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Instabilities in interacting quantum field theories in non-Minkowskian spacetimes

L. H. Ford*

Institute of Field Physics, Department of Physics and Astronomy, University of North Carolina, Chapel Hill, North Carolina 27514

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Instabilities produced by one-loop quantum effects in quantum field theories in curved spacetime or in flat spacetime with boundaries are discussed. A criterion for the existence of instability is proposed and applied to various models.

I. INTRODUCTION

The global structure of spacetime can be of considerable importance in governing the behavior of quantum field theories. For example, nontrivial spacetime topology or the imposition of boundary conditions will in general cause the energy of the vacuum state to be nonzero (the Casimir effect). One-loop quantum processes in interacting field theories are also altered by global considerations.[1-8] Furthermore, the spacetime topology can determine the number of field configurations which may exist; a nonsimply connected spacetime admits twisted scalar and spinor fields in addition to the usual (untwisted) fields which exist in a simply connected spacetime.[9,10]

Flat spacetime with periodic identification of points in one spatial direction ($S^1 \times R^3$) exhibits all of the above features. The periodicity alters the self-energy of a self-interacting scalar field and can cause a massless field to acquire a mass which depends upon the periodicity length.[1,4] Similarly, the coupling of photons to the vacuum polarization produced by electrons in $S^1 \times R^3$ can change the propagation of electromagnetic waves and cause certain modes to propagate as though photons were massive.[8] The associated mass can be tachyonic (imaginary), and in Ref. 8 it was suggested that violations of causality might result. However, as is discussed in Sec. II, a tachyonic mass in a wave equation does not result in superluminal propagation. Thus no causality violation need occur provided that the mass is associated with waves rather than particles. It may, on the other hand, lead to an instability if there are modes for which $\mathrm{Im}\,\omega > 0$. In a classical theory, the existence of exponentially growing modes represents instability against small perturbations. In a quantum theory, such modes are also a sign of instability, that the postulated vacuum state is not in fact the physical vacuum. The purpose of this paper is to investigate various models to search for instabilities induced by one-loop quantum corrections.

In the context of Minkowski-space scalar field theories, it was argued by Coleman and Weinberg[11] that such quantum corrections could lead to instabilities which are interpreted as producing symmetry breaking. In the context of field theories in more general spacetimes, these effects have been considered recently by Shore,[12] who discusses massless scalar electrodynamics in de Sitter spacetime, and by Toms[6] who discusses interacting scalar fields in $S^1 \times R^3$ and in the Einstein universe.

In Sec. II the effects of one-loop quantum corrections upon interacting scalar fields are discussed. A method is proposed to test for the stability of such theories. This method has the advantage that it can be applied to models in which an effective potential does not exist. In Sec. III this method is applied to scalar fields in $S^1 \times R^3$. Scalar fields in the Einstein universe are discussed in Sec. IV. Interacting scalar fields which are confined between two parallel plates are investigated in Sec. V. In Sec. VI quantum electrodynamics in $S^1 \times R^3$ is considered.

II. STABILITY OF INTERACTING SCALAR FIELDS

Let us first consider a pair of real scalar fields ϕ and ψ in an arbitrary spacetime which are described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\alpha \phi\, \partial^\alpha \phi - m^2 \phi^2) + \tfrac{1}{2}(\partial_\alpha \psi\, \partial^\alpha \psi - M^2 \psi^2) - \tfrac{1}{2} g\, \phi^2 \psi^2 . \tag{1}$$

The associated equations of motion are

$$\Box \phi + m^2 \phi + g\, \psi^2 \phi = 0 \tag{2}$$

and

$$\Box \psi + M^2 \psi + g\, \phi^2 \psi = 0 , \tag{3}$$

where $\Box = \nabla_\mu \nabla^\mu$, the wave operator in this spacetime.

Suppose that ϕ is quantized, but that ψ is a classical field. If $\langle \phi^2 \rangle$ denotes the (finite) expectation value in an appropriate quantum state, then the propgation of the ψ field is described by

$$\Box\psi + M^2\psi + g\langle\phi^2\rangle\psi = 0. \qquad (4)$$

The effect of the coupling to the ϕ field is equivalent to the introduction of a space- and time-dependent mass. It is possible to have $\langle\phi^2\rangle < 0$, so if

$$M^2 + g\langle\phi^2\rangle < 0, \qquad (5)$$

this "mass" is tachyonic.

A tachyonic mass does not, in the present context, imply violation of causality. The wave equation

$$\Box\psi + \mu^2\psi = 0 \qquad (6)$$

possesses the same characteristics, regardless of the sign of μ^2; these characteristics are just the light cones in the spacetime in question. A disturbance propagates causality on or within the forward light cone. This was apparently first demonstrated in Minkowski spacetime by Ehrenfest[13] and is discussed by several recent authors.[14-16] A corresponding theorem also holds in curved spacetime and for μ^2 nonconstant.[17,18] This result is perhaps surprising because of the fact that a classical particle with a tachyonic mass clearly travels a spacelike world line and can be used to violate causality.[19] This suggests that field theories in which tachyonic masses arise will not have a particle interpretation. A tachyonic mass may also lead to an instability of the classical field theory if Eqs. (4) or (6) possess exponentially growing solutions. In this paper, our attention will be restricted to static spacetimes, so that the normal modes have exponential time dependence.

