Post-post-Newtonian deflection of light by the Sun

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Using a generalized metric, we calculate the post-post-Newtonian (ppN) contribution to the deflection of starlight by the Sun as well as the post-Newtonian deflections due to the Sun's gravitational quadrupole moment and angular momentum. For light rays grazing the Sun, the magnitudes of these deflections predicted by general relativity are, respectively, about 11, 0.2, and 0.7 μarcsec. A covariance analysis shows that the ppN contribution could be measured with an uncertainty under 20% with an optical interferometer, in Earth orbit, that can determine the separation of stars with an uncertainty of 1 μarcsec.

I. INTRODUCTION

The first measurement of the gravitational deflection of starlight by the Sun verified an important quantitative prediction of general relativity.[1] However, even modern light-deflection measurements[2] have not been sufficiently accurate to distinguish any contribution other than the post-Newtonian (pN). In 1974, it was suggested[3] that an optical interferometer deployed in Earth orbit could be capable of measurement with accuracy sufficient to determine the post-post-Newtonian (ppN) contribution to the deflection of starlight. One approach[4] envisioned for such instrumentation could allow measurement of the angular separation of stars on the celestial sphere to within 1 μ arcsec.

In this paper, we present the ppN contribution to the light deflection and also give the predicted deflections attributable to the Sun's gravitational oblateness and rotation. Finally, we mention the results from a covariance analysis to determine the detectability of this ppN contribution with the proposed interferometer.

II. DEFLECTION CAUSED BY A POINT MASS

We first calculate the deflection of starlight by the Sun, considered as a point mass. For the microarcsecond level of accuracy, the gravitational effects of finite wavelength and polarization are negligible as is the deflection at optical wavelengths due to the solar corona. Ray optics are therefore sufficient. Following standard procedure, we write the generalized Schwarzschild metric as

$$d\tau^2 = A(r)dt^2 - B(r)d\vec{x}\cdot d\vec{x}, \tag{1}$$

where

$$A(r) = 1 - \frac{2M}{r} + \frac{2\beta M^2}{r^2} + O\left(\frac{M^3}{r^3}\right) \tag{2}$$

and

$$B(r) = 1 + \frac{2\gamma M}{r} + \frac{3\epsilon M^2}{2r^2} + O\left(\frac{M^3}{r^3}\right). \tag{3}$$

Here M is the mass of the Sun and r is the coordinate distance from its center; we use units such that $G=c=1$. The parameters β, γ, and ϵ are each unity in the general theory of relativity.

Calculating the deflection angle in the usual manner,[5] we find for the ppN contribution

$$\Delta\vec{\theta}_{\text{ppN}} = \pi(2+2\gamma-\beta+\tfrac{3}{4}\epsilon)\frac{M^2}{r_0^2}\hat{r}_0, \tag{4}$$

where $\Delta\vec{\theta}_{\text{ppN}}$ represents the correction to the unit vector that extends from the observer in the apparent direction of the source. The vector $\vec{r}_0$ (unit vector $\hat{r}_0$) extends from the center of the Sun to the point closest to the Sun on the line tangent to the ray path at the source. Note that the combination $(2+2\gamma-\beta)$ in Eq. (4) also appears in the pN expression for the advance of planetary perihelia.

For general relativity, the magnitude of $\Delta\vec{\theta}_{\text{ppN}}$ for rays grazing the solar limb is about 11 μarcsec and that of the (new) ϵ term about 2 μarcsec.

III. DEFLECTION CAUSED BY THE QUADRUPOLE MOMENT OF THE MASS DISTRIBUTION

The deflection caused by the quadrupole moment of the mass distribution of the Sun is sufficiently small that it can be treated as an additive vector correction. To determine this correction, we use the generalization of the equation[6] from the general theory of relativity for the time dependence of the unit three-vector $\hat{u}$ tangent to the trajectory of a photon traveling through a region characterized by the Newtonian gravitational potential Φ:

$$\frac{d\hat{u}}{dt} \simeq (1+\gamma)\hat{u}\times(\hat{u}\times\vec{\nabla}\Phi), \tag{5}$$

where, for the accuracy we need, $\hat{u}$ can be replaced on the right-hand side by the unit vector tangent to the undeflected trajectory. The quadrupole contribution, or, more precisely, the contribution Φ_2 of the second zonal harmonic of the gravitational potential of the Sun is[7]

