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Large nonperturbative effects for small x_E in lepton-proton scattering

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A rigorous nonperturbative approach to radiative corrections in charged-lepton scattering is derived from a rearrangement of the QED perturbation series. The new series is covariant and involves no cutoff. Successive terms, corresponding to an increasing number of "oblique" photons, incorporate the emission of an arbitrary number of collinear photons. The infrared spectrum of an oblique photon is shown to be suppressed. The zero- and one-oblique-photon terms are computed and numerical results for the elastic contribution are presented. In the very inelastic regime, small x_E and large Q_E^2, we find an elastic contribution to the cross section much larger than that predicted by standard perturbation theory. We note that our results may modify somewhat the current picture of scaling violations, at least in the very inelastic regime.

I. INTRODUCTION

An outstanding problem in practical QED radiative corrections is the following: how to reconcile the soft-photon nonperturbative treatment, essential for example in describing radiative tails, with the perturbative calculation of hard-photon emission, believed to be justified. In the usual approach[1,2] to this problem, one turns to some energy cutoff which defines the boundary between hard and soft photons. Since soft-photon contributions, which involve the infrared divergences, exponentiate, one sums up these contributions to all orders, whereas hard-photon effects are computed perturbatively. The drawback of this approach, which was very important in proving cancellation of infrared divergences, is the unphysical, noncovariant, cutoff dependence of the final results. This cutoff dependence is awkward when both soft- and hard-photon effects are important.

The main contribution of this paper is a new, practical scheme for the radiative corrections in charged-lepton scattering. Let us quote what has been already achieved by this work.

(a) The approach is rigorous in the sense that our series is nothing but the QED perturbation series in a rearranged form.

(b) The theory does not involve any cutoff on photon energy.

(c) All infrared divergences are canceled out.

(d) There is no limitation for the applicability of the method.

(e) For very inelastic $\mu+p$ scattering, the *elastic contribution* we obtain using our method is much larger than what is predicted by Mo and Tsai's[3] one-photon-emission formula, which does not properly take into account important higher-order multiphoton emission.

Let us now summarize the basic ideas we rely on and the methods we use in this paper. It is well known that, at high energy, real-photon emission is mainly concentrated along the directions of the charged particles. This fact leads us to separate the emitted photons into *collinear* and *oblique* photons. The important point is that collinear photons *exhaust the real-photon infrared divergences*, and their effects are included *to all order in perturbation theory* through the use of the spectral weight function we have derived in a previous work.[4,5] Oblique photons, whose emission probabilities are given by a set of precise rules, are treated one at a time, that is, perturbatively. Let us note that oblique photons are always accompanied by an arbitrary number of collinear photons. Thus, the radiative tail of any process is automatically included. This is the essence of our rearranged perturbation series.

In Sec. II we present first the guidelines of our approach and then we derive the oblique-photon expansion, in analogy with the method of Grammer and Yennie[2] (GY). In Sec. III, we give the explicit zero- and one-oblique-photon terms for applications to charged-lepton-proton scattering. The physical meaning of an oblique photon is clarified. Section IV is devoted to the analysis of the numerical results for the elastic contribution, arising from zero- and one-oblique-photon terms and to the comparison with standard perturbation-theory results. In Sec. IV, we summarize the results obtained thus far and enumerate refinements of the program which remain to be done. Finally, a short appendix is devoted to the azimuthal integration of the lepton tensor.

II. OBLIQUE-PHOTON EXPANSION SERIES

We present first the motivations of the oblique-photon expansion series and we introduce the necessary ingredients. The second part of this section is devoted to the detailed derivation of this expansion.

A. Guidelines of the approach

In this paper, we concentrate on lepton- (electron or muon) proton scattering, although the generalization of our method to similar processes is straightforward. Also, our considerations will be limited to the one-photon-exchange approximation and we shall neglect all radiative corrections to the hadron vertex. The slight change of notations from Ref. 6 is done for consistency with the more numerous symbols needed in the treatment of oblique-photon emission.

Let p and p' be the momenta of the incident and scattered lepton E, E' the corresponding laboratory energies, and θ the scattering angle. The proton and lepton mass are denoted by M and m, respectively. Note that nowhere is m neglected; useful approximate forms of some formulas, valid at high energy, are preceded by the symbol $\overset{m}{\simeq}$. We denote by $Q_E{}^2$ and x_E the squared momentum transfer and Feynman variable (E is for experimental) defined by

$$Q_E{}^2 = -q_E{}^2 = -(p-p')^2 = 2(EE' - pp'\cos\theta - m^2) \overset{m}{\simeq} 4EE'\sin^2\tfrac{1}{2}\theta \tag{2.1}$$

and

$$x_E = \frac{Q_E{}^2}{2M(E-E')}\,. \tag{2.2}$$

We introduce two positive-energy light-cone momenta l and l', which almost coincide with p and p' at high energy, by

$$l = \frac{p - rp'}{1+r} \overset{m}{\simeq} p\,, \quad l' = \frac{p'-rp}{1+r} \overset{m}{\simeq} p'\,. \tag{2.3}$$

The kinematical variable r, which is such that

$$l^2 = l'^2 = 0\,, \quad l_0 \geq 0\,, \quad l_0' \geq 0\,, \tag{2.4}$$

is given by

$$r = \frac{1-u_B}{1+u_B} \overset{m}{\simeq} \frac{m^2}{Q_E{}^2}\,. \tag{2.5}$$

Here, u_B is the lepton velocity in the Breit frame which is given by

$$u_B = \frac{p_B}{E_B} = \left(\frac{Q_E{}^2}{Q_E{}^2 + 4m^2}\right)^{1/2}. \tag{2.6}$$

Consider now the momentum k of an emitted photon, "drawn" on the positive-energy light cone. When the lepton mass can be neglected, there is a large probability for k to be *collinear* with l or l'; this is of course the well known peaking approximation[7] which emphasizes the p and p' peaks. These peaks arise from $k\cdot p$ and $k \cdot p'$ denominators corresponding to the propagation of the lepton after or before photon emission, respectively.

When the emitted photon is soft, $k_0 \to 0$, k goes down to the light-cone vertex and we encounter the infrared divergence. In this limit, the relative probability density in k space, summed over photon polarizations, is given by

$$-\frac{\alpha}{2\pi^2}(j^\rho j_\rho)\delta_+(k^2)\,, \tag{2.7}$$

where α is the fine-structure constant, j^ρ the classical current

$$j^\rho = \frac{p'^\rho}{k\cdot p'} - \frac{p^\rho}{k \cdot p}\,, \tag{2.8}$$

and

$$\delta_+(k^2) \equiv \delta(k^2)\theta(k_0) = \frac{1}{2\omega_k}\delta(k_0 - k)\,, \tag{2.9}$$

where $\omega_k = |\vec{k}|$.

Let us rephrase some of the results of our work in Ref. 4 on which we rely in this paper. As a starting point, the peaking of the emitted photon at high energy leads us to attach to the emission of one photon the relative probability density

$$\alpha I_1(k) \equiv \alpha\bar{A}\int\frac{d\sigma}{\sigma}\delta^4(k-\sigma l) + \alpha\bar{A}\int\frac{d\sigma'}{\sigma'}\delta^4(k - \sigma' l')\,, \tag{2.10}$$

where $\bar{A} = A/2$ in the notation of Yennie, Frautschi, and Suura (YFS),[1]

$$A = 2\bar{A} = \frac{\omega_k{}^2}{4\pi^2}\int d\Omega_k(-j^\rho j_\rho)\,, \tag{2.11}$$

and, explicitly,

$$\alpha\bar{A} = \frac{\alpha}{\pi}\left(\frac{1+r^2}{1-r^2}\ln r^{-1} - 1\right) \overset{m}{\simeq} \frac{\alpha}{\pi}\left(\ln\frac{Q_E{}^2}{m^2} - 1\right). \tag{2.12}$$

The above considerations allow us to distinguish, among the light-cone photons, *collinear* and *oblique* photons.[8] The important point is that collinear photons exhaust the infrared divergences arising from real-photon emission and their effects can be included to all orders in perturbation theory through the introduction of the spectral weight function

$$E_\lambda(p,p';K) = \left(\frac{\lambda^2 e^{2\gamma}}{Q_E{}^2}\right)^{-\alpha\bar{A}}\Gamma^{-2}(\alpha\bar{A})e^{\alpha F(r)} \times \int_0^\infty \frac{d\sigma\, d\sigma'}{(\sigma\sigma')^{1-\alpha\bar{A}}}\delta^4(K - \sigma l - \sigma' l')\,. \tag{2.13}$$

Here, Γ and γ are Euler's function and constant, $F(r)$ is a normalization function given explicitly in

Ref. 4, and K is the effective momentum of the collinear photons. As we shall see shortly, the photon mass regulator λ which appears in this equation cancels immediately, once virtual radiative corrections are included nonperturbatively. Here, we shall rely on YFS or GY analysis, although we use the vertex function we introduced in Ref. 4. For spacelike momentum transfer, this function reads

$$F_{es} = e^{\alpha \bar{A} \gamma} \Gamma(1+\alpha \bar{A}) e^{\alpha B + \alpha/2\pi}, \tag{2.14}$$

where B is the function defined by YFS. Note, however, that F_{es} is essentially $e^{\alpha B}$ at high energy. Equation (2.14) will be the starting point for virtual radiative corrections in this paper.

As in Ref. 6, we shall introduce a spectral function *free from infrared divergences*, which takes into account the contribution of collinear photons and also part of the virtual-photon effects represented by Eq. (2.14). This function is given by

$$\hat{E}(p,p';K) = E_\lambda(p,p';K) F_{es}^2 = (\alpha\bar{A})^2 e^{\alpha \hat{F}(r)} \int_0^\infty d\sigma \int_0^\infty d\sigma' (\sigma\sigma')^{\alpha\bar{A}-1} \times \delta^4(K - \sigma l - \sigma' l'), \tag{2.15}$$

where

$$\hat{F}(r) = F(r) + 2B + \frac{1}{\pi} - A \ln \frac{\lambda e^\gamma}{Q_E} \overset{m}{\simeq} \frac{1}{\pi} \ln \frac{Q_E}{m} - \frac{\pi}{6}. \tag{2.16}$$

The form of $\hat{F}(r)$ which takes into account the complete lepton-mass dependence is given in Ref. 6.

In some kinematical domains, for example the vicinity of the elastic peak, the emitted photons are necessarily soft owing to phase-space limitations. In this case, the collinear photons alone, which are also soft, represent an excellent approximation to the physical process. When we vary slightly the kinematics in such a way that more phase space opens to the emitted photons, we can still describe the process in terms of collinear photons, provided we include a "hard factor" derived from comparison with standard perturbation theory. A full account of this approach as well as explicit computations using this nonperturbative technique is given in Ref. 6.

For recent high-energy $\mu + p$ experiments[9,10] in the very inelastic regime (small x_E and large Q_E^2), the emitted photons can no longer be considered as soft and the above approximation becomes totally inadequate, at least for the contribution of the elastic peak. In this regime, radiative corrections are very large owing to hard bremsstrahlung, in which the squared momentum t, transferred by the exchanged photon, can be considerably lower than Q_E^2. This process leads to a "t peak" where simple kinematics shows that the minimum value of t is given approximately by $t_{\min} \simeq M^2 x_E^2$. On the other hand, a collinear photon implies $t = t_p$ or $t = t_{p'}$, which are the values of t for the p and p' peaks, respectively. Consequently, as emphasized by Mo and Tsai,[3] there are large discrepancies between the peaking approximation and the exact one-photon-emission formula in the very inelastic region, especially for the elastic contribution. In Ref. 6, we emphasized that in the very inelastic region, hard bremsstrahlung becomes a basic process by itself and, as such, it must be radiatively corrected. In other words, at least for small x_E and large Q_E^2, we must take into account additional photon emission on top of the bremsstrahlung process. These "corrections" can, and indeed do, modify the already large hard-bremsstrahlung cross section. Here, the separation between hard and soft photon is unclear and, in any case, nonperturbative effects are too much involved to be computed by the known techniques. This is not a minor problem since inadequate radiative corrections to the data will give misleading radiatively corrected structure functions.