This instability at the classical level suggests that the quantum theory will also be unstable. Hence the quantized ψ field, which is stable at the tree-graph level, becomes unstable as a result of one-loop quantum corrections. The existence of an instability does not, by itself, determine what the correct, stable theory should be. One familiar example of the effects of instability is the Goldstone model[20] of spontaneous symmetry breaking; the theory in which the vacuum expectation value of the field is zero is unstable, but a nonzero vacuum expectation value can lead to a stable theory. In more complicated situations, the outcome is not so clear. For example, if space is not homogeneous, there is no reason to assume that the vacuum expectation value will be a constant. Even if the space is homogeneous, the symmetry-breaking solution may be very nontrivial. An example is given by Avis and Isham[21] who treat the analog of the Goldstone model for a twisted scalar field in $S^1 \times R$. The models of Goldstone and of Avis and Isham differ from those considered here in that the former have a tachyonic mass term in the Lagrangian at the classical level.

Another field theory which is of interest here is the $\lambda\phi^4$ model

$$\mathcal{L} = \tfrac{1}{2}(\partial_\alpha\phi\,\partial^\alpha\phi - m^2\phi^2) - \tfrac{1}{12}\lambda\phi^4 \qquad (7)$$

for which the equation of motion is

$$\Box\phi + m^2\phi + \tfrac{1}{3}\lambda\phi^3 = 0. \qquad (8)$$

The stability of this model is often discussed in Minkowski-space field theories by means of the effective potential.

For our purposes it will be convenient to develop a different approach. We will work to first order in λ, corresponding to one-loop processes. Let ϕ_0 be a free-field operator

$$\phi_0 = \sum_j (a_j F_j + a_j^\dagger F_j^*), \qquad (9)$$

where the $\{F_j\}$ are a complete set of positive-norm solutions of

$$\Box F_j + m^2 F_j = 0. \qquad (10)$$

Let $|z\rangle$ be a coherent state[22] in which only the mode j is excited:

$$a_j|z\rangle = z|z\rangle, \qquad (11)$$

where z is some complex number and $\langle z|z\rangle = 1$. It may be shown that

$$\langle z|\phi_0^3|z\rangle = (\langle z|\phi_0|z\rangle)^3 + 3\langle z|\phi_0|z\rangle\langle\phi_0^2\rangle_0, \qquad (12)$$

where $\langle\,\rangle_0$ denotes the vacuum expectation value. If

$$\Phi = \langle z|\phi|z\rangle, \qquad (13)$$

then

$$\langle z|\phi^3|z\rangle = \langle z|\phi_0^3|z\rangle + O(\lambda)$$
$$= \Phi^3 + 3\langle\phi_0^2\rangle_0\Phi + O(\lambda). \qquad (14)$$

We require that Φ be sufficiently small that the Φ^3 term on the right-hand side of Eq. (14) may be neglected. Then we have, to first order in λ,

$$\Box\Phi + m^2\Phi + \lambda\langle\phi_0^2\rangle_0\Phi = 0. \qquad (15)$$

The criterion of stability which will be adopted here is the requirement that this equation has no exponentially growing solutions. This is a necessary, but not sufficient, requirement for absolute stability. In cases where an effective potential may be defined, it is equivalent to the requirement that the vacuum be a local minimum of the effective potential, but it says nothing about whether the vacuum is a global minimum. As will be discussed below, it is possible to apply this criterion in situations where the effective potential is not well defined. The motivation for the use of coherent states is that such states represent classical field excitations in the limit that $|z| \to \infty$. Thus the stability criterion used here is a generalization of classical stability which allows for the inclusion of one-loop quantum corrections.

A more general model is that given by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\alpha \phi \partial^\alpha \phi - m^2\phi^2) + \tfrac{1}{2}(\partial_\alpha \psi \partial^\alpha \psi - M^2\psi^2) - \tfrac{1}{12}\lambda_1 \phi^4 - \tfrac{1}{12}\lambda_2 \psi^4 - \tfrac{1}{2} g \phi^2 \psi^2 . \tag{16}$$

If

$$\psi = \langle z | \psi | z \rangle , \tag{17}$$

then to first order in λ_1, λ_2, g, and ψ, we have

$$\Box \psi + M^2 \psi + g \langle \phi_0^2 \rangle_0 \psi + \lambda_2 \langle \psi_0^2 \rangle_0 \psi = 0 , \tag{18}$$

where ψ_0 is a free field operator which satisfies

$$\Box \psi_0 + M^2 \psi_0 = 0 . \tag{19}$$

A similar equation holds for Φ:

$$\Box \Phi + m^2 \Phi + g \langle \psi_0^2 \rangle_0 \Phi + \lambda_1 \langle \phi_0^2 \rangle_0 \Phi = 0 . \tag{20}$$

We consider the theory to be stable, in the restricted sense discussed above, if neither Eq. (18) nor Eq. (20) possesses any unstable solutions.

The formal expectation values of ϕ_0^2 and ψ_0^2 are of course infinite, but the singular parts may be absorbed by renormalization of M^2 and m^2. We will henceforth assume that this renormalization has been performed, so that M and m are the renormalized masses and $\langle \phi_0^2 \rangle_0$ and $\langle \psi_0^2 \rangle_0$ are the finite expectation values. Similarly, we regard g, λ_1, and λ_2 as the renormalized coupling constants, although to the order to which we work here no coupling-constant renormalization is required.

