Stationary electromagnetic fields around black holes. III. General solutions and the fields of current loops near the Reissner-Nordström black hole

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The stationary multipole solutions of the Einstein-Maxwell equations representing coupled electromagnetic and gravitational perturbations of the extreme Reissner-Nordström black hole are simple rational functions of the radial coordinate. These solutions are used to construct the axially symmetric field of a current loop in the equatorial plane and of a small current loop on the polar axis. The magnetic lines of force are calculated numerically and exhibited graphically. The field representing a general Reissner-Nordström black hole in an asymptotically uniform magnetic field is also given, and its relation to the exact solutions of Ernst describing Kerr-Newman black holes in Melvin's magnetic universe is discussed.

I. INTRODUCTION

In the first two papers of this series[1] the solutions representing the electromagnetic fields of general stationary sources in the vicinity of the Schwarzschild and the Kerr black holes were given. As examples, the fields of point charges, charged rings, current loops and magnetic dipoles, not necessarily located in axisymmetric positions, were constructed explicitly. The solution representing the Kerr black hole placed in a uniform magnetic field was also given, without assuming the alignment of the direction of the field and the axis of symmetry of the black hole.

When a black hole carries a net charge of which the contribution to the geometry cannot be neglected, coupled electromagnetic and gravitational perturbations have to be treated concurrently. Recently, we compared and developed in detail various methods of analyzing the interacting perturbations of the Reissner-Nordström black hole[2]: A complete account of the Hamiltonian approach, initiated by Moncrief,[3] was given and its relation to Zerilli's treatment[4] and to the methods using the Newman-Penrose formalism,[5] was established. Among other things, all metric and electromagnetic field perturbations with both odd and even parity were found in terms of Moncrief's gauge-invariant canonical variables in the Regge-Wheeler gauge; and the $l=1$ perturbations were also analyzed in detail. In the present paper, we shall employ these results for constructing the field of a current loop placed axisymmetrically in the equatorial plane and of a small current loop (magnetic dipole) placed axisymmetrically on the polar axis. The source terms will be expressed by using Zerilli's[4] results. We shall also derive the electromagnetic and gravitational fields occurring when the Reissner-Nordström black hole is placed in an originally uniform magnetic field. It will be indicated that the corresponding exact solution of Ernst[6] should be identified not with the Reissner-Nordström black hole but rather with the Kerr-Newman black hole in Melvin's magnetic universe.

The magnetic fields of given sources will be illustrated by lines of force, constructed numerically. In particular, in the case of a small loop on the polar axis the lines reveal an unexpected structure. If the hole carries an extreme charge, the magnetic lines of force do not cross the horizon. By inspecting the behavior of the fields around Kerr black holes,[1] one learns that the same is true in the case of extremely rotating black holes. It remains to be seen whether this feature is connected with some general ("dielectric", "conductive") properties of black holes.

The simple combinations of Moncrief's gauge-invariant canonical variables satisfy wave equations with effective potential barriers.[3,2] The general stationary multipole solutions of these equations (with zero source terms) are simple rational functions of the radial coordinate[7] for both the odd- and the even-parity perturbations of the extreme Reissner-Nordström black hole ($e^2=M^2$). In the case of a general Reissner-Nordström black hole ($e^2<M^2$), the solutions are much more complicated, being given in terms of certain series of hypergeometric functions (as the solutions of Heun's equation with four regular singular points). Therefore, we shall restrict ourselves to the fields of current loops in the vicinity of the extreme Reissner-Nordström black hole; only in the case of the Reissner-Nordström black hole immersed in an asymptotically uniform magnetic field has the solution a simple form for an arbitrary value of the hole's charge. The general features of the resulting fields and of correponding lines of force, in particular those caused by the coupling of electromagnetic and gravita-

tional perturbations, should, however, be well illustrated by the case of the extremely charged black hole.

The stationary electromagnetic fields of some special sources in the Reissner-Nordström background have been considered in several papers[8] previously. None of these treatments is really consistent, however, since the interaction between electromagnetic and gravitational perturbations has not been taken into account.

II. ODD-PARITY PERTURBATIONS OF THE REISSNER-NORDSTRÖM BLACK HOLE

Since we shall assume that the current loops carry no macroscopic charge and have a negligible rest mass, it is sufficient to consider only odd-parity perturbations. Starting from Moncrief's Hamiltonians[3] for the $l \geq 2$ and $l=1$ odd-parity perturbations, we can derive the Hamiltonian equations which imply the decoupled equations for gauge-invariant functions $\mathcal{P}_l^{\pm}(l\geq 2)$ and $\mathcal{P}_1^{+}(l=1)$ in the form[2]

$$\mathcal{P}^{\pm}_{l,r^*r^*} - Br^{-2}[l(l+1) - 3Mr^{-1} + 4e^2r^{-2} \pm \sigma r^{-1}]\mathcal{P}^{\pm}_l = \pm 8\pi[(l-1)(l+2)(\sigma \pm 3M)(2\sigma)^{-1}]^{1/2}By_l\,, \quad (2.1)$$

$$\mathcal{P}^{+}_{1,r^*r^*} - Br^{-2}[2+4e^2r^{-2}]\mathcal{P}^{+}_1 = 8\pi By_1\,. \quad (2.2)$$

Here,

$$dr/dr^* = B = 1 - 2Mr^{-1} + e^2r^{-2}\,,$$
$$\sigma = [9M^2 + 4e^2(l-1)(l+2)]^{1/2}\,, \quad (2.3)$$

and the source terms on the right-hand sides of (2.1) and (2.2) were derived by comparing Zerilli's results[4] [cf., in particular, Zerilli's Eq. (19)] and the results of Ref. 2. The stationarity of the fields and sources is assumed in (2.1) and (2.2) already so that all functions depend only on the radial coordinate. In the case of axisymmetric current loops the source term reads (for all $l\geq 1$)

$$y_l = -2\pi r^2[l(l+1)]^{-1}\int_0^{\pi}\sin^2\theta\frac{\partial Y_{l0}}{\partial\theta}j^{\varphi}(r,\theta)\,d\theta\,, \quad (2.4)$$

where j^{φ} is the azimuthal component of the four-current.

