Second-order contribution to the gravitational deflection of light

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We calculate the gravitational deflection of light to second order in the Newtonian constant G. For a light ray passing the Sun with impact parameter b, the deflection in general relativity is $(4GM_\odot/bc^2)\times[1+(15\pi/16)(GM_\odot/bc^2)]$ radians.

The gravitational deflection of light has played an important historic role in the verification of general relativity (GR). In terms of the parametrized-post Newtonian (PPN) parameter γ [see Eqs. (3)–(5) below] the most recent results of Fomalont and Sramek[1] give $(\gamma+1)/2=1.015\pm0.011$, in agreement with the GR prediction $\gamma=1$. Although the value of γ obtained from radar delay is more precise (Ref. 2) $(\gamma+1)/2=1.000\pm0.002$, the gravitational deflection of light is nonetheless important because it may lead to a test of GR to $O(G^2)$ where G is the Newtonian constant of gravitation. A method for carrying out such a test has been proposed recently by Reasenberg and Shapiro.[3] The purpose of this paper is to calculate the second-order contribution to the deflection angle ϕ and in the process develop a simple iterative method that allows ϕ to be calculated to arbitrary order in G. The $O(G^2)$ contributions to the radar time delay involve additional problems and will be presented elsewhere.

The calculation of second-order effects can be simplified to some extent by adopting a picture in which a gravitational field influences the propagation of electromagnetic radiation by imparting to space an effective index of refraction $n(r)$. Such a picture arises from the observation that Maxwell's equations in the presence of gravity[4]

$$F^{\mu\nu}{}_{;\mu}=-J^\nu\,, \tag{1a}$$

$$F_{\mu\nu;\lambda}+F_{\nu\lambda;\mu}+F_{\lambda\mu;\nu}=0\,, \tag{1b}$$

can be recast in the form

$$\vec{\nabla}\cdot(\epsilon\vec{E})=\rho\,,\quad \vec{\nabla}\times(\vec{B}/\mu)=\vec{J}+\partial(\epsilon\vec{E})/\partial t\,, \tag{2a}$$

$$\vec{\nabla}\cdot\vec{B}=0\,,\quad \vec{\nabla}\times\vec{E}+\partial\vec{B}/\partial t=0\,, \tag{2b}$$

where $\epsilon=\epsilon(x)$ and $\mu=\mu(x)$ are determined by the metric tensor $g_{\mu\nu}(x)$. It follows that a gravitational field can be viewed as endowing a Minkowskian region of space-time with an index of refraction $n=\sqrt{\epsilon\mu}$. For a static spherically symmetric geometry $g_{\mu\nu}(x)$ is specified in isotropic coordinates by writing

$$ds^2=f(r)(dx^2+dy^2+dz^2)+g_{00}(r)(dx^0)^2\,,\qquad r=(x^2+y^2+z^2)^{1/2}\,, \tag{3}$$

in which case $n(r)=\epsilon=\mu=[-f(r)/g_{00}(r)]^{1/2}$. This result can be verified by noticing that since $ds^2=0$ for photons we can divide Eq. (3) through by $f(r)$ to give

$$0=ds^2=dx^2+dy^2+dz^2-[-g_{00}(r)/f(r)](dx^0)^2\,. \tag{4}$$

In the form of Eq. (4) the photon is viewed as propagating in a Minkowskian space-time but with a local index of refraction given by the previous result. To lowest order in $\Phi=GM_\odot/rc^2$, $f(r)$ and $g_{00}(r)$ can be expanded in terms of the PPN parameters β,γ, and δ defined by

$$\begin{aligned} f(r)&=1+2\gamma\Phi+\tfrac{3}{2}\delta\Phi^2+O(\Phi^3)\,,\\ -g_{00}(r)&=1-2\Phi+2\beta\Phi^2+O(\Phi^3)\,, \end{aligned} \tag{5a}$$

$$\begin{aligned} n(r)&=1+(\gamma+1)\Phi+(\tfrac{3}{2}+\gamma-\tfrac{1}{2}\gamma^2-\beta+\tfrac{3}{4}\delta)\Phi^2+O(\Phi^3)\\ &\equiv 1+\frac{A}{r}+\frac{B^2}{r^2}\,. \end{aligned} \tag{5b}$$

Given $n(r)$ in Eq. (5b), the calculation of the gravitational deflection of light reduces to a classical optics problem in which the contributions from successively higher powers of Φ (or G) can be evaluated by a straightforward iterative procedure.