Because we will consider only one-loop processes, the infinities which arise are identical to those in Minkowski-space field theories. In two or more loop diagrams this is not the case; there are nonlocal infinities which arise from the combination of the singular part of one loop with the finite part of another loop. However, when all diagrams which contribute to a given process are summed, such nonlocal infinities have been found to cancel in all models for which explicit calculations have been performed.[4,6,23-27] It has been argued by Banach[28] that this cancellation is a general feature of field theories in a locally flat, but topologically nontrivial spacetime, provided that the theory is renormalizable in Minkowski spacetime. However, a proof for the case of nonflat spacetime has not yet been given.

III. SCALAR FIELDS IN $S^1 \times R^3$

Let us first consider scalar fields in $S^1 \times \boldsymbol{R}^3$, locally flat spacetime with periodic identification of points in one spatial direction. Take the z axis in a rectangular coordinate system to be the axis of periodicity and L to be the periodicity length, so that the points (t, x, y, z) and $(t, x, y, z+L)$ are identified.

This spacetime admits two types of scalar fields. The untwisted scalar fields satisfy periodic boundary conditions

$$\phi(z) = \phi(z+L) \tag{21}$$

and are cross sections of a product fiber bundle over $S^1 \times \boldsymbol{R}^3$. The twisted scalar fields satisfy antiperiodic boundary conditions

$$\tilde{\phi}(z) = -\tilde{\phi}(z+L) \tag{22}$$

and are cross sections of a nonproduct bundle over $S^1 \times R^3$ which has the topology of a Möbius strip.

Let us first consider the case of a pair of untwisted fields which are described by the Lagrangian Eq. (16). In Appendix A it is shown that

$$\langle \phi_0^2 \rangle_0 = 2L^{-2} f_{-1/2}(mL/2\pi) , \tag{23}$$

where

$$f_\lambda(\xi) = \int_\xi^\infty \frac{(u^2 - \xi^2)^{-\lambda}\, du}{e^{2\pi u} - 1} . \tag{24}$$

In the case $m=0$, we have

$$\langle \phi_0^2 \rangle_0 |_{m=0} = (12L^2)^{-1} . \tag{25}$$

Because $\langle \phi_0^2 \rangle_0 \geq 0$ for all m and L, the effect of the radiative corrections in this case is to enhance rather than to destroy the stability of the theory. That is, for fixed L, the terms in Eqs. (18) and (20) involving $\langle \phi_0^2 \rangle_0$ and $\langle \psi_0^2 \rangle_0$ act as nontachyonic mass terms, and there are no unstable modes. We assume that g, λ, and λ_2 are nonnegative; otherwise the theory would be instable at the classical level.

Let us now consider the case of two twisted scalar fields $\tilde{\phi}$ and $\tilde{\psi}$, which are described by the Lagrangian Eq. (16), with ϕ replaced by $\tilde{\phi}$ and ψ replaced by $\tilde{\psi}$. From Appendix A we have that

$$\langle \tilde{\phi}_0^2 \rangle_0 = 2L^{-2}[\tfrac{1}{2} f_{-1/2}(mL\pi^{-1}) - f_{-1/2}(mL/2\pi)] \tag{26}$$

and similarly for $\langle \tilde{\psi}_0^2 \rangle_0$. Furthermore, $\langle \tilde{\phi}_0^2 \rangle_0 < 0$ for all m and L, and for fixed L the minimum value occurs at $m=0$. The first statement follows from the fact that

$$\tfrac{1}{2}f_{-1/2}(\xi)-f_{-1/2}\xi)=-\tfrac{1}{4}\xi^2\int_1^\infty \frac{(t^2-1)^{1/2}}{e^{\pi\xi t}-1}$$

and the second may be verified by numerical calculation. Because $f_{-1/2}(0)=1/24$, this minimum value is

$$\langle\tilde{\phi}_0^2\rangle_0\,|_{m=0}=-(24L^2)^{-1}. \qquad (27)$$

If $M=m=0$, then the analog of Eq. (20) for $\tilde{\phi}=\langle z|\,\tilde{\phi}\,|z\rangle$ is

$$\Box\tilde{\phi}-(24\,L^2)^{-1}(g+\lambda_1)\tilde{\Phi}=0. \qquad (28)$$

The solutions of this equation are plane waves of frequency

$$\omega_{\vec{k}}=[\vec{k}^2-(24\,L^2)^{-1}(g+\lambda_1)]^{1/2}, \qquad (29)$$

where $\vec{k}=(k_x,k_y,k_z)$, $-\infty<k_x,\ k_y<\infty$, and

$$k_z=\pi L^{-1}(2n+1),\quad n=0,\ \pm1,\ \pm2,\ \ldots. \qquad (30)$$

The allowed values of k_z follow from the antiperiodicity of $\tilde{\phi}$. The lowest eigenfrequency is

$$\omega_0=(2\sqrt{6}\,L)^{-1}(24\pi^2-g-\lambda_1)^{1/2}. \qquad (31)$$

Thus if

$$g+\lambda_1\leq 24\,\pi^2, \qquad (32)$$

there are no complex eigenfrequencies and no unstable modes. Similarly, Eq. (18) for $\Psi=\langle z\,|\psi|\,z\rangle$ will have no unstable solutions provided that

$$g+\lambda_2\leq 24\,\pi^2. \qquad (33)$$

However, Eqs. (32) and (33) are necessarily satisfied for values of g, λ_1, and λ_2 such that first-order perturbation theory is applicable, i.e., g, λ_1, and $\lambda_2\lesssim 1$. Furthermore, if the masses M and m are nonzero the effect is to enhance the stability of the theory. Hence we conclude that this model is stable, in spite of the fact that the effect of radiative corrections in this case is to produce a tachyonic mass.

