2c)$$

and

$$\tilde{\chi}_0 = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\gamma_{10'}\,, \qquad (A3a)$$

$$2\tilde{\chi}_1 = \frac{\partial}{\partial r}\gamma_{11'} - \frac{1}{r}(\eth\gamma_{10'} - \bar{\eth}\gamma_{01'}) - [\gamma_{01'}, \gamma_{10'}]\,, \qquad (A3b)$$

$$\tilde{\chi}_2 = \left(\frac{\partial}{\partial r} - \frac{\partial}{\partial u} + \frac{1}{r}\right)\gamma_{01'} - \frac{1}{r}\eth\gamma_{11'} + [\gamma_{11'}, \gamma_{01'}]\,. \qquad (A3c)$$

$\eth$ and $\bar{\eth}$ are defined as before but now using η and $\bar{\eta}$. $\gamma_{11'}$, $\gamma_{01'}$, and $\gamma_{10'}$ are $s = 0, 1, -1$. Actually the equations can be greatly simplified by noting that the remaining gauge freedom, keeping $\gamma_{00'} = 0$, allows from (A2a) the additional gauge condition

$$\gamma_{01'} = \gamma_a m^a = 0\,. \qquad (A4)$$

(b) Also using (A1) we write out the self-dual equations for the null-plane coordinates and associated tetrad (2.13),

$$\frac{\partial}{\partial v}\gamma_{01'} = 0\,, \qquad (A5a)$$

$$\frac{\partial}{\partial v}\gamma_{11'} - \frac{\partial}{\partial w}\gamma_{10'} + \frac{\partial}{\partial \bar{w}}\gamma_{01'} - [\gamma_{10'}, \gamma_{01'}] = 0\,, \qquad (A5b)$$

$$\frac{\partial}{\partial \bar{w}}\gamma_{11'} - \frac{\partial}{\partial u}\gamma_{10'} + [\gamma_{11'}, \gamma_{10'}] = 0\,, \qquad (A5c)$$

and

$$\tilde{\chi}_0 = \frac{\partial}{\partial v}\gamma_{10'}\,, \qquad (A6a)$$

$$2\tilde{\chi}_1 = \frac{\partial}{\partial v}\gamma_{11'} + \frac{\partial}{\partial w}\gamma_{10'} - \frac{\partial}{\partial \bar{w}}\gamma_{01'} - [\gamma_{01}, \gamma_{10'}]\,, \qquad (A6b)$$

$$\tilde{\chi}_2 = \frac{\partial}{\partial w}\gamma_{11'} - \frac{\partial}{\partial u}\gamma_{01'} + [\gamma_{11'}, \gamma_{01'}]\,. \qquad (A6c)$$

Again from the remaining gauge freedom and (A5a) we can set

$$\gamma_{01'} = 0 \qquad (A7)$$

with a large simplification of the remaining equations.

[1] M. F. Atiyah and R. Ward, Commun. Math. Phys. 55, 117 (1977).

[2] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld, and Yu. I. Manin, Phys. Lett. 65A, 185 (1978).

[3] A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. 59B, 85 (1975).

[4] A. I. Akhiezer and V. B. Berestetsky, *Quantum Electrodynamics* (Interscience, New York, 1965).

[5] R. Geroch, Lecture Notes on Quantum Field Theory, University of Chicago (unpublished).

[6] E. T. Newman, Phys. Rev. D 18, 2901 (1978).

[7] E. T. Newman and R. Penrose, J. Math. Phys. 7, 863 (1966).

[8] G. Sparling, University of Pittsburgh report (unpublished).

[9] J. Goldberg, Phys. Rev. D 20, 1909 (1979).

[10] A. R. Exton, E. T. Newman, and R. Penrose, J. Math. Phys. 10, 1566 (1969).

[11] Actually the I can be replaced by a $GL(n, C)$ function of the x^a, it being the gauge freedom. The use of I was a gauge choice made for simplicity.

[12] Actually the zeros occur on the two lines in the $(\zeta, \tilde{\zeta})$ plane $\tilde{\zeta} = \tilde{\eta}$ and $\tilde{\zeta} = 1/\eta$. However, for the purposes of regularity of $F(x^a, \zeta, \tilde{\zeta})$ on the real S^2 $(\zeta; \tilde{\zeta} = \bar{\zeta})$ we restrict ourselves to the two points on the lines given by (5.17a) and (5.17b).

[13] M. Ludvigsen, E. T. Newman, and K. P. Tod, University of Pittsburgh report (unpublished).

[14] R. Penrose, Gen. Relativ. Gravit. 7, 31 (1976).