Knowing the solutions of (2.1) and (2.2), we can determine all metric and electromagnetic perturbations after specifying the gauge. We shall choose the Regge-Wheeler gauge, supplemented by appropriate gauge conditions for the $l=1$ perturbations (see Ref. 2 for details). Moncrief's canonical variables π_f and π_1 are (in any gauge) given by the relations ($l\geq 2$)

$$\pi_f = [(l-1)(l+2)]^{-1/2}\hat{\pi}_f\,,$$
$$\pi_1 = l(l+1)r^{-1}\hat{\pi}_g + 2l(l+1)[(l-1)(l+2)]^{-1/2}er^{-2}\hat{\pi}_f\,, \quad (2.5)$$
$$\hat{\pi}_f = (2\sigma)^{-1/2}[(\sigma+3M)^{1/2}\mathcal{P}^{+}_l - (\sigma-3M)^{1/2}\mathcal{P}^{-}_l]\,,$$
$$\hat{\pi}_g = (2\sigma)^{-1/2}[(\sigma-3M)^{1/2}\mathcal{P}^{+}_l + (\sigma+3M)^{1/2}\mathcal{P}^{-}_l]\,.$$

In the case of dipole perturbations, Moncrief's variable π_f is connected with $\mathcal{P}^{+}_1$ (denoted by $\mathcal{P}_f$ in Ref. 2) by

$$\pi_f = \mathcal{P}^{+}_1 - Ce/3Mr\,, \quad (2.6)$$

where constant C corresponds to adding a small angular momentum to the hole.[2] Except for Sec. IV, we set $C=0$. In the Regge-Wheeler gauge the general stationary and axisymmetric perturbations of the electromagnetic field tensor with odd parity can be expressed in the form[2]

$$\delta F_{r\varphi} = \sum_{l=1}^{\infty}\frac{1}{2}\frac{d\pi_f}{dr}\sin\theta\frac{\partial Y_{l0}}{\partial\theta}\,, \quad (2.7)$$

$$\delta F_{\theta\varphi} = -\sum_{l=1}^{\infty}\tfrac{1}{2}l(l+1)\pi_f\sin\theta\, Y_{l0}\,, \quad (2.8)$$

where π_f's are given by (2.5) and (2.6). The only nonvanishing component of the metric perturbations is given by[2]

$$\delta g_{t\varphi} = \sum_{l=1}^{\infty} h_0\sin\theta\frac{\partial Y_{l0}}{\partial\theta}\,, \quad (2.9)$$

where

$$h_0 = -B[(l-1)(l+2)]^{-1}(r\hat{\pi}_g)_{,r} \quad \text{for } l\geq 2\,, \quad (2.10)$$
$$= -r^2\int_{\infty}^{r}2er^{-4}\mathcal{P}^{+}_1\,dr \quad \text{for } l=1\,. \quad (2.11)$$

Let us note that in the case of dipole odd-parity perturbations, one can always choose the gauge in which not only $\delta g_{\theta\theta} = \delta g_{\varphi\varphi} = \delta g_{\theta\varphi} = 0$ (the Regge-Wheeler gauge conditions for odd-parity perturbations with $l\geq 2$), but also either $\delta g_{t\varphi}$ or $\delta g_{r\varphi}$ vanish. Since $\delta g_{t\theta} = \delta g_{r\theta} = 0$ due to the axial symmetry, there remains only one nonzero component of $\delta g_{\mu\nu}$. We require $\delta g_{r\varphi}=0$; however, there still exists a gauge freedom which changes $\delta g_{t\varphi}$ to $\delta g_{t\varphi} - f(t)r^2\sin^2\theta$, f being an arbitrary function of time (see Ref. 2, Sec. 3B for the details about gauge transformations of dipole odd-parity perturbations). Let us emphasize that none of the $\delta g_{t\varphi}$'s given explicitly in the following contains such a removable term. Furthermore, it is easy to see[9] that the combination $[\delta g_{t\varphi}/(r^2\sin^2\theta)]_{,r} - [\delta g_{r\varphi}/(r^2\sin^2\theta)]_{,t}$ is a gauge invariant which reduces just to $[\delta g_{t\varphi}/(r^2\sin^2\theta)]_{,r} = -(2e/r^4)\mathcal{P}^{+}_1$ with our choice of gauge. Any gauge transformation changing $\delta g_{t\varphi}$ in a nontrivial manner [not just by $f(t)r^2\sin^2\theta$] would lead to a time-dependent $\delta g_{r\varphi}$. An analogous statement is true for $l\geq 2$ perturbations. Henceforth, all our assertions about the only nonvanishing component $\delta g_{t\varphi}$ should be understood in this sense.

III. THE FIELDS OF CURRENT LOOPS IN THE VICINITY OF THE EXTREME REISSNER-NORDSTRÖM BLACK HOLE

We shall now solve Eqs. (2.1) and (2.2), with the sources representing current loops, in the case of the extreme Reissner-Nordström black hole ($e^2=M^2$). Putting $\xi=(r-M)/M$, Eqs. (2.1) and (2.2) assume the form

$$\begin{aligned}\mathcal{P}^{\pm}_{l,\xi\xi}&+2[\xi(\xi+1)]^{-1}\mathcal{P}^{\pm}_{l,\xi}\\&-\xi^{-2}[l(l+1)+4(\xi+1)^{-2}\pm(2l^{-2}_{+4})(\xi+1)^{-1}]\mathcal{P}^{\pm}_{l}=g^{\pm}_{l}\,,\\&\qquad l\geqslant 2\,,\end{aligned}\tag{3.1}$$

$$\mathcal{P}^{+}_{1,\xi\xi}+2[\xi(\xi+1)]^{-1}\mathcal{P}^{+}_{1,\xi}-\xi^{-2}[2+4(\xi+1)^{-2}]\mathcal{P}^{+}_{1}=g^{+}_{1}\,,\tag{3.2}$$

where

$$g^{\pm}_{l}=\pm 8\pi[(l-1)(l+2)]^{1/2}[(l^{+2}_{-1})/(2l+1)]^{1/2}\times[(\xi+1)/\xi]^2M^2y_l\,,\tag{3.3}$$

$$g^{+}_{1}=8\pi[(\xi+1)/\xi]^2M^2y_1\,,\tag{3.4}$$

and y_l, y_1 are functions of ξ to be determined from (2.4).