The above problem is completely solved within our theory. We take into account the contribution of collinear photons to all orders in perturbation theory and, in a precise technical sense, we expand in the number n_σ of oblique photons. The t peak can be reached beginning with $n_\sigma = 1$. Without any cutoff, it turns out that an oblique photon is mainly hard since the soft part of its spectrum, $d\omega/\omega$, is suppressed. As we shall see, the emission of oblique photons is always accompanied by an arbitrary number of collinear photons. In this sense, radiative tails are automatically incorporated in the cross-section formula.

B. Rearrangement of the perturbation series

We shall now derive a rearrangement of the cross-section perturbation series in the one-photon-exchange approximation. Although differing in presentation and objective, this derivation is quite similar to the GY method.[2]

The differential cross section for lepton-proton scattering with emission of n unobserved photons is given by

$$\frac{d\sigma_n}{d\Omega' dE'} = \frac{\alpha^2 p'}{p} \frac{1}{n!} \int \frac{d^4 q}{(q^2)^2} \delta^4\Big(p - q - p' - \sum_{i=1}^n k_i\Big) \prod_{i=1}^n \frac{d^4 k_i \delta_+(k_i^2)}{(2\pi)^3} W^{NR}_{\mu\nu}(P,q) \rho^{\nu\dagger}_{\nu_1 \cdots \nu_n}(p,p';k_1,\ldots,k_n) \times \rho^\mu_{\mu_1 \cdots \mu_n}(p,p';k_1,\ldots,k_n) \epsilon_1^{\mu_1} \cdots \epsilon_n^{\mu_n} \epsilon_1^{\nu_1} \cdots \epsilon_1^{\nu_n}. \tag{2.17}$$

Here $W_{\mu\nu}^{\rm NR}$ is the usual nonradiative (NR) proton structure tensor, ρ^ν is the amplitude for the emission of n photons of momenta k_i and polarization ϵ_i, and q is the momentum of the exchanged photon. For $n=0$ there is of course no k integration. Let us note the following properties of ρ^ν: (a) symmetry in photon momenta, and (b) gauge invariance, leading to the Ward identities

$$k^{\mu_i}\rho^\nu_{\mu_1\cdots\mu_n}(p,p';k_1,\ldots,k_n)=0\,. \tag{2.18}$$

In what follows we take the explicit convention of expressing ρ^ν as a function of $p,p';k_1,\ldots,k_n$, considered as independent variables. This extends its definition outside the support of the δ^4 functions in Eq. (2.17). Equation (2.18) allows us to sum over photon polarization using

$$\sum_{\rm pol}\epsilon_i^\mu\epsilon_i^\nu=-g_i^{\mu\nu} \tag{2.19}$$

and Eq. (2.17) becomes

$$\frac{d\sigma_n}{d\Omega' dE'}=\frac{\alpha^2p'}{pn!}\int\frac{d^4q}{(q^2)^2}\,\delta^4\Big(p-q-p'-\sum_{i=1}^n k_i\Big)W_{\mu\nu}^{\rm NR}(P,q)\times\Big[\prod_{i=1}^n d^4k_i\Big]T^{\mu\nu}_{(n)\lambda}(p,p';k_1,\ldots,k_n)\,. \tag{2.20}$$

The *lepton tensors* $T_{(n)\lambda}$ are simply related to the amplitudes ρ through

$$T^{\mu\nu}_{(n)\lambda}(p,p';k_1,\ldots,k_n)=\rho^{\nu\dagger}_{\nu_1\cdots\nu_n}\,\rho^\mu_{\mu_1\cdots\mu_n}\times\prod_{i=1}^n\frac{[-g^{\mu_i\nu_i}\delta_+(k_i^2)]}{(2\pi)^3}\,. \tag{2.21}$$

The λ index attached to the $T^{\mu\nu}_{(n)\lambda}$ reminds us of the infrared divergence of these tensors which is due to virtual radiative corrections. To get rid of the Lorentz contractions in Eq. (2.20), let us write the standard form of the nonradiative hadron structure tensor,

$$W^{\rm NR}_{\rho\sigma}(P,q)=\frac{1}{M^2}\Big(P_\rho-\frac{\nu}{q^2}q_\rho\Big)\Big(P_\sigma-\frac{\nu}{q^2}q_\sigma\Big)W_2^{\rm NR}(q^2,\nu)-\Big(g_{\rho\sigma}-\frac{q_\rho q_\sigma}{q^2}\Big)W_1^{\rm NR}(q^2,\nu)\,, \tag{2.22}$$

where $\nu=P\cdot q$. Since the lepton tensors verify the following Ward identities,

$$q_\mu T^{\mu\nu}_{(n)\lambda}(p,p';k_1,\ldots,k_n)\delta^4\Big(p-q-p'-\sum k_i\Big)=0\,, \tag{2.23}$$

and also similar identities with $q_\mu\to q_\nu$, we can write Eq. (2.20) as

$$\frac{d\sigma_n}{d\Omega' dE'}=\frac{\alpha^2p'}{pn!}\int\frac{d^4q}{(q^2)^2}\,\delta^4\Big(p-q-p'-\sum k_i\Big)\prod_{i=1}^n d^4k_i\sum_{j=1}^2 W_j^{\rm NR}(q^2,\nu)\,T_{j(n)\lambda}(p,p';k_1,\ldots,k_n) \tag{2.24}$$

with the following definitions:

$$T_{1(n)\lambda}\equiv -g_{\mu\nu}T^{\mu\nu}_{(n)\lambda},\quad T_{2(n)\lambda}\equiv\frac{P_\mu P_\nu}{M^2}T^{\mu\nu}_{(n)\lambda}\,. \tag{2.25}$$

The following analysis is most easily understood by representing $T^{\mu\nu}_{(n)\lambda}$ or $T_{j(n)\lambda}$ as a fermion loop cut on both lines with n photons, also cut, going across. Photons arising from virtual radiative corrections, of course not cut, can be included. Now the basic ingredient in the rearrangement of Eq. (2.24) is Grammer and Yennie's[2] decomposition

$$g_i^{\mu\nu}\delta_+(k_i^2)=C_i^{\mu\nu}+O_i^{\mu\nu}\,, \tag{2.26}$$

with

$$C_i^{\mu\nu}=b(p,p';k_i)k_i^\mu k_i^\nu\,, \tag{2.27}$$

$$O_i^{\mu\nu}=g_i^{\mu\nu}-b(p,p';k_i)k_i^\mu k_i^\nu\,. \tag{2.28}$$

The function b, suitably chosen later, depends on the fermion lines pair (p_e,p_a) emitting and absorbing the photon of momentum k_i. It is precisely this dependence which distinguishes Eq. (2.26) from a trivial gauge transformation.

To sum up the contribution of the collinear (C) photons, we shall use the Ward identities represented in Fig. 1. The graphs include the propagators of the fermion lines. The dashed line represents a gauge photon, that is, a photon with a k^μ polarization vector. After multiplication of these identities by the inverse propagators $S^{-1}(p')$ and $S^{-1}(p)$ and insertion between free-fermion wave functions, the right-hand side of these identities reduces to the second and first term, respectively.

Let us now expand every real photon which contributes to $T_{(n)\lambda}$, in collinear and oblique parts according to Eqs. (2.26)–(2.28), beginning for example by photon n. The oblique contribution of k_n will be noted

$$T_{(n)\lambda}(p,p';k_1,\ldots,k_{n-1},O_n)\,.$$

The contribution of the collinear part can be computed since it is obtained by attaching in all possible ways to $T_{(n-1)\lambda}$, which is proportional to $\rho^\dagger_{(n-1)}\rho_{(n-1)}$, a gauge photon k_n. For a fixed emission point, we can sum up over absorbed photons belonging to *the same line* using Fig. 1 identities and then over emission from the same line, these photons having the same b factor. Thus we get the following identities:

$$T_{(n)\lambda}(p,p',k_1,\ldots,k_n) = T_{(n)\lambda}(p,p';k_1,\ldots,k_{n-1},O_n) + \alpha I(k_n) T_{(n-1)\lambda}(p,p';k_1,\ldots,k_{n-1}), \quad (2.29)$$

where

$$\alpha I(k) = -\frac{e^2}{(2\pi)^3}[b(p,p;k) + b(p',p';k) - 2b(p,p';k)], \quad (2.30)$$

equivalent to the GY identities.[2]

For later reference, let us reduce one more photon. Using twice Eq. (2.30), we obtain

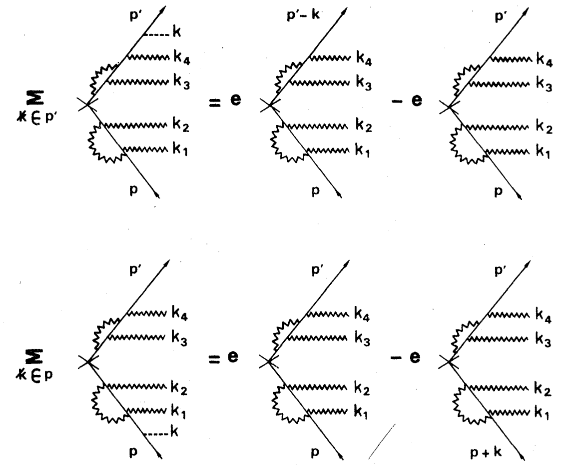

FIG. 1. Ward identities in graphical form. The graphs include the propagators of the external fermion lines.

$$\begin{aligned} T_{(n)\lambda}(p,p';k_1,\ldots,k_n) &= T_{(n)\lambda}(p,p';k_1,\ldots,k_{n-2},O_{n-1},O_n) + \alpha I(k_{n-1}) T_{(n-1)\lambda}(p,p';k_1,\ldots,k_{n-2},\overline{O}_{n-1},O_n) \\ &\quad + \alpha I(k_n) T_{(n-1)\lambda}(p,p';k_1,\ldots,k_{n-2},O_{n-1},\overline{O}_n) \\ &\quad + \alpha I(k_n)\alpha I(k_{n-1}) T_{(n-2)\lambda}(p,p';k_1,\ldots,k_{n-2},\overline{O}_{n-1},\overline{O}_n). \end{aligned} \quad (2.31)$$

The arguments marked with an overbar are in fact missing. We note that every missing argument is transmitted to a factor αI. By iteration, we can reduce all photons in succession with the result

$$T_{(n)\lambda}(p,p';k_1,\ldots,k_n) = \sum_{r=0}^{n} \alpha^r \sum_{1\le i_1<i_2\cdots<i_r\le n} I(k_{i_1})\cdots I(k_{i_r}) T_{(n-r)\lambda}(p,p';O_1,\ldots,\overline{O}_{i_1}\cdots\overline{O}_{i_r}\cdots O_n). \quad (2.32)$$

We return now to Eq. (2.24). Using the Fourier representation of the δ function,

$$\delta^4\Big(p-q-p'-\sum_i k_i\Big) = \int \frac{d^4x}{(2\pi)^4} e^{i(p-q-p')x} \prod_i e^{-ik_i x}, \quad (2.33)$$

and Eq. (2.32), we write Eq. (2.24) as

$$\begin{aligned} \frac{d\sigma_n}{d\Omega'\,dE'} &= \frac{\alpha^2 p'}{p\,n!} \int \frac{d^4x}{(2\pi)^4} e^{i(p-q-p')x} \frac{d^4q}{(q^2)^2} W_j^{\mathrm{NR}}(q^2,\nu) \\ &\quad \times \sum_{r=0}^{n} \alpha^r \sum_{1\le i_1<i_2\cdots<i_r\cdots\le n} \int \Big(\prod_{i=1}^{n} d^4k_i\, e^{-ik_i x}\Big) I(k_{i_1})\cdots I(k_{i_r}) \\ &\quad \times T_{j(n-r)\lambda}(p,p';O_1,\ldots,\overline{O}_{i_1},\ldots,\overline{O}_{i_r},\ldots,O_n) \end{aligned} \quad (2.34)$$