Another model which may be constructed in $S^1\times R^3$ is that of a twisted scalar field $\tilde{\psi}$ coupled to an untwisted field ϕ. Take the Lagrangian to be Eq. (16) with ψ replaced by $\tilde{\psi}$. Thus Eq. (20) becomes

$$\Box\Phi+m^2\,\Phi+g\langle\tilde{\psi}_0^2\rangle_0\,\Phi+\lambda_1\langle\phi_0^2\rangle_0\Phi=0. \qquad (34)$$

The terms in Eq. (34) proportional to λ_1 and to g are due to the self-energy processes shown in Figs. 1(a) and 1(b), respectively. The corresponding eigenfrequencies are

$$\omega_{\vec{k}}=(\vec{k}^2+m^2+g\langle\tilde{\psi}_0^2\rangle_0+\lambda_1\langle\phi_0^2\rangle_0)^{1/2}, \qquad (35)$$

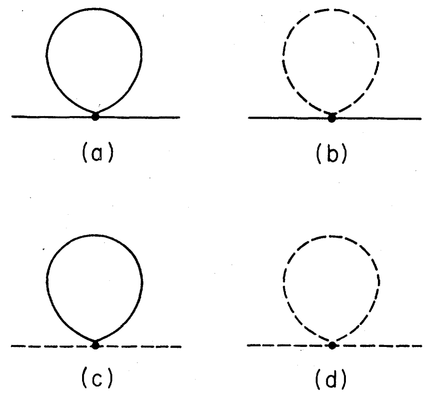

FIG. 1. Self-energy correction of an untwisted scalar field coupled to (a) itself, (b) a twisted scalar field, and of a twisted scalar field coupled to (c) an untwisted scalar field, (d) itself.

where $k_z=2\pi n L^{-1}$, $n=0,\ \pm1,\ \pm2,\ \ldots,$ and $-\infty<k_x,\ k_y<\infty$. Here $\langle\phi_0^2\rangle_0$ is given by Eq. (23) and

$$\langle\tilde{\psi}_0^2\rangle_0=2\,L^{-2}[\tfrac{1}{2}f_{-1/2}(ML/\pi)-f_{-1/2}(ML/2\pi)]. \qquad (36)$$

If

$$m^2+g\langle\tilde{\psi}_0^2\rangle_0+\lambda_1\langle\phi_0^2\rangle_0<0, \qquad (37)$$

then there will be unstable modes. In particular, if $M=m=0$, then the theory is unstable if

$$g>2\,\lambda_1. \qquad (38)$$

This agrees with the result of Toms.[6]

This instability arises in the untwisted field as a result of its interaction with the twisted scalar field. The twisted field $\tilde{\psi}$ does not exhibit any instabilities. The equation for $\tilde{\Psi}$, which contains contributions from the processes shown in Figs. 1(c) and 1(d), has no unstable modes provided that

$$\pi^2L^{-2}\geq M^2+g\langle\phi_0^2\rangle_0+\lambda_2\langle\tilde{\psi}_0^2\rangle_0. \qquad (39)$$

This is always satisfied if g and $\lambda_2\leq 1$. The reason for this asymmetry lies in the fact that the free untwisted field may have eigenfrequencies which are arbitrarily close to zero and hence may be forced to become imaginary as a result of a small perturbation. On the other hand, the free twisted field has a minimum eigenfrequency which is greater than zero even in the massless limit.

IV. SCALAR FIELD IN THE EINSTEIN UNIVERSE

In this section we consider interacting scalar fields in the Einstein universe, a space of constant curvature and topology $R\times S^3$. There is only one type of scalar field possible in this spacetime, the untwisted scalar fields. Let us consider two such fields described by the Lagrangian

$$\mathfrak{L}=\tfrac{1}{2}(\partial_\alpha\phi\partial^\alpha\phi-m^2\phi^2-\xi_1R\phi^2)+\tfrac{1}{2}(\partial_\alpha\psi\partial^\alpha\psi-M^2\psi^2-\xi_2R\psi^2)-\tfrac{1}{12}\lambda_1\phi^4-\tfrac{1}{12}\lambda_2\psi^4-\tfrac{1}{2}g\phi^2\psi^2\,, \quad (40)$$

where R is the scalar curvature and ξ_1 and ξ_2 are constants. The analogs of Eqs. (18) and (20) are now, respectively,

$$\Box\Psi+(M^2+\xi_2R+g\langle\phi_0^2\rangle_0+\lambda_2\langle\psi_0^2\rangle_0)\Psi=0 \quad (41)$$

and

$$\Box\Phi+(m^2+\xi_1R+g\langle\psi_0^2\rangle_0+\lambda_1\langle\phi_0^2\rangle_0)\Phi=0\,. \quad (42)$$