The fundamental system of the solutions of the homogeneous equations corresponding to (3.1) and (3.2) reads[7]

$$\begin{aligned}\mathcal{P}^{+(\mathrm{I})}_{l}&=[l(\xi+1)]^{-1}\varphi^{+(\mathrm{I})}_{l}\,,\\\mathcal{P}^{+(\mathrm{II})}_{l}&=[(l+1)(l+2)(2l+3)(\xi+1)]^{-1}\varphi^{+(\mathrm{II})}_{l}\,,\\\mathcal{P}^{-(\mathrm{I})}_{l}&=[(l-1)l(2l-1)(\xi+1)]^{-1}\varphi^{-(\mathrm{I})}_{l}\,,\\\mathcal{P}^{-(\mathrm{II})}_{l}&=[(l+1)(\xi+1)]^{-1}\varphi^{-(\mathrm{II})}_{l}\,,\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}\varphi^{+(\mathrm{I})}_{l}&=\xi^{l+1}[l+2+l\xi]\,,\\\varphi^{+(\mathrm{II})}_{l}&=\xi^{-(l+2)}[l(l+1)(2l+1)+3l(l+2)(2l+1)\xi\\&\quad+3l(l+2)(2l+3)\xi^2\\&\quad+(l+1)(l+2)(2l+3)\xi^3]\,,\\\varphi^{-(\mathrm{I})}_{l}&=\xi^{l-1}[l(l+1)(2l+1)+3(l-1)(l+2)(2l+1)\xi\\&\quad+3(l-1)(l+1)(2l-1)\xi^2\\&\quad+(l-1)l(2l-1)\xi^3]\,,\\\varphi^{-(\mathrm{II})}_{l}&=\xi^{-l}[l-1+(l+1)\xi]\,.\end{aligned}\tag{3.6}$$

(The solutions given in Ref. 7 are here multiplied by convenient factors depending on l.) Functions $\mathcal{P}^{+}_{l}$ are meaningful for both $l\geqslant 2$ and $l=1$, but $\mathcal{P}^{-}_{l}$ are not defined for $l=1$. We note that if we formally put $l=1$ into the left-hand side of (3.1) for $\mathcal{P}^{+}_{l}$, we recover the left-hand side of (3.2). At infinity,

$$\mathcal{P}^{\pm(\mathrm{I})}_{l}\sim\xi^{l+1}\,,\quad\mathcal{P}^{\pm(\mathrm{II})}_{l}\sim\xi^{-l}\,,\tag{3.7}$$

whereas at the horizon ($\xi=0$),

$$\mathcal{P}^{\pm(\mathrm{I})}_{l}\sim\xi^{l\pm1}\,,\quad\mathcal{P}^{\pm(\mathrm{II})}_{l}\sim\xi^{-(l+1\pm1)}\,.\tag{3.8}$$

Functions $\mathcal{P}_l$ determine observable quantities, for example, both the coordinate and the tetrad gauge-invariant perturbations of the appropriate combinations of the Newman-Penrose scalars.[2,10] By requiring the observable quantities to be well behaved at infinity and at the horizon, we see that, at infinity, only solutions $\mathcal{P}^{\pm(\mathrm{II})}_{l}$ are admissible, while solutions $\mathcal{P}^{\pm(\mathrm{I})}_{l}$ are well behaved at the horizon.

By using standard procedure,[1] we obtain the solution of inhomogeneous equations (3.1) and (3.2):

$$\begin{aligned}\mathcal{P}^{\pm}_{l}(\xi)=&-\mathcal{P}^{\pm(\mathrm{I})}_{l}(\xi)\int_{\infty}^{\xi}\frac{\mathcal{P}^{\pm(\mathrm{II})}_{l}(\eta)g^{\pm}_{l}(\eta)}{W(\mathcal{P}^{\pm(\mathrm{I})}_{l},\mathcal{P}^{\pm(\mathrm{II})}_{l},\eta)}d\eta\\&+\mathcal{P}^{\pm(\mathrm{II})}_{l}(\xi)\int_{0}^{\xi}\frac{\mathcal{P}^{\pm(\mathrm{I})}_{l}(\eta)g^{\pm}_{l}(\eta)}{W(\mathcal{P}^{\pm(\mathrm{I})}_{l},\mathcal{P}^{\pm(\mathrm{II})}_{l},\eta)}d\eta\,,\end{aligned}\tag{3.9}$$

where

$$W(\mathcal{P}^{\pm(\mathrm{I})}_{l},\mathcal{P}^{\pm(\mathrm{II})}_{l},\eta)=-(2l+1)[(\eta+1)/\eta]^2$$

is the Wronskian of $\mathcal{P}^{\pm(\mathrm{I})}_{l},\mathcal{P}^{\pm(\mathrm{II})}_{l}$ at the point η. Assuming the sources to be located between ξ_1 and ξ_2 such that $0<\xi_1<\xi_2<\infty$, the solutions satisfying the correct boundary conditions at the horizon and at infinity can be written in the form