Owing to the symmetry of $T_{j(n-r)}$ in the momenta $k_{i_1},\ldots,k_{i_r}$, all terms of Eq. (2.34) with fixed r give equal contributions. The number of such terms being

$$\frac{n(n-1)\cdots(n+1-r)}{r!} = \frac{n!}{r!\,(n-r)!}, \quad (2.35)$$

Eq. (2.34) becomes

$$\frac{d\sigma_n}{d\Omega'\,dE'} = \frac{\alpha^2 p'}{p} \sum_{r=0}^{n} \frac{\alpha^r}{r!} \int \frac{d^4x}{(2\pi)^4} e^{i(p-q-p')x} \frac{d^4q}{(q^2)^2} W_j^{\mathrm{NR}}(q^2,\nu)[\tilde{I}(x)]^r T^{O}_{j(n-r)\lambda}(p,p';x). \quad (2.36)$$

Here,

$$\tilde{I}(x) = \int d^4k\, I(k)\, e^{-ikx}, \quad (2.37)$$

$$T^{O}_{j(s)\lambda}(p,p';x) = \frac{1}{s!} \int \Big(\prod_{i=1}^{s} d^4k_i\, e^{-ik_i x}\Big) T^{O}_{j(s)\lambda}(p,p';k_1,\ldots,k_n), \quad (2.38)$$

and we have used the abbreviation

$$T^{O}_{j(n-r)\lambda}(p,p';k_1,\ldots,k_n) = T_{j(n-r)\lambda}(p,p';O_1,\ldots,\overline{O}_{i_1},\ldots,\overline{O}_{i_r},\ldots,O_n) \quad (2.39)$$

for the lepton tensor components corresponding to oblique-photon emission. Upon summation over unobserved photons, we get from Eq. (2.36)

$$\frac{d\sigma}{d\Omega' dE'} = \sum_{n=0}^{\infty} \frac{d\sigma_n}{d\Omega' dE'} = \frac{\alpha^2 p'}{p} \int \frac{d^4x}{(2\pi)^4} e^{i(p-q-p')x} \frac{d^4q}{(q^2)^2} W_j^{\rm NR}(q^2,\nu) \tilde{E}_\lambda(p,p';x) \sum_{m=0}^{\infty} \tilde{T}^{\rm O}_{j(m)\lambda}(p,p';x) \qquad (2.40)$$

after permutation of r and n summations. Here, $\tilde{E}_\lambda$ is the spectral weight function[4,5] in x space, defined by

$$\tilde{E}_\lambda(p,p';x) = \exp[\alpha \tilde{I}(x)]\,. \qquad (2.41)$$

Equation (2.40) is more transparent in momentum space. The spectral weight function, which takes care of the emission of an arbitrary number of collinear photons of effective momentum K, being defined by

$$E_\lambda(p,p';K) = \int \frac{d^4x}{(2\pi)^4} e^{iKx} \tilde{E}(p,p',x) = \int \frac{d^4x}{(2\pi)^4} e^{iKx+\alpha\tilde{I}(x)}\,, \qquad (2.42)$$

Eq. (2.40) becomes

$$\frac{d\sigma}{d\Omega' dE'} = \frac{\alpha^2 p'}{p} \int \frac{d^4q}{(q^2)^2} d^4Q\, W_j^{\rm NR}(q^2,\nu) \times E_\lambda(p,p';p-q-p'-Q) \times \sum_{m=0}^{\infty} T^{\rm O}_{j(m)\lambda}(p,p';Q)\,. \qquad (2.43)$$

In the last equation $T^{\rm O}_{j(m)\lambda}(p,p';Q)$ are the inclusive lepton tensor components corresponding to the emission of m oblique photons of total momentum Q which, from Eq. (2.38), are given by

$$T^{\rm O}_{j(m)\lambda}(p,p';Q) = \int \frac{d^4x}{(2\pi)^4} e^{iQx} T^{\rm O}_{j(m)\lambda}(p,p';x) = \int \left(\prod_{i=1}^{m} d^4k_i\right) T^{\rm O}_{j(m)\lambda}(p,p';k_1,\ldots,k_n) \times \delta^4\Big(Q-\sum_{i=1}^{m} k_i\Big)\,. \qquad (2.44)$$

Another form of Eq. (2.43), which uses Eq. (2.44), appears as

$$\frac{d\sigma}{d\Omega' dE'} = \frac{\alpha^2 p'}{p} \sum_{m=0}^{\infty} \int \frac{d^4q}{(q^2)^2} \left[\prod_{i=1}^{m} d^4k_i\right] T^{\rm O}_{j(m)\lambda}(p,p';k_1,\ldots,k_n) \times W_j^{\rm NR}(q^2,\nu) E_\lambda \times E_\lambda\Big(p,p';p-q-p'-\sum_i k_i\Big) \qquad (2.45)$$

At this point, we follow a different course from GY. The b function, which was left arbitrary up to now, is chosen such that $\alpha I(k)$, defined in Eq. (2.30), is now the collinear function of Eq. (2.10):

$$\alpha I_1(k) = \alpha \bar{A} \int \frac{d\sigma}{\sigma} \delta^4(k-\sigma l) + \alpha\bar{A} \int \frac{d\sigma'}{\sigma'} \delta^4(k-\sigma' l')$$

According to Eqs. (2.10) and (2.11), the symmetric form of the b function is then

$$b(p,p;k) = \left[\int d\Omega_k \frac{m^2}{(\hat{k}\cdot p)^2}\right] \int \frac{d\sigma}{2\sigma} \delta^4(k-\sigma l)\,, \qquad (2.46a)$$

$$b(p',p';k) = \left[\int d\Omega_k \frac{m^2}{(\hat{k}\cdot p')^2}\right] \int \frac{d\sigma'}{2\sigma'} \delta^4(k-\sigma' l')\,, \qquad (2.46b)$$

and

$$b(p,p';k) = b(p',p;k) = \left[\int d\Omega_k \frac{p\cdot p'}{(\hat{k}\cdot p)(\hat{k}\cdot p')}\right] \times \tfrac{1}{2}\left[\int \frac{d\sigma}{2\sigma}\delta^4(k-\sigma l) + \int \frac{d\sigma'}{2\sigma'} \delta^4(k-\sigma' l')\right], \qquad (2.46c)$$

where $\hat{k}^\mu = k^\mu/\omega_k$ and the Ω_k integrals have been computed in going from (2.11) to (2.12).

Equation (2.43) or (2.45) must be complemented by similarly analyzing virtual-photon contributions to the lepton tensors. This has been done by Grammer and Yennie and we shall, throughout this paper, rely on their results, except for the replacement of $e^{\alpha B}$ by F_{es} given in Eq. (2.14). F_{es} has the correct threshold behavior for timelike momentum transfer,[4] but $e^{\alpha B}$ and F_{es} differ very little at high energy. Of course, both functions exhaust virtual-photon infrared divergences. When $F_{es}{}^2$ is lumped with E_λ, infrared divergences cancel and Eq. (2.45) can be written in the form

$$\begin{aligned}\frac{d\sigma}{d\Omega' dE'} = {} & \frac{\alpha^2 p'}{p} \int \frac{d^4q}{(q^2)^2} W_j^{\rm NR}(q^2,\nu) T_{j(0)}(p,p') \hat{E}(p,p';p-q-p') \\ & + \frac{\alpha^2 p'}{p} \int \frac{d^4q}{(q^2)^2} d^4k\, W_j^{\rm NR}(q^2,\nu) T^{\rm O}_{j(1)}(p,p';k) \hat{E}(p,p';p-q-p'-k) \\ & + \sum_{n_{\rm O}=2}^{\infty} \alpha^2 \frac{p'}{p} \int \frac{d^4q}{(q^2)^2} \left(\prod_{i=1}^{n_{\rm O}} d^4k_i\right) W_j^{\rm NR}(q^2,\nu) T^{\rm O}_{j(n_{\rm O})}(p,p';k_1,\ldots,k_{n_{\rm O}}) \hat{E}\Big(p,p';p-q-p'-\sum_i k_i\Big)\,, \end{aligned} \qquad (2.47)$$

where $\hat{E}$ is given explicitly in Eq. (2.15) and

$$T^{O}_{j(n)}(p,p';k_1,\ldots,k_n)=T^{O}_{j(n)\lambda}(p,p';k_1,\ldots,k_n)/F_{es}^2 \tag{2.48}$$

are infrared-convergent lepton tensors. To lowest order, every one of these tensors can be computed in the tree approximation. The next order, infrared divergenceless virtual radiative corrections, are known only for the simplest lepton tensor $T_{j(0)}$ from our knowledge of lepton $e-m$ vertices to third order.

The first term of Eq. (2.47), which describes an arbitrary number of collinear photon emission, was discussed in detail in Ref. 6. The next term, corresponding to one oblique photon and an arbitrary number of collinear photons was taken care of approximately. In particular, we have shown that when the t peak is not very conspicuous we can take into account, approximately, the effect of the second term of Eq. (2.47) by multiplying the zero-photon lepton tensor $T_{j(0)}$ by

$$H \overset{m}{\simeq} (1-\sigma+\tfrac{1}{2}\sigma^2)(1+\sigma'+\tfrac{1}{2}\sigma'^2), \tag{2.49}$$

where σ and σ' are the parameters appearing in Eq. (2.15). Strictly speaking, oblique photons are merely corrections to collinearity, especially in the quasielastic domain ($x_E \simeq 1$) where H can be less than one and the second term of Eq. (2.48) is negative.

The above approximation breaks down when the lepton loses a large fraction of its energy in the scattering process. Since $t_{\min}$ has its lowest value when the hadron vertex is elastic, this breakdown occurs first for the elastic contribution. In other words, when the t peak becomes important, we have to compute exactly the first two terms of Eq. (2.47). This is done in the following sections. From a close examination of our results, we shall claim that two-oblique-photon emission, whose contribution could be computed with some labor, is not expected to appreciably modify our results.

III. METHOD OF TREATMENT OF OBLIQUE-PHOTON EMISSION

A. Summary of previous results and kinematics

For later use, let us first present a summary of zero-oblique-photon-emission treatment, discussed in detail in Ref. 6 and represented here by the first term of Eq. (2.47). To lowest order, the lepton tensor corresponding to zero-photon emission is of course well known and is given by

$$T^{\mu\nu}_{(0)}=\tfrac{1}{2}\,\mathrm{Tr}[\gamma^\mu(\not p'+m)\gamma^\nu(\not p+m)]$$
$$=2(m^2-pp')g^{\mu\nu}+2(p^\mu p'^\nu+p^\nu p'^\mu) \tag{3.1}$$

and, from Eq. (2.25),

$$T_{1(0)}(p,p')=2(Q_E^2-2m^2),\quad T_{2(0)}=4EE'-Q_E^2, \tag{3.2}$$

where Q_E^2 is defined in Eq. (2.1). The infrared-convergent parts of higher-order virtual radiative corrections to $T_{j(0)}$ are easily derived from the known expressions of the electric and magnetic lepton vertices which, in our notation, read[4,6]

$$F_e=F_{es}V_H$$
$$\equiv F_{es}\left\{1+\frac{\alpha}{\pi}\left[\frac{(1+r)}{2(1-r)}\ln r^{-1}-1\right]+O(\alpha^2)\right\}, \tag{3.3}$$

$$F_2=F_{es}\frac{2r}{1-r^2}\ln r^{-1}[1+O(\alpha)]. \tag{3.4}$$

Taking these corrections into account, Eq. (3.2) is modified to

$$T_{1(0)}=2(Q_E^2-2m^2)V_H^2(1+rC_1), \tag{3.5}$$

$$T_{2(0)}=(4EE'-Q_E^2)V_H^2(1+rC_2), \tag{3.6}$$

where

$$C_1=\frac{(1+r)}{(1-r)(1-4r+r^2)}\times\left[\frac{\alpha}{\pi}\ln r^{-1}-\frac{1}{8}\left(\frac{\alpha}{\pi}\right)^2\frac{(1+r)}{(1-r)}\ln^2 r\right], \tag{3.7}$$

$$C_2=\frac{(E+E')^2}{4EE'-Q_E^2}\left[\frac{2\alpha}{\pi}\frac{\ln r}{(1-r^2)}+\left(\frac{\alpha}{\pi}\right)^2\frac{\ln^2 r}{4(1-r)^2}\right]. \tag{3.8}$$

We note that these corrections are small compared with those already taken into account via F_{es}.