$$\Phi_2 = \frac{MJ_2R^2}{2r^3}(3\sin^2\delta - 1), \tag{6}$$

where J_2 is the dimensionless coefficient of the second zonal harmonic, R is the radius of the Sun, and δ and r are, respectively, the latitude of the field point and its distance from the center of the Sun. A straightforward calculation that involves only elementary integrals yields

$$\Delta\vec{\theta}_{J_2} \simeq -\frac{2(1+\gamma)}{r_0^3}MJ_2R^2\{[1-2(\hat{r}_0\cdot\hat{k})^2-(\hat{e}_s\cdot\hat{k})^2]\hat{r}_0 + 2(\hat{r}_0\cdot\hat{k})[\hat{k}\cdot(\hat{e}_s\times\hat{r}_0)]\hat{e}_s\times\hat{r}_0\}, \tag{7}$$

where $\Delta\vec{\theta}_{J_2}$ is the predicted deflection due to the second zonal harmonic, $\hat{e}_s$ is the unit vector in the direction from the center of the Sun to the star, and $\hat{k}$ is the unit vector in the direction of the axis of assumed symmetry of the Sun, or, equivalently, of its axis of maximum moment of inertia.

If the Sun is rotating uniformly, with the angular velocity observed for its surface, and if $\gamma \simeq 1$, then J_2 will be of order 10^{-7}, and a typical value of $\Delta\vec{\theta}_{J_2}$ would be 0.2 μarcsec for a photon trajectory that just grazes the solar limb.

IV. DEFLECTION CAUSED BY ROTATION

The angular momentum of the Sun is sufficiently small that we may treat its effect on the deflection as we did that of the quadrupole moment. The relevant equation,[6] appropriately generalized, for the effect of the rotation on the deflection is

$$\frac{d\vec{u}}{dt} \simeq (1+\gamma)\hat{u}\times\left[\vec{\nabla}\times\left(\frac{\vec{r}\times\vec{L}}{r^3}\right)\right], \tag{8}$$

where $\vec{u}$ is the three-velocity of the photon, $\vec{r}$ is the vector from the origin to the position of the photon, and $\vec{L}$ is the angular momentum vector of the Sun. With the same approximations as were used in Sec. III, we obtain

$$\Delta\vec{\theta}_L \simeq \frac{2(1+\gamma)}{r_0^2}\{[(\hat{e}_s\times\hat{r}_0)\cdot\vec{L}]\hat{r}_0 + [\hat{r}_0\cdot\vec{L}](\hat{e}_s\times\hat{r}_0)\}. \tag{9}$$

For starlight that grazes the limb of the Sun, we find for a typical geometry and for $\gamma \simeq 1$ that $\Delta\theta_L \sim 0.7$ μarcsec for an assumed value[8] of $L \sim 2 \times 10^{48}$ g cm^2 sec^{-1}, based on uniform rotation of the Sun. If measurements of angle could be made with uncertainties well under one μarcsec, then one could determine $\vec{L}$, given both the validity of the generalized metric and the value of γ.

V. COVARIANCE ANALYSIS

How accurately can the ppN contribution to the deflection be determined with an optical interferometer that can measure angular separations of stars to within one μarcsec? To answer this question, we carried out an extensive covariance analysis[9] in which we estimated the coordinates of the stars, the elements of the heliocentric orbit of the interferometer, parameters of a simple model of possible calibration errors of the instrument, and the appropriate coefficients of the pN and the ppN terms in the total deflection. The results show that, for reasonable scenarios of observations extending over about two weeks, such an interferometer could determine the ppN contribution with an uncertainty of under 20%, barely sufficient to detect the ϵ term.

The ppN contribution to the round-trip time delay of light signals traveling between planets was also calculated and a covariance analysis performed[9]; the results show that the observational effects are too small to be detected with equipment likely to be affordable in the next decade.

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[6]See Ref. 5, Chap. 9. Note, however, that a factor of 2 was omitted from the right-hand side of Eq. (9.2.7).

[7]See, for example, J. M. A. Danby, *Fundamentals of Celestial Mechanics* (Macmillan, New York, 1962).

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