$$\begin{aligned}\mathcal{P}^{\pm}_{l}(\xi)&=A^{\pm}_{l}\mathcal{P}^{\pm(\mathrm{I})}_{l}(\xi)\quad\text{for }0<\xi<\xi_1\\&=B^{\pm}_{l}\mathcal{P}^{\pm(\mathrm{II})}_{l}(\xi)\quad\text{for }\xi_2<\xi\,,\end{aligned}\tag{3.10}$$

in which the coefficients A and B are given by

$$\begin{aligned}A^{\pm}_{l}&=-(2l+1)^{-1}\int_{\xi_1-\epsilon}^{\xi_2+\epsilon}\mathcal{P}^{\pm(\mathrm{II})}_{l}(\xi)\xi^2(\xi+1)^{-2}g^{\pm}_{l}(\xi)d\xi\,,\\B^{\pm}_{l}&=-(2l+1)^{-1}\int_{\xi_1-\epsilon}^{\xi_2+\epsilon}\mathcal{P}^{\pm(\mathrm{I})}_{l}(\xi)\xi^2(\xi+1)^{-2}g^{\pm}_{l}(\xi)d\xi\,.\end{aligned}\tag{3.11}$$

(A small positive constant ϵ only indicates that ξ_1 and ξ_2 are to be included in the integration.)

Consider now a thin current loop with a total zero charge symmetric around the axis $\theta=0,\pi$. The four-current of the loop located at ξ_0, θ_0 has the only nonvanishing component

$$j^{\varphi}=\mathcal{I}\xi[M(\xi+1)]^{-3}(\sin\theta)^{-1}\delta(\xi-\xi_0)\delta(\theta-\theta_0)\,,\tag{3.12}$$

where $\mathcal{I}$ is the total current in the loop as measured in local nonrotating frames along the loop. We shall focus on two special cases: (i) the current loop in the equatorial plane ($\theta_0=\pi/2$), (ii) the infinitesimal current loop around the axis $\theta=0,\pi$, characterized by the magnetic dipole moment $\mathfrak{M}=\mathcal{I}\pi r_0{}^2\sin^2\theta_0$. In the second case, the limit

$\theta_0 \to 0$ is performed in the final expressions (see Ref. 1 for a detailed treatment of such sources in the Schwarzschild and the Kerr backgrounds).

Substituting now from (3.12) into (2.4), and regarding (3.3) and (3.4), we can calculate coefficients A and B given by (3.11). The final results read as follows.

(i) The current loop in the equatorial plane: $A_l^{\pm}$, $B_l^{\pm}$ vanish for l even. For $l=2k+1$, $k=1,2,\ldots$,

$$A^{\pm}_{2k+1} = \mp 8\pi^{3/2}\frac{[(l+2)(l-1)]^{1/2}}{(l+1)(2l+1)}(l^{+2}_{-1})^{1/2}\frac{(-1)^k}{4^k} \times \binom{2k}{k}\frac{\xi_0}{\xi_0+1}\mathcal{P}_l^{\pm(\mathrm{II})}(\xi_0)M\mathcal{I}\,,$$

$$B^{\pm}_{2k+1} = \mp 8\pi^{3/2}\frac{[(l+2)(l-1)]^{1/2}}{(l+1)(2l+1)}(l^{+2}_{-1})^{1/2}\frac{(-1)^k}{4^k} \times \binom{2k}{k}\frac{\xi_0}{\xi_0+1}\mathcal{P}_l^{\pm(\mathrm{I})}(\xi_0)M\mathcal{I}\,,\tag{3.13}$$

$$A_1^{+} = -\frac{4}{\sqrt{3}}\pi^{3/2}\frac{\xi_0}{\xi_0+1}\mathcal{P}_1^{+(\mathrm{II})}(\xi_0)M\mathcal{I}\,,$$

$$B_1^{+} = -\frac{4}{\sqrt{3}}\pi^{3/2}\frac{\xi_0}{\xi_0+1}\mathcal{P}_1^{+(\mathrm{I})}(\xi_0)M\mathcal{I}\,.$$

(ii) The infinitesimal loop with dipole moment $\mathfrak{M}$ around the axis $\theta=0,\pi$:

$$A_l^{\pm} = \mp 4\pi^{1/2}\frac{[(l+2)(l-1)]^{1/2}}{2l+1}(l^{+2}_{-1})^{1/2}\frac{\xi_0}{(\xi_0+1)^3} \times \mathcal{P}_l^{\pm(\mathrm{II})}(\xi_0)(\mathfrak{M}/M)\,,$$

$$B_l^{\pm} = \mp 4\pi^{1/2}\frac{[(l+2)(l-1)]^{1/2}}{2l+1}(l^{+2}_{-1})^{1/2}\frac{\xi_0}{(\xi_0+1)^3} \times \mathcal{P}_l^{\pm(\mathrm{I})}(\xi_0)(\mathfrak{M}/M)\,,\tag{3.14}$$

$$A_1^{+} = -4(\pi/3)^{1/2}\frac{\xi_0}{(\xi_0+1)^3}\mathcal{P}_1^{+(\mathrm{I})}(\xi_0)(\mathfrak{M}/M)\,,$$

$$B_1^{+} = -4(\pi/3)^{1/2}\frac{\xi_0}{(\xi_0+1)^3}\mathcal{P}_1^{+(\mathrm{I})}(\xi_0)(\mathfrak{M}/M)\,.$$

Finally, by calculating all π_f's and $\hat{\pi}_g$'s from (2.5) and (2.6), we can express the perturbations of the electromagnetic field and of the metrics in the form (2.7)–(2.11).