We shall denote by $\tilde{E}$ and $\tilde{E}'$ the energy components of the four-vectors l and l'. From Eq. (2.3) we get

$$\tilde{E}=\frac{E-rE'}{1+r}\overset{m}{\simeq}E,\quad \tilde{E}'=\frac{E'-rE}{1+r}\overset{m}{\simeq}E'. \tag{3.9}$$

Besides $E=P\cdot p/M$ and Q_E^2, one needs one more "experimental" invariant which is chosen as

$$\nu_E=P\cdot(p-p')=P\cdot(l-l')=M(E-E')$$
$$=M(\tilde{E}-\tilde{E}') \tag{3.10}$$

or, alternatively,

$$w_E^2=(P+p-p')^2=2\nu_E-Q_E^2+M^2. \tag{3.11}$$

Let K be the total momentum carried off by the emitted collinear photons which, from Eq. (2.15), is represented by

$$K=\sigma l+\sigma' l'. \tag{3.12}$$

Without oblique-photon emission, the spacelike momentum transfer u carried off by the exchanged photon is given by

$$u = p - p' - K = l - l' - K = l(1-\sigma) - l'(1+\sigma') . \quad (3.13)$$

The fundamental invariants, which incorporate collinear-photon emission only, are defined as

$$\overline{Q}^2 = -u^2 = Q_E^2(1-\sigma)(1+\sigma') , \quad (3.14a)$$

$$\overline{\nu} = P \cdot u \equiv M\Delta E = \nu_E - \sigma M\tilde{E} - \sigma' M\tilde{E}' , \quad (3.14b)$$

and

$$\overline{w}^2 = (P+u)^2 = M^2 + 2M\Delta E - \overline{Q}^2 = w_E^2 - \sigma A_E - \sigma' A_{E'} + \sigma\sigma' Q_E^2 . \quad (3.14c)$$

Here A_E and $A_{E'}$ are the following useful combinations:

$$A_E = 2M\tilde{E} - Q_E^{\,2} , \quad (3.15)$$

$$A_{E'} = 2M\tilde{E}' + Q_E^2 \quad (3.16)$$

and we have used the equation

$$(l-l')^2 = (p-p')^2 = -Q_E^2 . \quad (3.17)$$

Setting Eq. (2.15) into Eq. (2.47), the zero-oblique-photon contribution to the differential cross section can be written in the form

$$\frac{d\sigma^{n_0=0}}{d\Omega' dE'} = F_0\left[W_2^{n_0=0}(1+rC_2) + 2W_1^{n_0=0}\frac{(Q_E^2 - 2m^2)}{4EE' - Q_E^2}(1+rC_1)\right] . \quad (3.18)$$

Here F_0 is the Mott cross section,

$$F_0 = \alpha^2 \frac{p'}{p}\frac{(4EE' - Q_E^2)}{Q_E^{\,4}} \underset{m}{\simeq} \frac{\alpha^2\cos^2\theta/2}{4E^2\sin^4\theta/2} \quad (3.19)$$

and

$$W^{n_0=0} = (\alpha\overline{A})^2 e^{\alpha\hat{F}(r)} V_H^{\,2}\int_0^\infty d\sigma\int_0^\infty d\sigma' (\sigma\sigma')^{\alpha\overline{A}-1} \times \frac{W^{\rm NR}(\overline{w}^2, \overline{Q}^2)}{(1-\sigma)^2(1+\sigma')^2} , \quad (3.20)$$

where we have used Eq. (3.14a).

In the (σ, σ') plane, the boundary of the integration domain in Eq. (3.20) being defined by

$$\sigma \geq 0 , \quad \sigma' \geq 0 \quad \text{and} \quad \overline{w}^2 \geq M^2 , \quad \overline{\nu} \geq 0 , \quad (3.21)$$

we make the change of variables $(\sigma, \sigma') \to (\overline{w}^2, \tau)$, where τ is a variable which varies between 0 and 1 along a constant $\overline{w}$ line,

$$\sigma = \frac{(w_E^{\,2} - \overline{w}^2)(1-\tau)}{A_E[1 - z(\overline{w}^2)\tau]} , \quad (3.22)$$

$$\sigma' = \frac{(w_E^{\,2} - \overline{w}^2)\tau}{A_{E'}} , \quad (3.23)$$

where

$$z(\overline{w}^2) = \frac{Q_E^{\,2}(w_E^{\,2} - \overline{w}^2)}{A_E A_{E'}} . \quad (3.24)$$

Equation (3.20) results in the form

$$W^{n_0=0} = \frac{(\alpha\overline{A})^2 e^{\alpha\hat{F}(r)}}{(A_E A_{E'})^{\alpha\overline{A}}} V_H^{\,2}\int_0^1 \frac{d\tau}{[\tau(1-\tau)]^{1-\alpha\overline{A}}}\int_{M^2}^{w_E^{\,2}} \frac{d\overline{w}^2 W^{\rm NR}(\overline{w}^2, \overline{Q}^2)}{(w_E^{\,2} - \overline{w}^2)^{1-\alpha A}[1 - z(\overline{w}^2)\tau]^{\alpha\overline{A}}(1-\sigma)^2(1+\sigma')^2} . \quad (3.25)$$

We note that while the exponents of the $\tau = 0$ and $\tau = 1$ singularities are decreased by $\alpha\overline{A}$, that of the $\overline{w}^2 = w_E^{\,2}$ singularity is decreased by αA.

When we consider the elastic contribution to the structure functions, this last equation simplifies. The elastic, nonradiative structure functions are expressed in terms of G_E and G_M, the electric and magnetic proton form factors by the well-known formulas

$$W_{1\,\rm el}^{\rm NR} = M\delta\left(\frac{\overline{w}^2 - M^2}{2}\right)\hat{W}_1(\overline{Q}^2) \equiv M\delta\left(\frac{\overline{w}^2 - M^2}{2}\right)\frac{\overline{Q}^2}{4M^2} G_M^{\,2}(\overline{Q}^2) , \quad (3.26)$$

$$W_{2\,\rm el}^{\rm NR} = M\delta\left(\frac{\overline{w}^2 - M^2}{2}\right)\hat{W}_2(\overline{Q}^2) \equiv M\delta\left(\frac{\overline{w}^2 - M^2}{2}\right)\frac{[G_E^{\,2}(\overline{Q}^2) + G_M^{\,2}(\overline{Q}^2)\overline{Q}^2/(4M^2)]}{[1 + \overline{Q}^2/(4M^2)]} \quad (3.27)$$

For numerical applications in this paper, we shall use the dipole expressions of the proton form factors

$$G_E^{\rm proton} = \frac{G_M^{\rm proton}}{2.793} = \frac{1}{(1 + \overline{Q}^2/0.71\ {\rm GeV}^2)^2} . \quad (3.28)$$

Using Eqs. (3.26) and (3.27) in Eq. (3.25) we get

$$W_{\rm el}^{n_0=0} = \frac{(\alpha\overline{A})^2 e^{\alpha\hat{F}(r)} 2MV_H^{\,2}}{(w_E^{\,2} - M^2)^{1-\alpha A}(A_E A_{E'})^{\alpha\overline{A}}}\int_0^1 \frac{d\tau[\tau(1-\tau)]^{\alpha\overline{A}-1}\hat{W}(\overline{Q}^2)}{[1 - z(M^2)\tau]^{\alpha\overline{A}}(1-\sigma)^2(1+\sigma^2)^2} , \quad (3.29)$$

where z, σ, and σ' are given by Eqs. (3.22)–(3.24) with $\overline{w}^2 = M^2$.

Let us end this summary by noting that a very efficient method for effecting the τ and $\overline{w}$ integrations is the Gauss-Jacobi quadrature described in Ref. 6, which we use frequently in this paper.

B. Practical form of one-oblique-photon emission

We study now the second term of Eq. (2.47). The lepton tensor corresponding to one-oblique-photon emission is known only in the tree approximation. Higher-order infrared-convergent virtual radiative corrections, not known for the moment, are expected to give small corrections to the cross section we shall compute, in analogy with the case of the zero-oblique-photon cross section we have presented [see Eqs. (3.3)–(3.8) above]. The reason is that these corrections, unlike additional photon emission, do not modify the kinematics of the basic process; they are neglected throughout this paper.

The ordinary lepton tensor, corresponding to one-photon emission, is given by

$$T^{\mu\nu}_{(1)}=\frac{e^2}{(2\pi)^3}\,\delta_+(k^2)\,\tfrac{1}{2}\sum_{(\epsilon)}\mathrm{Tr}\left\{(\not p'+m)\left[\frac{\not\epsilon(\not p'+\not k+m)\gamma^\mu}{2k\cdot p'}-\frac{\gamma^\mu(\not p-\not k+m)\not\epsilon}{2k\cdot p}\right]\right.$$
$$\left.\times(\not p+m)\left[\frac{\gamma^\nu(\not p'+\not k+m)\not\epsilon}{2k\cdot p'}-\frac{\not\epsilon(\not p-\not k+m)\gamma^\nu}{2k\cdot p}\right]\right\},\tag{3.30}$$

and, explicitly, reads[3,6]

$$T^{\mu\nu}_{(1)}=\frac{e^2}{(2\pi)^3}\,\delta_+(k^2)\{2(-j^2)[(m^2-p\cdot p')g^{\mu\nu}+(p^\mu p'^\nu+p^\nu p'^\mu)]+R^{\mu\nu}_{(1)}\},\tag{3.31}$$

where

$$R^{\mu\nu}_{(1)}=-2g^{\mu\nu}\left[(2p\cdot p'+m^2)\left(\frac{1}{k\cdot p'}-\frac{1}{k\cdot p}\right)+m^2\left(\frac{k\cdot p'}{(k\cdot p)^2}-\frac{k\cdot p}{(k\cdot p')^2}\right)+\frac{k\cdot p}{k\cdot p'}+\frac{k\cdot p'}{k\cdot p}\right]$$
$$+2(p^\mu p'^\nu+p^\nu p'^\mu)\left(\frac{1}{k\cdot p'}-\frac{1}{k\cdot p}\right)+\frac{4p'^\mu p'^\nu}{k\cdot p'}-\frac{4p^\mu p^\nu}{k\cdot p}-2\left(\frac{p'^\mu k^\nu+p'^\nu k^\mu}{k\cdot p}\right)\left(\frac{p\cdot p'}{k\cdot p'}-\frac{m^2}{k\cdot p}-1\right)$$
$$+2\left(\frac{p^\mu k^\nu+p^\nu k^\mu}{k\cdot p'}\right)\left(\frac{p\cdot p'}{k\cdot p}-\frac{m^2}{k\cdot p'}+1\right)-\frac{4m^2k^\mu k^\nu}{(k\cdot p)(k\cdot p')}\tag{3.32}$$

and

$$-j^2=\frac{2p\cdot p'}{(k\cdot p)(k\cdot p')}-\frac{m^2}{(k\cdot p)^2}-\frac{m^2}{(k\cdot p')^2}\,.\tag{3.33}$$

It is well known that the classical current term, proportional to $(-j^2)$, leads to an infrared divergence while the "remainder" R gives an infrared-finite contribution. From Eqs. (2.29), (2.10), and (3.1) the corresponding lepton tensor is given by

$$T^{O\mu\nu}_{(1)}=T^{\mu\nu}_{(1)}-[2(m^2-p\cdot p')g^{\mu\nu}+2(p^\mu p'^\nu+p^\nu p'^\mu)]\left[\alpha\bar A\int\frac{d\sigma_1}{\sigma_1}\,\delta^4(k-\sigma_1 l)+\alpha\bar A\int\frac{d\sigma_1'}{\sigma_1'}\,\delta^4(k-\sigma_1' l')\right].\tag{3.34}$$

The physical meaning of this equation is obvious. The "counterterm" $\alpha I_1(k)\,T^{\mu\nu}_{(0)}$, which is subtracted from $T^{\mu\nu}_{(1)}$, avoids double counting of the p and p' peaks and, as we shall see shortly, cancels the infrared divergence. To be rigorous, we may assume temporarily that the W functions are arbitrary test functions and use them to control the infrared divergence of the separate terms of Eq. (3.34).