In Appendix B it is shown that

$$\langle\phi_0^2\rangle_0=(32\pi^{3/2})^{-1}[\mu^2a^{-2}\ln(l^2\mu^2a^{-2})-m^2\ln(l^2m^2)]-(2\pi^2a^2)^{-1}[f_{-1/2}(\mu)+\mu^2f_{1/2}(\mu)]\,, \quad (43)$$

where

$$\mu^2=a^2[m^2+(\xi_1-\tfrac{1}{6})R]\,, \quad (44)$$

l is an arbitrary renormalization length, and a is the radius of the universe. The expectation value $\langle\psi_0^2\rangle_0$ may be obtained from Eq. (43) by the substitutions $m\to M$, $\xi_1\to\xi_2$, and $l\to l'$, where l' is another renormalization length.

The eigenfrequencies of the solutions of Eq. (41) are

$$\omega_n=[n^2a^{-2}+M^2+(\xi_2-\tfrac{1}{6})R+g\langle\phi_0^2\rangle_0+\lambda_2\langle\psi_0^2\rangle_0]^{1/2}\,, \quad (45)$$

where $n=1,\ 2,\ \ldots$. In the Einstein universe

$$R=6/a^2\,. \quad (46)$$

Let us assume that

$$\xi_2\geq-\tfrac{1}{6}M^2a^2\,. \quad (47)$$

Otherwise there would be unstable modes in the classical theory. In the particular case that $\xi_1=\xi_2=\frac{1}{6}$, we have that

$$\langle\phi_0^2\rangle_0=-(2\pi^2a^2)^{-1}[f_{-1/2}(ma)+m^2a^2f_{1/2}(ma)] \quad (48)$$

and

$$\langle\psi_0^2\rangle_0=-(2\pi^2a^2)^{-1}[f_{-1/2}(Ma)+M^2a^2f_{1/2}(Ma)]\,, \quad (49)$$

so the renormalization lengths l and l' do not appear. As in the case of $S^1\times R^3$, these expectation values attain their minimum value in the massless limit

$$\langle\phi_0^2\rangle_0\geq-(2\pi^2a^2)^{-1}f_{-1/2}(0)=-\frac{1}{48\pi^2a^2}\,. \quad (50)$$

Consequently, if the theory is stable when $m=M=0$, it will be stable for all masses. The minimum eigenfrequency from Eq. (45) is now

$$\omega_0=a^{-1}[1-(g+\lambda_1)(48\pi^2)^{-1}]^{1/2}\,, \quad (51)$$

which is real if

$$g+\lambda_1\leq48\pi^2\,. \quad (52)$$

Thus within the domain of first-order perturbation theory, there are no instabilities if $\xi_1=\xi_2=\frac{1}{6}$. This agrees with the conclusion reached by Toms[6] by the effective potential method.

V. SCALAR FIELDS WHICH VANISH ON TWO PARALLEL PLATES

Consider scalar fields ϕ and ψ which are described by the Lagrangian Eq. (16) in flat spacetime and subject to the boundary conditions that they vanish on the planes defined by $z=0$ and $z=L$, i.e.,

$$\phi(z=0)=\phi(z=L)=\psi(z=0)=\psi(z=L)=0\,. \quad (53)$$

We will be concerned only with the field theory defined in the region $0\leq z\leq L$. If $m=M=0$, one may show that[2]

$$\langle\phi_0^2\rangle_0=\langle\psi_0^2\rangle_0=\frac{5+\cos2\pi zL^{-1}}{48L^2(\cos2\pi zL^{-1}-1)}\,. \quad (54)$$

These expectation values are negative at all points between the two plates and become singular at the boundaries:

$$\langle\phi_0^2\rangle=\langle\psi_0^2\rangle_0\sim-(16\pi^2\xi^2)^{-1}\,,\quad \xi\to0\,, \quad (55)$$

where $\xi=z$ or $L-z$.

The equation for Ψ in this case is Eq. (18) with $M=0$.[29] If we let

$$\Psi=f(z)\,e^{i(k_xx+k_yy)-i\omega t}\,, \quad (56)$$

this equation becomes

$$\frac{d^2f}{dz^2}+[\omega^2+k_x^2+k_y^2-(g+\lambda_2)\langle\phi_0^2\rangle_0]f=0\,. \quad (57)$$

Let $k_x=k_y=0$, as the minimum eigenfrequency and hence least stable mode is associated with these values. Then

$$\frac{d^2f}{dz^2}+[\omega^2-F(z)]f=0\,, \quad (58)$$

where

$$F=(g+\lambda_2)\langle\phi_0^2\rangle_0\,. \quad (59)$$

We may regard the term in F as small and apply first-order perturbation theory to find its effect upon the eigenfrequencies. The smallest eigenfrequency for the unperturbed ($F=0$) equation is

$$\omega_0 = \pi L^{-1} \tag{60}$$

associated with the mode

$$f_0 = \sin \pi z L^{-1} . \tag{61}$$

If we let ω_1^2 denote the first-order shift in ω^2 as a result of the perturbation, we have

$$\omega_1^2 = \int_0^L f_0^2 F(z)\, dz \left(\int_0^L f_0^2\, dz\right)^{-1} . \tag{62}$$