IV. THE REISSNER-NORDSTRÖM BLACK HOLE IN AN ASYMPTOTICALLY UNIFORM MAGNETIC FIELD

Equation (2.2) for $l=1$ odd-parity perturbations with a zero source term admits a simple polynomial solution for arbitrary e and M. The solution

$$\mathcal{P}_1^{+} = K(r^2 - 3e^2 r + 2e^4/Mr)\tag{4.1}$$

is well behaved at the horizon and it leads to the asymptotically uniform magnetic field with intensity H_0 if constant $K=-(4\pi/3)^{1/2}H_0$. By using (4.1), (2.6)–(2.11), and writing constant C in (2.6) as $(4\pi/3)^{1/2}6Ma$, with a being a small parameter, we find the general stationary and axisymmetric $l=1$ odd-parity perturbations, which are well behaved at the horizon, to have the form

$$\begin{aligned}\delta F_{r\varphi} &= -ear^{-2}\sin^2\theta + H_0(r - e^4/Mr^2)\sin^2\theta\,,\\ \delta F_{\theta\varphi} &= 2ear^{-1}\sin\theta\cos\theta\\ &\quad + H_0(r^2 - 3e^2 + 2e^4/Mr)\sin\theta\cos\theta\,,\\ \delta g_{t\varphi} &= -a(2Mr^{-1} - e^2 r^{-2})\sin^2\theta\\ &\quad + 2eH_0(r - e^2/r + e^4/2Mr^2)\sin^2\theta\,.\end{aligned}\tag{4.2}$$

Setting $H_0=0$, the total field and the metric represent a slowly rotating Kerr-Newman black hole (with angular momentum per unit mass equal to a). With $a=0$ [i.e., with zero C entering relation (2.6)], expressions (4.2) give the perturbations describing the Reissner-Nordström black hole immersed in the weak, asymptotically uniform magnetic field of strength H_0.

Now it is clear that our solution (4.2) is not meaningful for arbitrarily large r, because a uniform field in an asymptotically flat spacetime would contain an infinite amount of energy which should bring the spacetime into a closure. Indeed, there exists the exact solution of Ernst[6] describing a Reissner-Nordström black hole immersed in the Melvin universe which is not asymptotically flat. However, if $|H_0 M| \ll 1$, there is a region in the Ernst spacetime in which $r \gg M$, but $|rH_0| \ll 1$. In this region, the space-time is approximately flat and the magnetic field is uniform. Since our approximation is meaningful for all r satisfying the condition $|rH_0| \ll 1$, we expect a relation should exist between our solution and that of Ernst. Now assuming in Ernst's solution that the magnetic field is weak, we obtain [cf. Ernst's Eqs. (4.2)–(4.6)]

$$\delta F_{r\varphi} = H_0 r\sin^2\theta,\quad \delta F_{\theta\varphi} = H_0(r^2 - 3e^2)\sin\theta\cos\theta\,,\qquad \delta g_{t\varphi} = 2eH_0 r\sin^2\theta\,.\tag{4.3}$$

A comparison of (4.3) with (4.2) shows that Ernst's solution does not describe the Reissner-Nordström black hole in Melvin's magnetic universe; it represents the Kerr-Newman (rotating) black hole in that universe. Indeed, by choosing

$$a = -H_0 e^3/M\tag{4.4}$$

in (4.2), one obtains (4.3). Such a result is not so surprising if we remember that Ernst's method of constructing exact solutions of the Einstein-Maxwell equations, when applied to the uncharged Kerr metric in Melvin's universe,[11] yielded a slightly charged Kerr black hole, in fact, the end product of the process of charge accretion discussed by Wald.[12] It is thus tempting to conjecture that Ernst's method, applied to the Reissner-Nordström black hole, leads to the energetically most favorable situation in which the exterior magnetic field is weakened by the magnetic field caused by the appropriate rotation of the Reissner-Nordström black hole.

In fact, Ernst[11] also analyzed Kerr-Newman black holes immersed in Melvin's magnetic universe, and we can make sure that our solution (4.2) with $a = 0$ (supplemented by the field and the metric of the Reissner-Nordström background) coincides with the solution which Ernst considers as the Kerr-Newman balck hole with $a = H_0 e^3/M$, immersed in field H_0.

As is often stated, in general relativity one rarely knows whether there exists an exact solution corresponding to a solution obtained by an approximation method. The connection between our solution and Ernst's solutions in the case of asymptotically uniform magnetic fields gives some additional confidence in the solutions corresponding to the bounded sources of the field as those treated in Sec. III.

V. THE LINES OF FORCE

In order to have a more intuitive picture of the solutions analyzed in the preceding sections, we