Upon contraction of $T^{\mu\nu}_{(1)}$ according to Eq. (2.25), the oblique-lepton tensor components read

$$T^{O}_{1(1)}=2(Q_E{}^2-2m^2)\left[\frac{\alpha(-j^2)}{(2\pi^2)}\,\delta_+(k^2)-\alpha\bar A\int\frac{d\sigma_1}{\sigma_1}\,\delta^4(k-\sigma_1 l)-\alpha\bar A\int\frac{d\sigma_1'}{\sigma_1'}\,\delta^4(k-\sigma_1' l')\right]+\frac{\alpha}{(2\pi^2)}\,\delta_+(k^2)R_{1(1)}\,,\tag{3.35a}$$

$$T^{O}_{2(1)}=(4EE'-Q_E^2)\left[\frac{\alpha(-j^2)}{(2\pi^2)}\,\delta_+(k^2)-\alpha\bar A\int\frac{d\sigma_1}{\sigma_1}\,\delta^4(k-\sigma_1 l)-\alpha\bar A\int\frac{d\sigma_1'}{\sigma_1'}\,\delta^4(k-\sigma_1' l')\right]+\frac{\alpha}{(2\pi^2)}\,\delta_+(k^2)R_{2(1)},\tag{3.35b}$$

where

$$R_{1(1)}=8p\cdot p'\left(\frac{1}{k\cdot p'}-\frac{1}{k\cdot p}\right)+4\left(\frac{k\cdot p}{k\cdot p'}+\frac{k\cdot p'}{k\cdot p}\right)+4m^2\left[\frac{k\cdot p'}{(k\cdot p)^2}-\frac{k\cdot p}{(k\cdot p')^2}+\frac{1}{k\cdot p'}-\frac{1}{k\cdot p}\right]\tag{3.36a}$$

and

$$R_{2(1)} = -2\left[(2p\cdot p'+m^2)\left(\frac{1}{k\cdot p'}-\frac{1}{k\cdot p}\right)+m^2\left(\frac{k\cdot p'}{(k\cdot p)^2}-\frac{k\cdot p}{(k\cdot p')^2}\right)+\frac{k\cdot p}{k\cdot p'}+\frac{k\cdot p'}{k\cdot p}\right]$$
$$-4EE'\left(\frac{1}{k\cdot p}-\frac{1}{k\cdot p'}\right)-\frac{4E^2}{k\cdot p}+\frac{4E'^2}{k\cdot p'}-4E'\frac{P\cdot k}{M}\left(\frac{p\cdot p'}{(k\cdot p)(k\cdot p')}-\frac{m^2}{(k\cdot p)^2}-\frac{1}{k\cdot p}\right)$$
$$+4E\frac{P\cdot k}{M}\left(\frac{p\cdot p'}{(k\cdot p)(k\cdot p')}-\frac{m^2}{(k\cdot p')^2}+\frac{1}{k\cdot p'}\right)-\frac{4m^2(P\cdot k)^2}{M^2(k\cdot p)(k\cdot p')}. \tag{3.36b}$$

Using again Eq. (2.15), the one-oblique-photon contribution to the differential cross section, the second term of Eq. (2.47), is

$$\frac{d\sigma^{n_O=1}}{d\Omega'\,dE'}=\frac{\alpha^2 p'}{p'}(\alpha\bar{A})^2 e^{\alpha\hat{F}(r)}\int_0^\infty d\sigma\int_0^\infty d\sigma'\,(\sigma\sigma')^{\alpha\bar{A}-1}\int\frac{d^4k}{(q^2)^2}\,W_j^{\rm NR}\,T^{O}_{j(1)}, \tag{3.37}$$

where now

$$q=u-k=l(1-\sigma)-l'(1+\sigma')-k. \tag{3.38}$$

Using Eq. (3.34), we can bring Eq. (3.37) to the form

$$\frac{d\sigma^{n_O=1}}{d\Omega'\,dE'}=\frac{\alpha^2p'}{p}(\alpha\bar{A})^2e^{\alpha\hat{F}(r)}\int\frac{d\sigma\,d\sigma'}{(\sigma\sigma')^{1-\alpha\bar{A}}}\left[\int\frac{d^4k}{(q^2)^2}\,W_j^{\rm NR}\,T_{j(1)}-\alpha\bar{A}\int\frac{d\sigma_1}{\sigma_1}\,\frac{W_j^{\rm NR}T_{j(0)}}{(q^2)^2}\bigg|_{k=\sigma_1 l}-\alpha\bar{A}\int\frac{d\sigma_1'}{\sigma_1'}\,\frac{W_j^{\rm NR}T_{j(0)}}{(q^2)^2}\bigg|_{k=\sigma_1' l'}\right], \tag{3.39}$$

where $T_{j(1)}$ are the invariants formed from the ordinary lepton tensor $T^{\mu\nu}_{(1)}$. It is convenient to present the last equation in a form similar to Eq. (3.18):

$$\frac{d\sigma^{n_O=1}}{d\Omega'dE'}=F_0\left[W_2^{n_O=1}+2W_1^{n_O=1}\,\frac{(Q_E^2-2m^2)}{4EE'-Q_E^2}\right], \tag{3.40}$$

where

$$W_j^{n_O=1}=(\alpha\bar{A})^2\,e^{\alpha\hat{F}(r)}\int_0^\infty d\sigma\,d\sigma'\,(\sigma\sigma')^{\alpha\bar{A}-1}\left[T_{j(0)}{}^{-1}\int\frac{d^4k}{(q^2/Q_E^2)^2}\,W_j^{\rm NR}T_{j(1)}-\alpha\bar{A}\int\frac{d\sigma_1}{\sigma_1}\,\frac{W_j^{\rm NR}}{(q^2/Q_E^2)^2}\bigg|_{k=\sigma_1 l}-\alpha\bar{A}\int\frac{d\sigma_1'}{\sigma_1'}\,\frac{W_j^{\rm NR}}{(q^2/Q_E^2)^2}\bigg|_{k=\sigma_1' l'}\right]. \tag{3.41}$$

Here, $T_{j(0)}$ are given in Eq. (3.2) and there is no summation over j in the right-hand side of this last equation.

The collinear or (σ,σ') integration is the same as in part A of this section. Thus, we need discuss only the k integral and the counterterms for a fixed (σ,σ') point. Since $W_j^{\rm NR}$ are functions of w^2, the squared mass produced at the hadron vertex, and t, the squared momentum carried off by the exchange photon, it is natural to use these quantities as integration variables in Eq. (3.41). In the laboratory frame with $\vec{u}$ as a polar axis, $\omega\equiv\omega_L$ and $x_k=\hat{u}\cdot\hat{k}$, these variables are given by

$$w^2=(P+q)^2=(P+u-k)^2=\bar{w}^2-2\omega(M+\Delta E-ux_k) \tag{3.42}$$

and

$$t=-q^2=-(u-k)^2=\bar{Q}^2+2\omega(\Delta E-ux_k). \tag{3.43}$$

Here, ΔE is the energy component of u.

$$\Delta E\equiv u_0=\tilde{E}(1-\sigma)-\tilde{E}'(1+\sigma') \tag{3.44}$$

and

$$u=|\vec{u}|=[\bar{Q}^2+(\Delta E)^2]^{1/2}. \tag{3.45}$$

From Eqs. (3.42) and (3.43), we note that the energy ω of the oblique photon is given by

$$\omega=\frac{\bar{Q}^2-t+\bar{w}^2-w^2}{2M}. \tag{3.46}$$

For the counterterms in Eq. (3.41), we shall use w^2 instead of σ_1 or σ_1' variables. Using Eq. (3.42), we get

$$w^2|_{k=\sigma_1 l}=(P+u-\sigma_1 l)^2=\bar{w}^2-2\sigma_1(M\tilde{E}+u\cdot l)=\bar{w}^2-\sigma_1[2M\tilde{E}-Q_E^2(1+\sigma')], \tag{3.47a}$$

$$
\begin{aligned}
w^2|_{k=\sigma_1' l'} &= (P+u-\sigma_1' l')^2 \\
&= \bar{w}^2 - 2\sigma_1'(M\tilde{E}' - u\cdot l') \\
&= \bar{w}^2 - \sigma_1'[2M\tilde{E}' + Q_E^2(1-\sigma)]\,. \qquad (3.47\text{b})
\end{aligned}
$$

The corresponding t variables represent the value of t for the $p(k\,\|\,l)$ and $p'(k\,\|\,l')$ peaks, with collinear photon emission incorporated. Using Eqs. (3.43) and (3.47) we obtain

$$
\begin{aligned}
t_{p(1)}(w^2) \equiv t|_{k=\sigma_1 l} &= -(u-\sigma_1 l)^2 \\
&= \bar{Q}^2 - \frac{Q_E^2(1+\sigma')(\bar{w}^2 - w^2)}{2M\tilde{E} - Q_E^2(1+\sigma')}
\end{aligned}
\qquad (3.48\text{a})
$$

and

$$
\begin{aligned}
t_{p'(1)}(w^2) \equiv t|_{k=\sigma_1' l'} &= -(u-\sigma_1' l')^2 \\
&= \bar{Q}^2 + \frac{Q_E^2(1-\sigma)(\bar{w}^2 - w^2)}{2M\tilde{E}' + Q_E^2(1-\sigma)}
\end{aligned}
\qquad (3.48\text{b})
$$

After the change of variables defined by Eqs. (3.42) and (3.43) is effected in the first term of Eq. (3.41), we are left with the azimuthal integration which is analytically effected in the Appendix. Defining

$$\frac{e^2}{(2\pi)^3}\,\delta_+(k^2)\,\mathcal{T}_{j(1)} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, T_{j(1)} \qquad (3.49)$$

and, similarly [compare Eq. (3.31)]

$$\mathcal{R}_{j(1)} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi R_{j(1)}, \qquad (3.50)$$

Eq. (3.41) becomes

$$W_j^{n_O=1} = (\alpha\bar{A})^2 e^{\alpha\hat{F}(r)} \int_0^\infty \frac{d\sigma\, d\sigma'}{(\sigma\sigma')^{1-\alpha A}} \int_{M^2}^{\bar{w}^2} dw^2\, \mathcal{S}_j\,. \qquad (3.51\text{a})$$