The first integral in Eq. (62) may be written as

$$(48 L^2)^{-1} \int_0^L \sin^2 \pi z L^{-1} \frac{5+\cos 2\pi z L^{-1}}{\cos 2\pi z L^{-1} - 1}\, dz$$

$$= -(96 L^2)^{-1} \int_0^L (5+\cos 2\pi z L^{-1})\, dz$$

$$= -5(96L)^{-1} . \tag{63}$$

Thus the lowest perturbed eigenfrequency is

$$\omega = (\omega_0^2 + \omega_1^2)^{1/2} = L^{-1}\left[\pi^2 - \tfrac{5}{48}(g+\lambda_2)\right]^{1/2} , \tag{64}$$

which is real for values of g and $\lambda_2 \leq 1$.

Although the massive theory has not been considered explicitly, one expects the presence of nonzero mass to enhance the stability. Hence we find that these boundary conditions do not lead to instabilities, in spite of the fact that $\langle \phi_0^2 \rangle_0 < 0$.

VI. QUANTUM ELECTRODYNAMICS IN $S^1 \times R^3$

In Ref. 8 the problem of formulating quantum electrodynamics in $S^1 \times R^3$ was considered. In this spacetime there are two types of spinors. In a particular representation, both types satisfy the usual flat-space Dirac equation; the untwisted spinors are characterized by periodic boundary conditions

$$\psi(z) = \psi(z) + L) \tag{65}$$

and the twisted spinors by antiperiodic boundary conditions

$$\tilde{\psi}(z) = -\tilde{\psi}(z+L) . \tag{66}$$

Photons may be coupled to either type of spinors.

In general, the effect of the one-loop vacuum polarization process of Fig. 2 is to cause certain photon modes to propagate with a nonzero effective mass. If the photons are coupled to twisted spinors, this mass is nontachyonic. However, if the photons are coupled to untwisted spinors, it may be tachyonic. In particular, photons propagating in a direction perpendicular to the direction of periodicity ($k_z = 0$) have an effective mass squared of

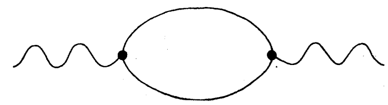

FIG. 2. One-loop vacuum polarization of photons coupled to a spinor field.

$$\mu^2 = -4e^2 L^{-2}\left[2f_{-1/2}(\alpha) + \alpha^2 f_{1/2}(\alpha)\right] \tag{67}$$

if the polarization vector is in the direction of periodicity $\hat{\epsilon} = \hat{z}$. Here

$$\alpha = mL/2\pi \tag{68}$$

and m and e are the mass and charge of the fermion, respectively. If the photon is in the other independent polarization state, with polarization vector orthogonal to the direction of periodicity, then the vacuum polarization does not effect the propagation, and the effective mass is zero. The quantity μ^2 given by Eq. (67) is negative for all values of m and L. If $L \ll m^{-1}$,

$$\mu^2 \sim -e^2/3L^2 \tag{69}$$

and if $L \gg m^{-1}$,

$$\mu^2 \sim -2^{-1/2} e^2 m^{3/2} \pi^{-3/2} L^{-1/2} e^{-mL} . \tag{70}$$

In Ref. 8 it was suggested that this tachyonic mass would lead to propagation outside of the light cone and hence violations of causality. This is not the case, as was discussed in Sec. II. It does, however, lead to an instability. The eigenfrequencies for photon modes with $k_z = 0$ and polarization $\hat{\epsilon} = \hat{z}$ are

$$\omega = (k_x^2 + k_y^2 + \mu^2)^{1/2} , \tag{71}$$

where $-\infty < k_x,\ k_y < \infty$. Thus there exist complex eigenfrequencies.

Hence we conclude that quantum electrodynamics in $S^1 \times R^3$ in which photons are coupled to untwisted spinors is unstable. If the photons are coupled to twisted spinors alone, no instabilities arise, but if they are coupled to both twisted and untwisted spinors simultaneously, then the theory is unstable. This follows from the fact that the effect of the combined vacuum polarization of untwisted and twisted spinors is to cause propagation with a tachyonic effective mass, as is discussed in Ref. 8. This is the case regardless of whether one combines the twisted and untwisted spinors by taking the spinor generating functional to be the sum of the twisted and untwisted functionals, as proposed by Avis and Isham,[30] or whether one takes it to be their product. The latter prescription is equivalent to taking a spinor

Lagrangian which is the sum of a twisted and an untwisted spinor Lagrangian.

VII. SUMMARY AND DISCUSSION

We have seen how one-loop quantum effects may influence the stability of a quantum field theory in a curved spacetime or in flat spacetime with boundaries. The method used to search for instabilities is that of testing whether equations of the form of (18) possess any exponentially growing solutions. The existence of such solutions indicate that there are coherent states in the theory whose particle content is increasing exponentially, which is unacceptable in a stable theory. As noted in Sec. II, the presence of an instability due to a tachyonic mass seems to be incompatible with a particle interpretation of the theory. Thus it is perhaps more accurate to state that unstable solutions indicate an inconsistency in the theory. Of course, the nonexistence of such solutions does not prove that the theory is fully stable in all respects; it does not rule out the possibility of tunnelling from a vacuum state associated with a local minimum of the effective potential to one associated with a global minimum. The present method seems to correspond to testing whether the vacuum in associated with a minimum or a maximum of the effective potential. In cases where both methods can be applied, they yield equivalent results. The method used here has the advantage over the effective potential method that it can be applied in cases where the latter cannot. An effective potential is defined only in scalar field theories where any constant field is consistent with the appropriate boundary conditions. This is not the case for twisted scalar fields or for fields which vanish on some boundary.