Consider the refraction of a ray passing through a thin spherical shell located at r, as shown in Fig. 1(a). From Snell's law, with $\theta=\theta_r+\theta_v$,

$$\begin{aligned} n(r)\sin\theta&=n(r+dr)\sin(\theta+d\phi)\\ &\simeq\left[n(r)+\frac{dn(r)}{dr}dr\right](\sin\theta+\cos\theta\,d\phi)\,. \end{aligned} \tag{6}$$

Neglecting second-order differentials we find the following differential equation for the angle $d\phi$:

$$d\phi=-\frac{1}{n(r)}\frac{dn(r)}{dr}\tan\theta\,dr\,. \tag{7}$$

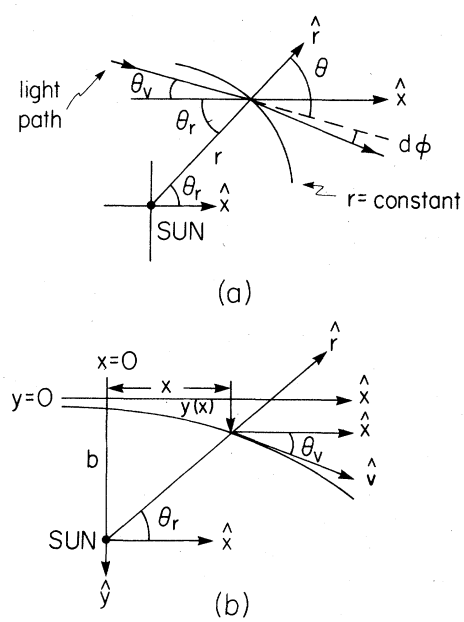

FIG. 1. Description of light bending in a gravitational field. (a) Refraction of light from a thin spherical shell at r. $\theta_v(\theta_r)$ denotes the angle between the velocity (radial) vector and the x axis, $\theta=\theta_r+\theta_v$, and $d\phi$ is the change in θ_v due to the thin shell. (b) Parametrization of the light path. The light ray is characterized by an impact parameter b, a displacement $y(x)$ from the undeflected path, and a local bending angle $\phi(x)=\theta_v$. The total angular deflection corresponds to $\phi(\infty)$.

To evaluate the scattering angle $\phi=\int d\phi$, we must parametrize the trajectory in such a way as to include the effects of the curved path. The following exact relations are useful (primes denote differentiation with respect to x):

$$r=\{x^2+[b-y(x)]^2\}^{1/2}, \quad r'=(x-by'+yy')/r\,, \tag{8a}$$

$$\tan\theta_r=(b-y)/x, \quad \tan\theta_v=\frac{dy/dt}{dx/dt}=y'=\tan\phi(x), \tag{8b}$$

$$\tan\theta=(b-y+xy')/(x-by'+yy')\,. \tag{8c}$$

Hence,

$$\frac{d\phi(x)}{dx}=-\frac{1}{rn(r)}\frac{dn(r)}{dr}(xy'+b-y)\,. \tag{9}$$

Since the leading contribution to dn/dr is of order G it follows that the right-hand side of Eq. (9) is always of one higher order than the left-hand side. This means that Eq. (9) can be solved iteratively for ϕ to arbitrary order in G. To $O(G^N)$ ($N=1, 2, \ldots$) we first solve for the bending angle to $O(G^{N-1})$ and then use this result to find the displacement $y(x)$ to $O(G^{N-1})$. We then insert this result into Eq. (9) to compute ϕ to order G^N. From Fig. 1 and Eq. (8b) we see that the scattering angle ϕ is just equal to the angle θ_v that the velocity vector makes with respect to the x axis. We can then use Eq. (8b) to express the trajectory $y^{(N)}(x)$ computed to a given order N in terms of the corresponding bending angle $\phi^{(N)}(x)$:

$$y^{(N)}(x)=\int_{-\infty}^{x}\frac{dy^{(N)}}{dx'}dx'=\int_{-\infty}^{x}\tan\phi^{(N)}(x')dx'\,. \tag{10}$$