Here

$$
\begin{aligned}
\mathcal{S}_j = {}& \frac{\alpha Q_E^4 T_{j(0)}^{-1}}{8\pi u M}\int_{t_m(w^2)}^{t_M(w^2)} dt\, \frac{W_j^{\rm NR}(w^2,t)\,\mathcal{T}_{j(1)}}{t^2} \\
&- \frac{\alpha\bar{A}Q_E^4}{\bar{w}^2 - w^2}\left[\frac{W_j^{\rm NR}(w^2,t_{p(1)})}{t_{p(1)}^2} + \frac{W_j^{\rm NR}(w^2,t'_{p(1)})}{t_{p'(1)}^2}\right]
\end{aligned}
\qquad (3.51\text{b})
$$

and the integration limits on the t variable result from Eqs. (3.42) and (3.43) and read

$$t_{m,M} = \bar{Q}^2 + \frac{(\bar{w}^2 - w^2)(\Delta E \mp u)}{M + \Delta E \mp u}\,. \qquad (3.52)$$

Consequently, the integration domain for the first term in Eq. (3.51b) is the ABC triangle of Fig. 2. Let us note that, in our change of variables, the origin of the light cone has been transformed into point C, while the $k=\sigma_1 l$ and $k=\sigma_1' l'$ lines are

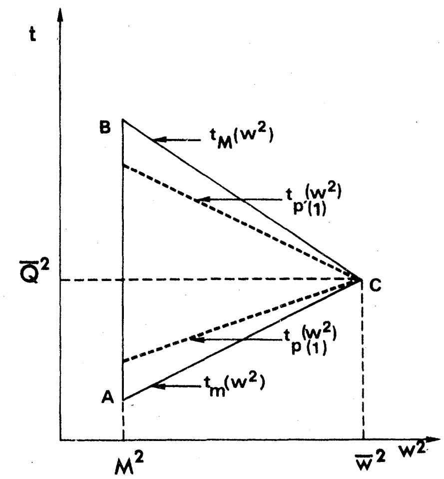

FIG. 2. Integration triangle in the (w^2,t) plane for oblique-photon emission (schematic). The lines $t_{p(1)}$ and $t_{p'(1)}$ correspond to the collinear counterterms supports.

represented by the lines $t_{p(1)}$ and $t_{p'(1)}$, respectively.

To demonstrate the absence of infrared divergence and also to gain physical insight into Eq. (3.51), we make the decomposition introduced in Eqs. (3.35) and (3.36) in which we separate the classical current contribution, which is of order ω^{-2}, from the remainder R which is of order ω^{-1} or higher. Furthermore, we effect on the integral representation of $\alpha\bar{A}$ (Eq. 2.11) the azimuthal-angle integration (in the laboratory frame with $\vec{u}$ as the polar axis) and we change the $x_k = \cos\theta_k$ variable to t. The result from Eq. (3.51b) is

$$
\begin{aligned}
\mathcal{S}_j = \frac{\alpha Q_E^2}{8\pi u M}\int_{t_m(w^2)}^{t_M(w^2)} dt \Bigg\{ & \frac{W_j^{\rm NR}(w^2,t)\,\mathcal{R}_{j(1)}}{t^2 T_{j(0)}} \\
&+ \frac{1}{\omega^2}\left(2p\cdot p' X - \frac{m^2 a}{S^3} - \frac{m^2 a'}{S'^3}\right) \\
&\times\left[\frac{W_j^{\rm NR}(w^2,t)}{t^2} - \frac{W_j^{\rm NR}(w^2,t_{p(1)})}{2t_{p(1)}^2} - \frac{W_j^{\rm NR}(w^2,t_{p'(1)})}{2t_{p'(1)}^2}\right]\Bigg\}.
\end{aligned}
\qquad (3.53)
$$

The combination $2p\cdot p' X - m^2a/S^3 - m^2a'/S^2$ is the result of the azimuthal integration and is given explicitly in the Appendix, while ω is given in Eq. (3.46).

By noting that the quantity in the square brackets of Eq. (3.53) vanishes at point C of Fig. 2, Eq. (3.53) demonstrates the infrared finiteness of the one-oblique-photon contribution and clarifies the role played by the counterterms. This is a basic result of our theory and has important corollaries.

In particular, it implies that the oblique-photon $d\omega/\omega$ spectrum is suppressed and thus, in practice, an oblique photon is automatically a hard photon without the need to recourse to any cutoff. Note that the converse is not true: collinear photons may also be hard. If the experimental variables are such that only the soft domain of phase space is available to the emitted photons, as it is case for $x_E \simeq 1$, the oblique-photon contribution is expected to become negligible.

Using the change of variables defined by Eqs. (3.22) and (3.23), Eq. (3.51a) becomes

$$W_j^{n_O=1} = (\alpha\bar{A})^2 e^{\alpha\hat{F}(r)} \int_{M^2}^{w_E^2} \frac{d\bar{w}^2}{(w_E^2 - \bar{w}^2)^{1-\alpha A}} \int_0^1 d\tau \frac{[A_E A_{E'} - \tau Q_E^2(w_E^2 - w^2)]^{-\alpha\bar{A}}}{[\tau(1-\tau)]^{1-\alpha\bar{A}}} \times \int_{M^2}^{\bar{w}^2} dw^2\, \mathcal{S}_j , \tag{3.54}$$

where $\mathcal{S}_j$ is given by Eq. (3.51b) or (3.53). On this last equation, we see that the one-oblique-photon contribution involves, for a continuous spectrum, a fourfold integral whereas the zero-oblique-photon contribution, given by Eqs. (318) and (3.25), involves a twofold integral only. The delicate integrations over $\bar{w}^2$ and τ are very efficiently effected using the Gauss-Jacobi integration method discussed and used already in Ref. 6. Here, we shall quote the formulas permitting the fixing of the notations.

Suppose we have to compute the integral

$$I(f;\alpha,\beta) \equiv B(\alpha,\beta)2^{-\alpha-\beta} \int_{-1}^{+1} dx\,(1-x)^{\alpha}(1+x)^{\beta} f(x) , \tag{3.55}$$

where $B(\alpha,\beta)$ is Euler's B function and the normalization is such that

$$I(1;\alpha,\beta) = 2. \tag{3.56}$$

The approximation for "degree" n to this integral is given by

$$I(f;\alpha,\beta) = \sum_{\nu=1}^{n} C_{\nu n}(\alpha,\beta) f(x_{\nu n}) , \tag{3.57}$$

where $x_{\nu n}$ are the roots of $P_n^{(\alpha,\beta)}(x)$, the Jacobi polynomial of degree n, and $C_{\nu n}$ the corresponding weights. These polynomials are orthogonal on the $[-1, 1]$ interval with weight function $(1-x)^{\alpha} \times (1+x)^{\beta}$. When $\alpha = \beta = 0$, we recover the well-known Gauss integration method where $P_n^{(\alpha,\beta)}$ reduces to a Legendre polynomial. In this case, $x_{\nu n}$ and $C_{\nu n}$ can be computed once and for all and are available in many computer libraries. Suppose now that $\alpha \to -1 + \alpha A$ and β is zero. In this case, the weight $(1-x)^{-1+\alpha A}$ is concentrated in the vicinity of $x=1$ and, if the function f is smooth enough, the main contribution to the integral comes from the vicinity of $x=1$. This fact is taken into account automatically by this method since, in this case, there is a root $x_{1n} = 1 - O(\alpha A)$ whose weight is $C_{1n} = 2 - O(\alpha A)$ out of a total weight of 2. A similar remark holds if $\alpha = \beta \to -1 + \alpha\bar{A}$. As for the Gauss method, the rapidity of convergence of the Gauss-Jacobi method depends on the smoothness of the function f; for a given f, the convergence is accelerated when αA *decrease*. Finally, to effect the $\bar{w}$ integration in Eq. (3.54) or (3.25) we have, for simplicity, scaled the whole interval $[M, w_E]$ to $[-1, 1]$. Of course, when $w_E \gg M$, one may avoid the computation of roots and weights of Jacobi polynomials of large degree by splitting the $\bar{w}$ integrals into two (or more) parts and using, after the appropriate scaling, the Gauss-Jacobi method only for the interval involving w_E.

Equations (3.25) and (3.54), which give, respectively, the zero- and one-oblique-photon contribution to the structure functions, summarize the results of this section.

IV. ELASTIC-CONTRIBUTION RESULTS

In this section, we present and discuss some numerical results illustrating the physical content of our theory. Generally speaking the relative order of magnitude of the zero- and one-oblique-photon contributions varies rapidly, at high energy, with the experimental point (x_E, Q_E^2) under consideration. For quasielastic scattering, $x_E \simeq 1$, we know already that the zero-oblique-photon process is dominant, since the emitted photons are necessarily soft. On the other hand, the one-oblique-photon process is expected to dominate in the very inelastic regime, for small x_E and large Q_E^2. Comparing Eqs. (3.25) and (3.54), we note that while the $(\tau, \bar{w}^2)$ or (σ, σ') integrations are common to both contributions, the one-oblique-photon process involves two more integrations delimited by the ABC triangle of Fig. 2. Of course, one may consider that Eq. (3.25) involves also the (t, w^2) integration, provided a factor

$$\delta(w^2 - \bar{w}^2)\,\delta(t - \bar{Q}^2)$$

is incorporated in this equation. This remark points to the fact that the zero-oblique-photon contribution is concentrated at point C of Fig. 2, while the leading contribution to the one- (or more) oblique photon arises from the vicinity of the line AC, that is for $t \sim t_m$, where t_m is given by Eq. (3.52). Using Eqs. (3.14c) and (3.45), we get after some elementary algebra

$$t_m(w^2)=\frac{\overline{Q}^4+(w^2-M^2)\overline{Q}^2(\Delta E+u)/M}{(u+\Delta E)^2[1+(\Delta E-u)/M]}\,. \tag{4.1}$$

The lowest value of t_m, $t_m(M^2)$ is attained at point A for the elastic contribution

$$t_m(M^2)=\frac{\overline{Q}^4}{(u+\Delta E)^2[1+(\Delta E-u)/M]}\sim\frac{\overline{Q}^4}{4(\Delta E)^2} \tag{4.2}$$

and the indicated approximation holds for $(\Delta E)^2 \gg \overline{Q}^2$. For $\sigma=\sigma'=0$, we see that the minimum value of t is

$$t_{\min}\sim M^2 x_E^2\,, \tag{4.3}$$

which is very small in comparison with Q_E^2, in the very inelastic regime. For the rest of this paper, we shall discuss numerically only the elastic contribution, arising from zero- and one-oblique-photon terms. In Ref. 6, we have presented numerical examples of the zero-oblique-photon contribution for continuous spectra and we intend to compare the corresponding one-oblique-photon contribution elsewhere.

For the elastic contribution, the zero-oblique-photon term is given by Eq. (3.29). The corresponding one-oblique-photon term is a particular case of Eqs. (3.54) and (3.53) or (3.51b) where $W_j^{\rm NR}(w^2,t)=2M\delta(w^2-M^2)\hat{W}_j(t)$ and we get

$$W_{j\,\rm el}^{n_O=1}=(\alpha\overline{A})^2 e^{\alpha\hat{F}(\tau)}\int_{M^2}^{w_E^2}\frac{d\overline{w}^2}{(w_E^2-\overline{w}^2)^{1-\alpha A}}\int_0^1 d\tau\,\frac{[A_E A_{E'}-\tau Q_E^2(w_E^2-\overline{w}^2)]^{-\alpha\overline{A}}}{[\tau(1-\tau)]^{1-\alpha\overline{A}}}\times\left\{\frac{\alpha Q_E^4}{4\pi u T_{j(0)}}\int_{t_m}^{t_M}dt\,\frac{\hat{W}_j(t)\,\mathcal{T}_{j(1)}}{t^2}-\frac{2M\alpha\overline{A}Q_E^4}{\overline{w}^2-M^2}\left[\frac{\hat{W}_j(t_{p(1)})}{t'^2_{p(1)}}+\frac{\hat{W}_j(t_{p'(1)})}{t^2_{p'(1)}}\right]\right\}. \tag{4.4}$$

Here, $t_{p(1)}$ and $t_{p'(1)}$ are given by Eqs. (3.48) while the $\mathcal{T}_j$ are given in the Appendix and, in all quantities we must put $w^2=M^2$.