The instabilities arise as a result of radiative quantum effects causing at least certain modes to acquire a tachyonic mass. However, a tachyonic mass does not by itself lead to an instability. If the lowest eigenfrequency of the free theory is separated from zero by a finite gap, as is the case for twisted scalar field in $S^1\times R^3$ or a scalar field in the Einstein universe, it will be able to resist the effect of small destabilizing perturbation and remain real.

Two models were found above to be unstable. One is the theory of an untwisted scalar field coupled to a twisted scalar field in $S^1\times R^3$ in which the coupling constants satisfy Eq. (38). In this case it is possible for the field to undergo spontaneous symmetry breaking and acquire a nonzero, constant vacuum expectation value. This is discussed further by Toms,[6] who shows that the effective potential has a minimum at nonzero values of $\langle\phi\rangle$.

The other unstable theory is quantum electrodynamics in $S^1\times R^3$ with photons coupled to untwisted spinors or to a combination of twisted and untwisted spinors. Here it is less clear what the stable configuration should be, although one might expect it to be associated with a nonzero vacuum expectation value of the field tensor $F_{\mu\nu}$. Of course, in this case one may avoid the instability if one couples photons only to twisted spinors.

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APPENDIX A

In this appendix we calculate $\langle\phi_0{}^2\rangle_0$ and $\langle\tilde{\phi}_0{}^2\rangle_0$, the expectation values of the square of an untwisted and a twisted scalar field, respectively, in $S^1\times R^3$. The formal vacuum expectation value of $\phi_0{}^2$ may be written as

$$\langle 0|\phi_0{}^2|0\rangle=(4\pi L)^{-1}\sum_{n=-\infty}^{\infty}\int_0^\infty dKK\,\omega^{-1}, \qquad (A1)$$

where

$$\omega=(\vec{k}^2+m^2)^{1/2}, \qquad (A2)$$

$K^2=k_x{}^2+k_y{}^2$, and $k_z=2\pi nL^{-1}$. This divergent quantity may be regularized by writing

$$\langle\phi_0{}^2\rangle_R=(4\pi L)^{-1}\sum_{n=-\infty}^{\infty}\int_0^\infty dKK\,\omega^{-\alpha}. \qquad (A3)$$

In general, there should be a renormalization length in Eq. (A3), as in Eq. (B4) below. In this case, however, it would not appear in our final results and is hence ignored. Let $\xi=mL/2\pi$ and

$$K=2\pi L^{-1}(n^2+\xi^2)^{1/2}\sinh u. \qquad (A4)$$

Then

$$\langle\phi_0{}^2\rangle_R=\tfrac{1}{2}(2\pi)^{1-\alpha}L^{\alpha-3}\int_0^\infty du\,\sinh u\,\cosh^{1-\alpha}u\sum_{n=-\infty}^{\infty}(n^2+\xi^2)^{1-\alpha/2}$$
$$=\tfrac{1}{4}(2\pi)^{1-\alpha}L^{\alpha-3}\Gamma(\tfrac{1}{2}\alpha-1)\,[\Gamma(\tfrac{1}{2}\alpha)]^{-1}F(\tfrac{1}{2}\alpha-1;\xi,0), \qquad (A5)$$

where

$$F(\lambda;\xi,b)=\sum_{n=-\infty}^{\infty}[(n+b)^2+\xi^2]^{-\lambda}. \tag{A6}$$

In Ref. 8 it is shown that

$$F(\lambda;\xi,b)=\pi^{1/2}\xi^{1-2\lambda}\Gamma(\lambda-\tfrac{1}{2})[\Gamma(\lambda)]^{-1} + 4\sin\pi\lambda f_\lambda(\xi,b), \tag{A7}$$

where $f_\lambda(\xi)=f_\lambda(\xi,0)$ is defined in Eq. (24). Thus

$$\langle\phi_0^2\rangle_R=\pi^{1/2}(16\pi^2)^{-1}m^{3-\alpha}\Gamma(\tfrac{1}{2}\alpha-\tfrac{3}{2})[\Gamma(\tfrac{1}{2}\alpha)]^{-1} + (2\pi)^{1-\alpha}L^{\alpha-3}\sin\pi(\tfrac{1}{2}\alpha-1)\Gamma(\tfrac{1}{2}\alpha-1) \times[\Gamma(\tfrac{1}{2}\alpha)]^{-1}f_{\alpha/2-1}(\xi). \tag{A8}$$