To evaluate $\phi^{(1)}$, the $O(G)$ contribution to ϕ, we neglect the term proportional to B^2 in Eq. (5b) which then gives

$$\frac{d\phi^{(1)}}{dx}\simeq\frac{A}{r^3}(b+xy'-y)\,. \tag{11}$$

To this order $y^{(0)}(x)\equiv 0$ and hence

$$\phi^{(1)}(x)=(A/b)[1+x(x^2+b^2)^{-1/2}]\,, \tag{12a}$$

$$\phi^{(1)}(\infty)=2A/b=2(\gamma+1)GM_\odot/bc^2\,, \tag{12b}$$

which is the standard result for $\phi^{(1)}$.

To evaluate $\phi^{(2)}(x)$ we begin by finding $y^{(1)}(x)$.

$$\begin{aligned} y^{(1)}(x)&=\int_{-\infty}^{x}dx'\tan\phi^{(1)}(x')\\ &\simeq\int_{-\infty}^{x}dx'\phi^{(1)}(x')\\ &=(A/b)[x+(x^2+b^2)^{1/2}]\,. \end{aligned} \tag{13}$$

From Eq. (8a) r is now given by

$$r\simeq(x^2+b^2)^{1/2}[1-by^{(1)}(x)(x^2+b^2)^{-1}]\,, \tag{14}$$

Hence, to $O(G^2)$ the deflection angle $\phi^{(2)}(\infty)$ is

$$\begin{aligned} \phi^{(2)}(\infty)&=b\int_{-\infty}^{\infty}dx[A(x^2+b^2)^{-3/2}+3A^2x(x^2+b^2)^{-5/2}\\ &\qquad+(A^2+2B^2)(x^2+b^2)^{-2}]\\ &=2A/b+(A^2+2B^2)\pi/2b^2\\ &=2(\gamma+1)GM_\odot/bc^2\\ &\quad+(\pi/4)(8-4\beta+8\gamma+3\delta)(GM_\odot/bc^2)^2\,. \end{aligned} \tag{15}$$

Numerically, for light grazing the Sun,[5]

$$\begin{aligned} \phi^{(2)}(\infty)=\{&\tfrac{1}{2}(\gamma+1)(1.7507)\\ &+\tfrac{1}{15}(8-4\beta+8\gamma+3\delta)(11\times10^{-6})\\ &-\tfrac{1}{4}(\gamma+1)^2(7.4\times10^{-6})\}\text{arcsec}\,. \end{aligned} \tag{16}$$

We see that for GR, corresponding to $\gamma=\beta=\delta=1$, the second-order effect contributes a deflection of ~3.5×10^{-6} arcsec. Since the current precision is[1] roughly 10^{-2} arcsec, it would appear that it may not be unrealistic to attempt to measure this contribution. From a theoretical point of view such a measurement would lead to a determination of the PPN parameter δ, when coupled with results for β and γ obtained from radar time delay and the precession of the perihelion of Mercury.

After completing this work we learned that the results of Eqs. (15) and (16) had been obtained previously some years ago by Shapiro and somewhat later by Epstein. We wish to thank Dr. Reasenberg for bringing the unpublished work of Professor Shapiro and Dr. Epstein to our attention. This work was supported in part by the U. S. Department of Energy.

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[4]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 125; C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 568; M. G. Bowler, *Gravitation and Relativity* (Pergamon, London, 1976), p. 58; A. Lightman and D. L. Lee, Phys. Rev. D 8, 364 (1973); F. J. Belinfante and J. Swihart, Ann. Phys. (N.Y.) 1, 196 (1957).

[5]Care must be taken to operationally define the light path when computing higher-order effects. On physical grounds grazing incidence corresponds to the minimum value of r in Eq. (8a), with $r_{\min} = R_\odot$. Minimizing r using Eq. (13) gives (to first order in G) $r_{\min} = b - A$. Hence for grazing incidence $b = R_\odot + A$ where A gives the first-order correction to the lowest-order value $b = R_\odot$. Inserting $b = R_\odot + A$ into (15) gives (16).