Table I represents a small fraction of our numerical results for the elastic contribution to $\mu+p$ scattering. Our choice of kinematics was inspired from Ref. 9 where the incident muon energy is $E=219$ GeV. The results are listed for fixed $\omega_E=1/x_E$ and varying Q_E^2. The first three columns give the kinematics of the experimental point. Columns four, five, and six give our results for $F_2^{\rm el}=\nu_E W_2^{\rm el}$, $F_1^{\rm el}=2MW_1^{\rm el}$, and the elastic contribution to the cross section. Column seven, labeled (1γ) gives, in particular, the result of the exact one-photon-emission cross section derived by Mo and Tsai.[3] The two lines corresponding to every experimental point are, respectively, the zero- and one-oblique-photon contributions computed from Eqs. (3.29) and (4.4). For the column labeled (1γ) the content of the two lines is as follows: The first line is the result of the peaking approximation in perturbation theory using Eq. (2.10). The same result is obtained[6] from Eq. (3.29) if the τ integral is computed in the Gauss-Jacobi method with degree $n=2$ and the roots are taken to be $\tau=0$ and $\tau=1$. The second line is the result of Eq. (3.41) with a δ function in place of the spectral function. In more physical terms, the second line is the difference between the exact (1γ) and the peaking cross sections.

The elastic contributions is thus the *sum* of the numbers appearing in the two lines; for the column labeled (1γ), the *sum* reproduces Mo and Tsai's[3] (1γ) cross section. Let us now point out what can be learned from this table.

(i) The first ω_E value illustrates the quasi-elastic regime. The negative signs of the one-oblique-photon contributions to F_2 and to the cross section are easily understood. In the vicinity of the elastic peak, the one-oblique-photon contribution can be incorporated[6] in the zero-oblique-photon term provided a hard factor H is included in Eq. (3.25). From Eq. (2.49), we see that $H\lesssim 1$ for the important domain $\sigma'\simeq 0$ and $0\leqslant\sigma\leqslant 1$. We note that for this argument to hold, one must assume that the lepton mass can be neglected and this does not seem to be justified for F_1.

(ii) As Q_E^2 increases, the one-oblique-photon contribution increases while the zero-oblique-photon contribution decreases. This behavior is steepest for larger ω_E.

(iii) For low Q_E^2, our predicted cross section is slightly *lower* than the standard one.

(iv) The most dramatic results appear in the very inelastic regime. Here, the predicted cross section is much *larger* than the standard one. Looking, for example, at the last two experimental points of this table, we see that our cross section, which is about four times larger than the standard one at $Q_E^2=1.7$ GeV^2, becomes about twelve times larger at $Q_E^2=2.2$ GeV^2

Such discrepancies of our results from the (1γ) formula of Mo and Tsai, for large ω_E and Q_E^2, were so unexpected that we spent some time to check our formulas and programs. Without entering in all technical details, let us briefly discuss the origin of the large cross section.
As we have said above, the $(\overline{w},\tau)$ or equivalently

TABLE I. Elastic contribution to $\mu+p$ scattering at $E=219$ GeV. For every kinematical configuration, the first and second lines are the zero- and one-oblique-photon contributions, respectively.

Q_E^2 (GeV^2)	E' (GeV)	θ (mrad)	$F_2^{el}=\nu_E W_2^{el}$ This work	$F_1^{el}=2MW_1^{el}$ This work	$\frac{d\sigma^{el}}{d\Omega' dE'}$ (nb/sr GeV) This work	(1γ)[a]
			$\omega_E=1.5$			
1	218.2	5	3.30×10^{-3}	6.86×10^{-3}	1.63×10^{4}	1.80×10^{4}
			-1.4×10^{-4}	2.2×10^{-3}	-6.7×10^{2}	-6.77×10^{2}
13	208.6	17	3.51×10^{-6}	2.87×10^{-6}	7.22×10^{-3}	7.99×10^{-3}
			-2.3×10^{-7}	9.7×10^{-5}	-3.2×10^{-4}	-1.5×10^{-4}
31	194.2	27	1.64×10^{-7}	1.19×10^{-7}	2.17×10^{-5}	2.36×10^{-5}
			1.53×10^{-7}	8.48×10^{-5}	7.40×10^{-5}	8.73×10^{-5}
49	179.8	35	3.22×10^{-8}	2.23×10^{-8}	9.27×10^{-7}	1.00×10^{-6}
			3.37×10^{-7}	8.30×10^{-5}	4.03×10^{-5}	4.70×10^{-5}
			$\omega_E=14.5$			
0.7	213.6	4	2.07×10^{-3}	5.23×10^{-4}	2.96×10^{3}	3.10×10^{3}
			-5.7×10^{-4}	4.11×10^{-3}	-8.2×10	-6.9×10
1.7	205.9	6	4.82×10^{-4}	8.01×10^{-5}	4.46×10	4.66×10
			1.4×10^{-5}	3.96×10^{-3}	1.3	2.0
14.2	109.3	24	1.89×10^{-5}	9.45×10^{-7}	8.49×10^{-4}	8.78×10^{-4}
			4.88×10^{-3}	5.13×10^{-3}	2.22×10^{-1}	1.78×10^{-1}
			$\omega_E=40$			
0.7	204.1	4	2.06×10^{-3}	1.87×10^{-4}	9.74×10^{3}	1.01×10^{3}
			1.5×10^{-4}	2.49×10^{-3}	7.39×10	1.51×10
2.7	161.5	9	3.29×10^{-4}	1.41×10^{-5}	1.70	1.74
			1.15×10^{-3}	2.57×10^{-3}	5.92	5.92
6.2	86.9	18	7.13×10^{-4}	1.47×10^{-5}	8.79×10^{-2}	9.02×10^{-2}
			2.82×10^{-2}	3.68×10^{-3}	3.48	1.85
8.2	44.2	29	7.77×10^{-3}	9.73×10^{-5}	1.07×10^{-1}	1.10×10^{-1}
			3.01×10^{-1}	6.67×10^{-3}	4.16	1.21
9.2	22.9	43	9.35×10^{-2}	7.82×10^{-4}	2.45×10^{-1}	2.50×10^{-1}
			1.68	1.27×10^{-2}	4.42	9.38×10^{-1}
			$\omega_E=60$			
0.7	196.6	4	2.14×10^{-3}	1.28×10^{-4}	6.25×10^{2}	6.42×10^{2}
			1.16×10^{-4}	1.95×10^{-3}	3.4×10	4.03×10
2.7	132.7	10	7.48×10^{-4}	1.89×10^{-5}	1.74	1.78
			4.63×10^{-3}	2.18×10^{-3}	1.07×10	8.91
5.2	52.7	21	9.62×10^{-3}	1.09×10^{-4}	4.94×10^{-1}	5.05×10^{-1}
			2.34×10^{-1}	4.36×10^{-3}	1.20×10	3.34
5.7	36.8	27	3.45×10^{-2}	3.08×10^{-4}	6.53×10^{-1}	6.66×10^{-1}
			6.90×10^{-1}	6.12×10^{-3}	1.31×10	2.78
6.2	20.1	37	2.46×10^{-1}	1.54×10^{-3}	1.15	1.18
			2.86	1.04×10^{-2}	1.34×10	2.02
			$\omega_E=160$			
0.2	201.9	2	4.77×10^{-3}	1.68×10^{-4}	2.37×10^{4}	2.41×10^{4}
			3.56×10^{-5}	1.10×10^{-3}	1.77×10^{2}	2.51×10^{2}
1.2	116.7	7	4.88×10^{-3}	6.21×10^{-5}	3.74×10	3.80×10
			1.36×10^{-2}	1.18×10^{-3}	1.04×10^{2}	5.62×10
1.7	74.1	10	1.55×10^{-2}	1.30×10^{-4}	1.68×10	1.71×10
			1.28×10^{-1}	1.60×10^{-3}	1.39×10^{2}	3.26×10
2.2	31.4	18	1.88×10^{-1}	8.27×10^{-4}	1.69×10	1.72×10
			2.1	3.15×10^{-3}	1.9×10^{2}	1.59×10

[a] This column is the result for a δ function replacing the spectral function. Note that the sum from the two lines in this column gives the Mo and Tsai (1γ) exact cross section.

the (σ, σ') integrations in Eqs. (4.4) are effected by the Gauss-Jacobi method, using twice Eq. (3.57) with degree approximations n and m. For fixed n and m, the roots of the relevant Jacobi polynomials generate $n\times m$ points in the $(\sigma, \underline{\sigma'})$ plane with weights $C_{\nu_m}(-1+\alpha A, 0)C_{\mu m}(-1+\alpha\bar{A}, -1+\alpha\bar{A})$. Then, for every (σ, σ') pair, the t integral is computed separately in the regions corresponding to the t, p, and p' peaks and in the region between the p and p' peaks. Apart from the counterterms which may be computed separately, the t integral reduces to Mo and Tsai's (1γ) formula when $\sigma = \sigma' = 0$ and this permitted a useful check.

In Table II, we present 2×8 (σ, σ') points with the corresponding t-integrals and courterterms for $W_2^{\rm el}$ at $E = 219$ GeV, $Q_E{}^2 = 2.2\,{\rm GeV}^2$, and $\omega_E = 160$. The weights attached to the $\bar{w}$ integration are given in the first line while the weights attached to the τ integration are, in this case, one for each line. The lines labeled t, p, $p-p'$, and p' give the results of the t integral in the corresponding domains, $p-p'$ being the contribution originating from the region between the p and p' peaks. The lines labeled C-p and C-p' give the results of the p and p' peaks counterterms and the line labeled s-total is the result of the total t integral minus the counterterms.

Let us note two important facts appearing in this table:

(a) Some (σ, σ') points, typically with large σ and small σ', overcompensate the handicap of a small weight by a very large contribution to $W_2^{\rm el}$.

(b) The most important contributions arise from the t peaks. It is possible to understand, technically, why the t integrals for large σ and small σ' values are much larger than for $\sigma = \sigma' = 0$. What happens is that for the $\sigma = \sigma' = 0$ part of T_2 all large terms of the form $Q_E^2(E^2 + E'^2)/[(k\cdot p)(k\cdot p')]$ cancel, leaving[6] instead $[t(E^2 + E'^2) - Q_E{}^4/4]/[(k\cdot p)(k\cdot p')]$ and this cancellation does not hold for the terms proportional to σ or σ'. These facts explain why, in the very inelastic regime, our results are much larger than those predicted by the (1γ) theory. We note that our $W_1^{\rm el}$ values are comparable to the (1γ) results, not presented here explicitly.

At this point, the reader may object that since the one-oblique-photon contribution to $W_2^{\rm el}$ is much larger than the zero-oblique-photon part, why not include the two-obique-photon contribution and so on. To answer this objection, we note that loosely speaking, the large contributions arise from the emission of a collinear photon whose momentum is $K = \sigma l$ besides the oblique photon of momentum k. When we come to the two-oblique-photon term, these large contributions will be suppressed since, according to Eq. (2.31),

$$\begin{aligned} T^{O}_{(2)}(p, p'; k_1, k_2) = T_{(2)}(p, p'; k_1, k_2) \\ - \alpha I(k_1)\, T_{(1)}(p, p'; k_2) \\ - \alpha I(k_2) T_{(1)}(p, p'; k_1) \\ - \alpha I(k_1)\alpha I(k_2) T_{(0)}(p, p')\,. \end{aligned} \tag{4.5}$$

To be sure, one would like numerical estimates

TABLE II. A 2×8 table of unweighted t integrals and counterterms contributing to $W_2^{\rm el}$ at $E = 219$ GeV, $Q_E{}^2 = 2.2$ GeV^2, and $\omega_E = 160$. The raw (τ) weight is 1 for each line and the column $(\bar{w})$ weight is as indicated.