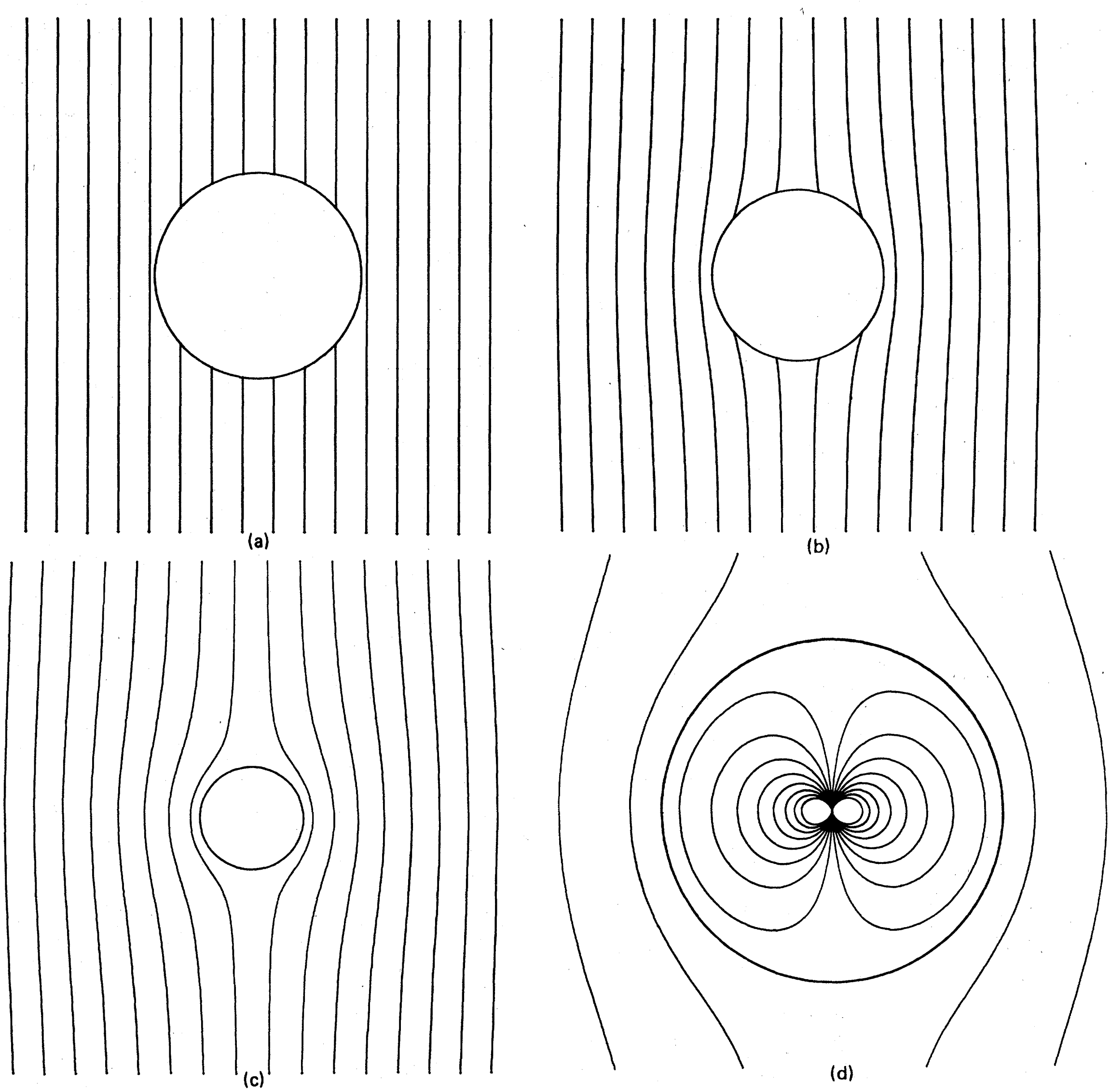

FIG. 1. Magnetic lines of force of a uniform field at infinity modified by the presence of the Reissner-Nordström black hole with (a) $e = 0$, (b) $e = \frac{3}{4}M$, (c) $e = M$. The lines are plotted in Schwarzschild-type coordinates. (d) gives the lines of force inside the hole with $e = M$.

shall construct the lines of force describing the magnetic fields of given sources. Since the magnetic field strength—the tangent to the line of force—is frame dependent, we have to choose the system of observers with respect to which the lines of force will be constructed. Since our resulting metrics are stationary rather than static ($\delta g_{t\varphi} \neq 0$), it is natural to construct the lines of force with respect to Bardeen's locally nonrotating frames (LNRF). We define magnetic lines of force as curves $x^\mu(\lambda)$ satisfying the equation

$$dx^\mu/d\lambda = - {*F^\mu}_\nu U^\nu = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}U_\nu F_{\rho\sigma}\,, \qquad (5.1)$$

where U_ρ is the four-velocity of LRNF, $F_{\alpha\beta}$ is the total electromagnetic field tensor. Since, however, only component $U_t \neq 0$ for LNRF, the background electric field $F^{(0)}_{rt} = e/r^2$ does not enter (5.1). In usual Schwarzschild-type coordinates, definition (5.1) yields

$$dr/d\theta = {*F^r}_t / {*F^\theta}_t = -\delta F_{\theta\varphi}/\delta F_{r\varphi} = rB^{1/2}H_{\hat r}/H_{\hat\theta}\,, \qquad (5.2)$$

where

$$H_{\hat r} = \delta F_{\theta\varphi}/(r^2 \sin\theta), \quad H_{\hat\theta} = -B^{1/2}\delta F_{r\varphi}/(r\sin\theta) \qquad (5.3)$$