The first term above is independent of L and is singular in the limit $\alpha\to 1$; the second term contains the L dependence and is finite at $\alpha=1$. The singular part may be absorbed by a mass renormalization. Then $\langle\phi_0^2\rangle$ is finite and defined up to an additive constant. If we require that the expectation value vanish as $L\to\infty$, then it is uniquely defined and given by the second term in Eq. (A8) evaluated at $\alpha=1$:

$$\langle\phi_0^2\rangle_0=2L^{-2}f_{-1/2}(\xi). \tag{A9}$$

The corresponding result for a twisted scalar field may be obtained readily. The analog of Eq. (A1) is

$$\langle 0|\tilde{\phi}_0^2|\rangle=(4\pi L)^{-1}\times\sum_{n=-\infty}^{\infty}\int_0^\infty dK\,K[K^2+m^2+\pi^2L^{-2}(2n+1)^2]^{-1/2}. \tag{A10}$$

Note that

$$\langle 0|\tilde{\phi}_0^2|0\rangle_L=\langle 0|\phi_0^2|0\rangle_{2L}-\langle 0|\phi_0^2|0\rangle_L. \tag{A11}$$

This relation also applies to the regularized and the renormalized expectation values. Hence, using Eq. (A9), one obtains Eq. (26). Equivalent results for both the untwisted and twisted cases were given by Toms.[4]

APPENDIX B

Here we calculate $\langle\phi_0^2\rangle_0$ in the Einstein universe. If ϕ_0 is a free field described by the wave equation

$$\Box\phi_0+m^2\phi_0+(\xi_1-\tfrac{1}{6})\phi_0=0 \tag{B1}$$

in the Einstein universe, then the formal expectation value of ϕ_0^2 is

$$\langle 0|\phi_0^2|0\rangle=(4\pi^2a^3)^{-1}\sum_{n=1}^{\infty}n^2\omega_n^{-1}. \tag{B2}$$

Here a is the radius of the universe and the eigenfrequencies are

$$\omega_n=[n^2a^{-2}+m^2+(\xi_1-\tfrac{1}{6})R]^{1/2},\quad n=1,2,\ldots \tag{B3}$$

with a degeneracy of n^2. The regularized expectation value is written as

$$\langle\phi_0^2\rangle_R=(4\pi^2a^3)^{-1}l^{1-\alpha}\sum_{n=1}^{\infty}n^2\omega_n^{-\alpha}, \tag{B4}$$

where l is an arbitrary length introduced to ensure that $\langle\phi_0^2\rangle_R$ has dimensions of (length)$^{-2}$.

Write

$$\langle\phi_0^2\rangle_R=a^{\alpha-3}l^{1-\alpha}(4\pi^2)^{-1}S, \tag{B5}$$

where

$$S=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}n^2(n^2+\mu^2)^{-\alpha/2} \tag{B6}$$

and μ is defined in Eq. (44). This sum may be expressed as

$$S=\tfrac{1}{2}[(\alpha-4)(\alpha-2)]^{-1}\left[\frac{\partial^2F(\frac{1}{2}\alpha^{-2};\mu,b)}{\partial b^2}\right]_{b=0} +\tfrac{1}{2}(\alpha-2)^{-1}F(\tfrac{1}{2}\alpha-1;\mu,0), \tag{B7}$$

where $F(\lambda;a,b)$ is defined in Eq. (A6). From Eq. (A7) we find that

$$S=\frac{2\pi^{1/2}\mu^{3-\alpha}\Gamma(\frac{1}{2}\alpha+\frac{1}{2})}{(\alpha-1)(\alpha-2)(\alpha-3)\Gamma(\frac{1}{2}\alpha-1)} +2\sin\pi(\tfrac{1}{2}\alpha-1)[f_{\alpha/2-1}(\mu)+\mu^2f_{\alpha/2}(\mu)]. \tag{B8}$$

The first term in Eq. (B8) contains the singular part of $\langle\phi_0^2\rangle_R$, which may be written as

$$\pi^{-3/2}(\alpha-1)^{-1}f(\alpha), \tag{B9}$$

where

$$f(\alpha)=\frac{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\mu^2a^{-2}}{(\alpha-2)(\alpha-3)\Gamma(\frac{1}{2}\alpha-1)}e^{-(\alpha-1)\ln(l\mu/a)}. \tag{B10}$$

We expand $f(\alpha)$ about $\alpha=1$ to obtain

$$f(\alpha)=-\mu^2(4\pi^{1/2}a^2)^{-1} +\mu^2a^{-2}(\alpha-1)[A+\tfrac{1}{8}\ln(l\mu/a)] +O((\alpha-1)^2), \tag{B11}$$

where A is a numerical constant.

As $\alpha \to 1$, we have that

$$\langle \phi_0^2 \rangle_R \sim -\frac{m^2 + (\xi_1 - \frac{1}{6})R}{8\pi^2(\alpha - 1)} + \pi^{-3/2}[m^2 + (\xi_1 - \tfrac{1}{6})R]\,[A + \tfrac{1}{32}\ln(l\mu/a)] - (2\pi^2 a^2)^{-1}[f_{-1/2}(\mu) + \mu^2 f_{1/2}(\mu)]\,. \quad \text{(B12)}$$

The pole term and the finite term proportional to A may be absorbed into renormalization of m^2 and ξ_1. The resulting renormalized expectation value, which is required to satisfy

$$\langle \phi_0^2 \rangle_0 \to 0 \text{ as } a \to \infty, \quad \text{(B13)}$$

is given by Eq. (43).

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*Present address: Department of Physics, Tufts University, Medford, Massachusetts 02155.

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