$C_{\bar{w}}(\nu, 8)$	1.8542	0.0614	0.0323	0.0208	0.0141	0.0093	0.0056	0.0023
σ	0.0005	0.0908	0.2705	0.4802	0.6614	0.7804	0.8371	0.8544
σ'	0.0000	0.0031	0.0092	0.0164	0.0226	0.0266	0.0286	0.0291
t	7.8×10^{-2}	1.9	1.7×10	5.4×10	7.7×10	3.5×10	3.6	7.6
p	1.2×10^{-1}	1.5×10^{-1}	3.7×10^{-1}	1.6×10^{-1}	2.6×10^{-1}	7.1×10^{-1}	2.9	2.6×10
p-p'	6.2×10^{-4}	8.7×10^{-4}	1.9×10^{-4}	6.5×10^{-3}	3.1×10^{-2}	1.8×10^{-1}	1.2	1.3×10
p'	2.5×10^{-5}	4.1×10^{-5}	1.2×10^{-4}	7.2×10^{-4}	7.5×10^{-3}	1.0×10^{-1}	1.3	1.7×10
C-p	1.0×10^{-1}	1.1×10^{-1}	1.4×10^{-1}	2.1×10^{-1}	3.9×10^{-1}	9.8×10^{-1}	3.8	3.3×10
C-p'	2.8×10^{-9}	9.0×10^{-9}	1.3×10^{-7}	6.8×10^{-6}	7.5×10^{-4}	6.1×10^{-2}	1.8	3.0×10
s total	9.6×10^{-2}	2.0	1.8×10	5.4×10	7.7×10	3.5×10	3.5	7.1×10^{-1}
σ	0.0000	0.0004	0.0013	0.0024	0.0033	0.0040	0.0043	0.0044
σ'	0.0035	0.6271	1.8688	3.3176	4.5692	5.3912	5.7826	5.9019
t	7.9×10^{-2}	7.9	3.4	5.6×10^{-1}	5.4×10^{-2}	2.0×10^{-3}	8.0×10^{-6}	2.6×10^{-7}
p	1.2×10^{-1}	1.2×10^{-2}	4.1×10^{-5}	8.6×10^{-7}	1.0×10^{-7}	6.1×10^{-8}	1.3×10^{-7}	9.3×10^{-7}
p-p'	6.1×10^{-4}	7.0×10^{-5}	2.8×10^{-7}	1.9×10^{-7}	3.9×10^{-8}	2.8×10^{-8}	6.2×10^{-8}	4.6×10^{-7}
p'	2.5×10^{-5}	3.7×10^{-6}	3.8×10^{-7}	7.8×10^{-8}	3.5×10^{-8}	3.4×10^{-8}	8.4×10^{-8}	6.4×10^{-7}
C-p	9.9×10^{-2}	4.7×10^{-3}	5.1×10^{-5}	1.3×10^{-6}	1.5×10^{-7}	8.2×10^{-8}	1.6×10^{-7}	1.2×10^{-6}
C-p'	2.8×10^{-9}	3.2×10^{-9}	4.2×10^{-9}	6.6×10^{-9}	1.3×10^{-8}	3.3×10^{-8}	1.3×10^{-7}	1.1×10^{-6}
s total	9.7×10^{-2}	7.9	3.4	5.6×10^{-1}	5.4×10^{-2}	2.0×10^{-3}	7.9×10^{-6}	-1.5×10^{-8}

of the two-oblique -photon effects. This involves a quite lengthy computation of the trace implicit in T_2 and one additional photon momentum integral which could, presumably, be done analytically since it affects the lepton tensor only. This program is under way.

V. DISCUSSION AND CONCLUSION

We have presented a rigorous nonperturbative method to effect the radiative corrections in lepton-proton scattering in the one-photon-exchange approximation and neglecting the radiative corrections to the hadron vertex. Every term in the rearranged perturbation series incorporate the emission of an arbitrary number of collinear photons. The $d\omega/\omega$ spectrum of an oblique photon is suppressed. We have applied this method to compute the elastic contribution arising from the zero- and one-oblique-photon terms. In the very inelastic regime, this elastic contribution is larger than expected from ordinary perturbation theory. We observed that this involves two hard photons, one oblique and one collinear.

From the theoretical point of view, our method resembles that of Grammer and Yennie for the rearrangement of the cross-section perturbation series. Using the spectral function derived in earlier work, we avoid the noncovariant separation between hard and soft photons. In the second work of Ref. 3, Tsai incorporates phenomenologically multiphoton effects by the method of equivalent radiators. We do not believe that our method is equivalent to the latter. Even if one considers that his exponentiation is similar to the use of the spectral function, the most important effects, associated with the kinematical modification of the lepton tensor, are missing in Tsai's method. Of course we have, in this paper, neglected all straggling and ionization effects which are negligible for $\mu+p$ in the kinematical domain we have considered.

Within this framework, it will be possible to check by explicit computation in the near future, that the two-oblique-photon contribution is small. Here, we have just presented a qualitative argument to explain the suppression of the large contributions encountered in the one-oblique-photon process. Also, finer details such as higher order, infrared finite virtual radiative corrections must be derived and incorporated in the one-oblique-photon emission. To go beyond the one-photon approximation, without being involved from the start with nonperturbative quantum chromodynamics we have demonstrated how to rearrange the QED perturbation series for charged-lepton scattering, $l+l' \to l+l'$, in the ladders and crossed-ladders approximation and we hope to present the results of this work soon. By extrapolating the result found in this problem, we expect that multiphoton exchange are not expected to significantly modify the results of this paper.

This theory predicts, for small x_E, larger radiative corrections than the Mo and Tsai, and Tsai[3] method used by experimentalists. However, until one demonstrates that the two-oblique-photon contribution is indeed small we cannot, convincingly, explain part of the scaling violations of the structure functions as originating from inadequate radiative corrections.

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APPENDIX

In this appendix, we give the details of the azimuthal integrations appearing in Eq. (3.50). We set

$$k\cdot p=\omega(a+b\cos\varphi)\ , \tag{A1a}$$

$$k\cdot p'=\omega(a'+b'\cos\varphi)\ , \tag{A1b}$$

where

$$a=E-p\cos\theta_p\cos\theta_k,\quad b=-p\sin\theta_p\sin\theta_k \tag{A2a}$$

and

$$a'=E'-p'\cos\theta_{p'}\cos\theta_k,\quad b'=-p'\sin\theta_{p'}\sin\theta_k. \tag{A2b}$$

Here, θ_p, $\theta_{p'}$, and θ_k are the angles of p and p' and k with the polar axis $\vec{u}$ in the laboratory frame. Simple geometrical considerations lead to

$$\cos\theta_p=\frac{p\tilde{x}-p'\cos\theta\tilde{y}}{u}\ ,\quad \cos\theta_{p'}=\frac{p\cos\theta\tilde{x}-p'\tilde{y}}{u} \tag{A3}$$

and

$$\sin\theta_p=\frac{p'\sin\theta\tilde{y}}{u}\ ,\quad \sin\theta_{p'}=\frac{p\sin\theta\tilde{x}}{u}\ , \tag{A4}$$

where

$$\tilde{x}=\frac{1-\sigma+r(1+\sigma')}{1+r}\overset{m}{\simeq}1-\sigma\ , \tag{A5}$$

$$\tilde{y}=\frac{1+\sigma'+r(1-\sigma)}{1+r}\overset{m}{\simeq}1+\sigma'\ . \tag{A6}$$

From these equations we observe that

$$\frac{b}{b'}=\frac{\tilde{y}}{\tilde{x}}\ . \tag{A7}$$

In effecting the φ integration in Eq. (3.50) we use the following formulas:

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{a+b\cos\varphi}=\frac{1}{S},\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{(a+b\cos\varphi)^2}=\frac{a}{S^3} \tag{A8a}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{(a+b\cos\varphi)(a'+b'\cos\varphi)} = \frac{1}{(a'b-ab')}\left(\frac{b}{S}-\frac{b'}{S'}\right) = \frac{(\tilde{y}a'+\tilde{x}a)}{SS'(yS'+\tilde{x}S)} \equiv X(a,b,a',b')\,, \tag{A8b}$$

where we define

$$S=(a^2-b^2)^{1/2},\quad S'=(a'^2-b'^2)^{1/2}\,. \tag{A9}$$

In Eq. (A8b) the advantage of the second form over the first, in a numerical program, is the absence of the spurious singularity which arises from the vanishing of the denominator $a'b-ab'$. Using Eqs. (A2), we can write Eqs. (A9) in the form

$$S=p[(\cos\theta_k-\frac{E}{p}\cos\theta_p)^2+\frac{m^2}{p^2}\sin^2\theta_p]^{1/2}\,, \tag{A10a}$$

$$S'=p'[(\cos\theta_k-\frac{E'}{p'}\cos\theta_{p'})^2+\frac{m^2}{p'^2}\sin^2\theta_{p'}]^{1/2} \tag{A10b}$$

Using Eqs. (3.36) into Eq. (3.50) we get after straightforward algebra,

$$\mathcal{T}_{1(1)}=2\,\frac{(Q_E{}^2-2m^2)}{\omega^2}\left(2p\cdot pX-\frac{m^2a}{S'^3}-\frac{m^2a'}{S'^3}\right)+\mathcal{R}_{1(1)}\,, \tag{A11a}$$

$$\mathcal{T}_{2(1)}=\frac{(4EE'-Q_E{}^2)}{\omega^2}\left(2p\cdot p'X-\frac{m^2a}{S^3}-\frac{m^2a'}{S'^3}\right)+\mathcal{R}_{2(1)}\,, \tag{A11b}$$

where

$$\mathcal{R}_{1(1)}=-4\,\frac{(2p\cdot p'+m^2)}{\omega}\left(\frac{1}{S}-\frac{1}{S'}\right)+\frac{4m^2}{\omega}(aa'-bb')\left(\frac{1}{S^3}-\frac{1}{S'^3}\right)+4Y\,,$$

$$\begin{aligned}\mathcal{R}_{2(1)}=&4\left[p\cdot p'\frac{(E-E')}{\omega}-m^2\right]X\\&+\frac{2}{\omega S}[2p\cdot p'+m^2-2E(E+E')+2E'\omega]\\&-\frac{2}{\omega S'}[2p\cdot p'+m^2-2E'(E+E')-2E\omega]\\&-\frac{2m^2}{\omega S^3}(aa'-bb'-2E'a)\\&+\frac{2m^2}{\omega S'^3}(aa'-bb'-2Ea')-2Y\,,\end{aligned} \tag{A.12b}$$

and

$$Y(a,b,a',b')=\frac{\tilde{y}}{\tilde{x}}+\frac{\tilde{x}}{y}+(a\tilde{x}-a'\tilde{y})\left(\frac{1}{\tilde{x}S'}-\frac{1}{\tilde{y}S}\right)\,. \tag{A13}$$

We note that ω is a function of t and w^2 given in Eq. (3.46). The variable x_k is also a function of t and w^2 as is seen from Eq. (3.44).

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[8]In our previous work (Refs. 4 and 6), we were using instead of collinear and oblique the terms soft and hard to conform with the literature "jargon" and also because we were not aware that the spectral function we introduced in Ref. 4 would be useful in applications where hard photons are likely to be emitted. We thank Professor M. Froissart for a discussion about this point and for suggesting this terminology.

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