are physical components of the magnetic field

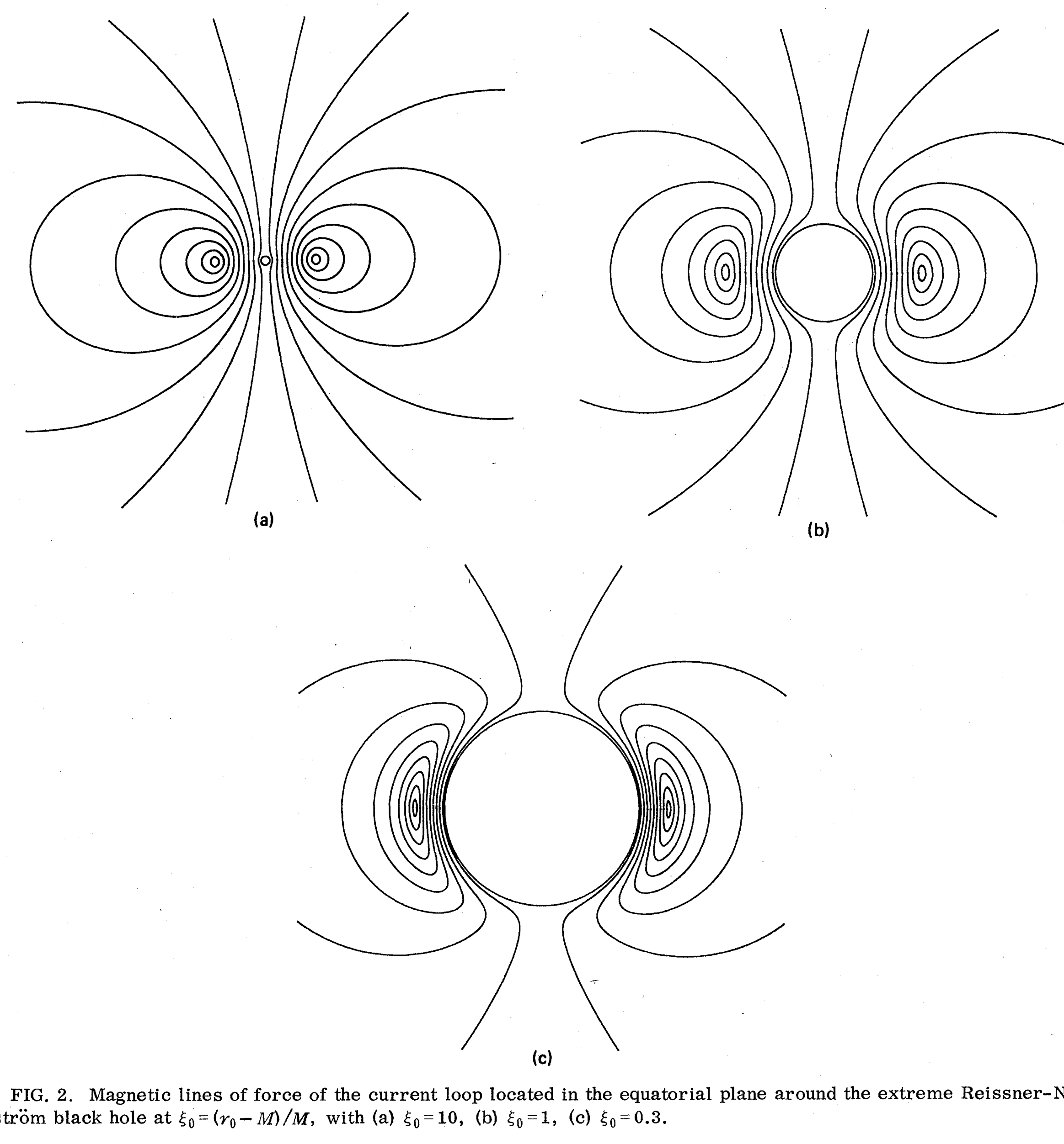

FIG. 2. Magnetic lines of force of the current loop located in the equatorial plane around the extreme Reissner-Nordström black hole at $\xi_0 = (r_0 - M)/M$, with (a) $\xi_0 = 10$, (b) $\xi_0 = 1$, (c) $\xi_0 = 0.3$.

in LNRF. [Notice that (5.2) really leads to the condition $\epsilon_{\hat{i}\hat{j}\hat{k}}H^{\hat{j}}dx^{\hat{k}}=0$ in LNRF.] Our definition of the magnetic lines of force reduces to that of Hanni and Ruffini[13] in their analysis of the Schwarzschild black hole in an asymptotically uniform field. (Of course, one can also consider the magnetic lines of force as tangents to the Lorentz force on a magnetic monopole at rest in LNRF, as Hanni and Ruffini do.)

Now, in the case of the Reissner-Nordström black hole in an asymptotically uniform magnetic field, when the magnetic field is given by (4.2) with $a=0$, Eq. (5.2) can be integrated analytically to yield

$$(r^2-3e^2+2e^4/Mr)\sin^2\theta=\text{const}. \qquad (5.4)$$

The lines of force are graphed in Fig. 1 for the Reissner-Nordström black hole with $e=0$, $e=\frac{3}{4}M$, and $e=M$. Since, in the extreme case, the radial coordinate is a space coordinate also under the horizon, we add the plot of the lines inside the black hole as well [Fig. 1(d)]. The lines intersect the horizon in general; however, as the extreme case is approached, they are pushed out. In the extreme case the magnetic field vanishes at the horizon, as seen from (4.2) and (5.3). We note that an asymptotically uniform magnetic field vanishes also at the horizon of an extremely rotating (Kerr) black holes[12,1] and its flux through

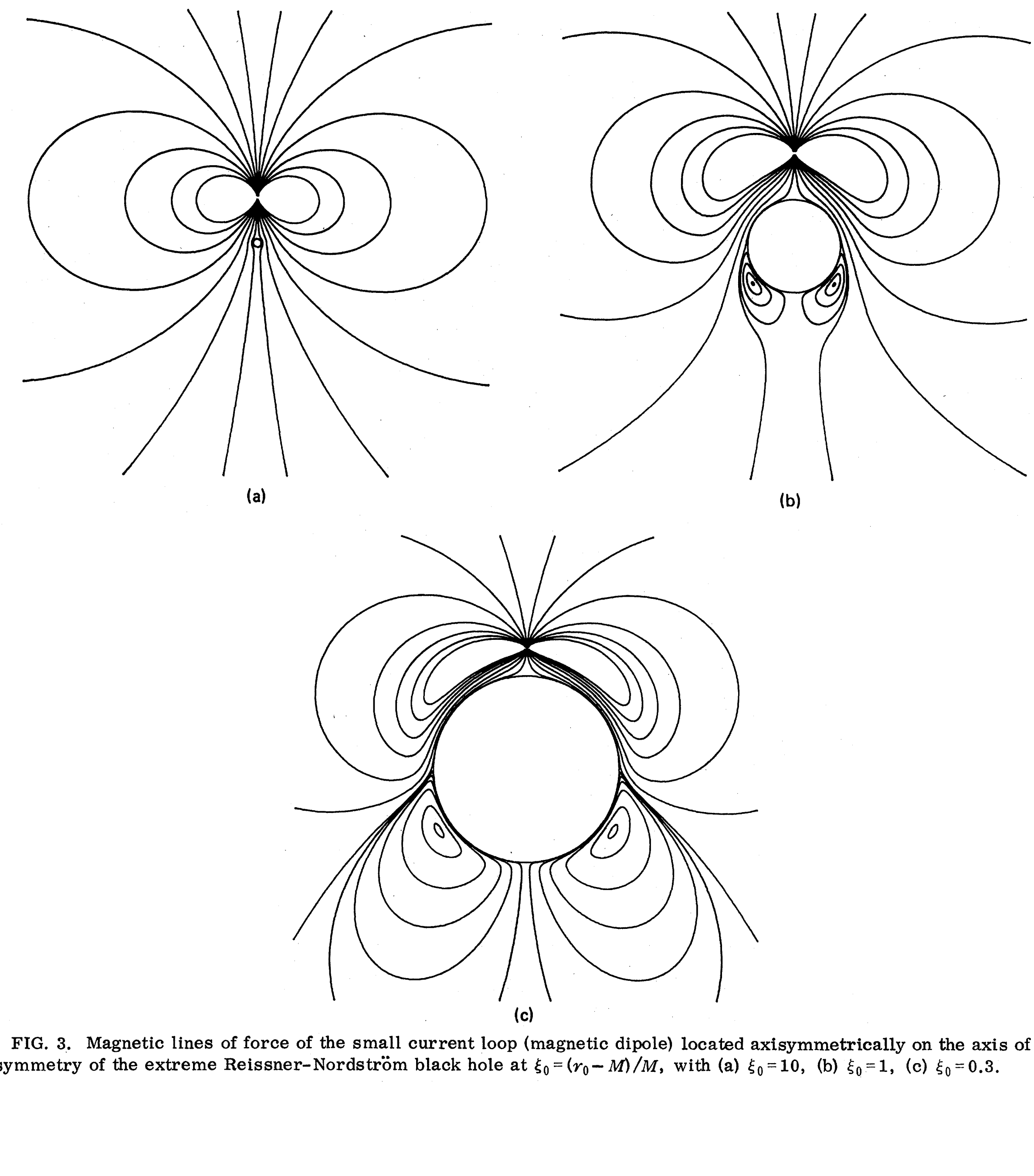

FIG. 3. Magnetic lines of force of the small current loop (magnetic dipole) located axisymmetrically on the axis of symmetry of the extreme Reissner-Nordström black hole at $\xi_0=(r_0-M)/M$, with (a) $\xi_0=10$, (b) $\xi_0=1$, (c) $\xi_0=0.3$.

one hemisphere of the horizon is zero as well.[14] However, it does not vanish at the horizon in the case of Ernst's (linearized) solution given by (4.3). This indicates again that this Ernst solution rather represents a Kerr-Newman black hole in Melvin's universe than a Reissner-Nordström black hole.

We should mention that since the lines of force are plotted in Schwarzschild-type coordinates (r, θ) one has to be careful when interpreting the behavior of the lines at the horizon. For example, the lines do not intersect the horizon perpendicularly; nevertheless, (5.3) implies that $H_{\hat{\theta}}=0$ at the horizon due to factor $B^{1/2}$. Hanni and Ruffini,[13] therefore, draw also an embedded diagram of the (r, θ) surface of the Schwarzschild spacetime; however, it is sufficient to realize that in the Schwarzschild coordinates the length of the radial basis vector as well as that of the magnitude of the magnetic field, get smaller and smaller as the horizon is approached. In any case, LNRF becomes unphysical at the very horizon and performing the boost to a freely falling frame, we find $\tilde{H}_{\hat{\theta}} \neq 0$. The vanishing of the magnetic field at the horizon of the extreme Reissner-Nordström black hole is independent of the choice of the frame in this sense–it is caused by the radial dependence of perturbations (4.2) (with $a=0$).

By using definition (5.2) and the solutions given in Secs. II and III [see (2.5), (2.7), (2.8), (3.5), (3.6), (3.10), (3.13), and (3.14)], magnetic lines of force of current loops in the equatorial plane (Fig. 2) and of small current loops (magnetic dipoles) located on the axis of symmetry (Fig. 3) of the extreme Reisnner-Nordström black hole have been constructed numerically.[15] As with an asymptotically uniform magnetic field, the lines are pushed out from the extreme Reissner-Nordström black hole (that the field vanishes at the horizon can easily be learned from the solutions above). An unexpected structure of the magnetic lines of force appears in the case of a magnetic dipole in the region "opposite" to the place where the dipole is located. This structure involving closed field lines not linking any current source is associated with the manner in which gravitational perturbations constitute, via the background Maxwell tensor, an effective source. The perturbed Maxwell equations

$$(\delta F_{\alpha\sigma})^{;\sigma} = 4\pi\delta j_\alpha + F^{(0)}_{\alpha\rho;\sigma}h^{\rho\sigma} + F^{(0)\rho\sigma}h_{\alpha\rho;\sigma} + F^{(0)\rho}_{\alpha}(h_\rho{}^\sigma{}_{;\sigma} - \tfrac{1}{2}h_\sigma{}^\sigma{}_{;\rho}),$$

where $\delta g_{\rho\sigma} \equiv h_{\rho\sigma}$, imply that there is an effective source of current even in regions devoid of any sources (i.e., $\delta j_\alpha = 0$). Clearly, when the dipole approaches quasistatically a charged black hole, its field decays in a more complicated manner than is the case of all $l \geq 1$ moments of a point charge which approaches a Schwarzschild black hole. We wish to analyze some physical aspects of these fields (as plasma horizons, for example) elsewhere. In a manner similar to that described in the present paper, one can also construct the field caused by rotating loops of uncharged matter around the extreme Reissner-Nordström black hole. The multipole solutions for coupled even-parity perturbations are almost as simple as in the case of odd-parity perturbations.[7] In another paper,[16] the solution for $l=1$ even-parity perturbations is used to analyze the motion of the charged black hole in an asymptotically uniform electric field.

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