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Radiative corrections to neutrino-induced neutral-current phenomena in the $SU(2)_L \times U(1)$ theory

W. J. Marciano
The Rockefeller University, New York, New York 10021

A. Sirlin*
The Institute for Advanced Study, Princeton, New Jersey 08540
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Weak corrections of order α to ν-induced neutral-current phenomena are studied in the $SU(2)_L \times U(1)$ theory. Calculations are carried out using a simple renormalization framework in which $\cos\theta_W = m_W/m_Z$ exactly and amplitudes are expressed in terms of G_μ, the universal constant of the weak interactions obtained from muon decay. To rigorously evaluate corrections to hadronic vertices, we employ the current-algebra formulation of radiative corrections. Our main emphasis is on large-momentum-transfer processes such as deep-inelastic scattering; however, we also discuss low-momentum transfers and ν-lepton interactions. We find that the weak radiative corrections to ν-hadron neutral-current scattering give rise to a universal renormalization factor $\rho_{\mathrm{NC}}^{(\nu;h)}$ multiplying the overall amplitude, a correction factor $\kappa^{(\nu;h)}(q^2)$ multiplying $\sin^2\theta_W$, and two new induced currents not present at the tree level. For nonexotic values of m_{ϕ_1} (Higgs-scalar mass) and m_t (t-quark mass), the corrections $\rho_{\mathrm{NC}}^{(\nu;h)} - 1$ and $\kappa^{(\nu;h)}(q^2) - 1$ turn out to be small over a wide range of momentum transfers. The smallness of these corrections is mainly due to the renormalization framework employed; but it is helped by a subtle partial cancellation between hadronic and bosonic contributions. Photonic corrections to the hadronic vertices are also briefly discussed in the leading-logarithm approximation of the quark-parton model. Detailed expressions for the ZZ, WW, γZ, and $\gamma\gamma$ self-energies along with a discussion of the effect of large m_t on these quantities are given. They play an important role in our renormalization scheme and are useful in the study of radiative corrections to many other processes of physical interest.

I. INTRODUCTION

The study of neutrino-induced neutral-current phenomena plays an important role in our attempts to understand weak interactions and nucleon structure. In particular, it gives significant information regarding modern spontaneously broken gauge theories and allows a precise determination of the weak-interaction angle θ_W. In turn, the value of θ_W provides accurate predictions for the masses[1-3] and decay properties[4] of the $W^\pm$ and Z intermediate vector bosons. It also has far reaching implications for grand unified models and in particular the predicted instability of the proton.[1]

One of the primary objectives of this paper is to study what may be called the "weak corrections of order α"[5] to neutrino-induced neutral-current phenomena. In the case of neutrino-hadron interactions, these are all the radiative corrections of order α with the exception of those amplitudes in which a photon is virtually emitted and absorbed by the hadrons or emitted by the hadrons as a real particle (photonic corrections). A similar definition holds for the weak corrections to neutrino-lepton scattering, substituting the charged leptons for the hadrons. The main emphasis of our work is the analysis of large-momentum-transfer reactions such as deep-inelastic scattering where experimental data has become most precise. However, many of our results are universal in that they apply to all neutrino-induced neutral-current phenomena characterized by $q^2, q\cdot P \ll m_W{}^2$ (q and P are the momentum transfer and initial-hadron or charged-lepton momentum, respectively). In particular, we also discuss neutrino-lepton interactions which are of special interest because [aside from hadronic contributions of $O(\alpha)$ to γZ mixing] they are free from strong-interaction effects.

The weak corrections of order α are studied in Secs. II and III within the framework of the $SU(2)_L \times U(1)$ theory of electroweak interactions proposed by Weinberg and Salam. For simplicity we assume here the minimal model with six quark and six lepton flavors and a single Higgs isodoublet. We rely heavily on the current-algebra formulation of radiative corrections.[6] This formalism has two important advantages: (i) it allows us to discuss weak radiative corrections of order α to actual processes involving physical nucleons rather than isolated quarks and (ii) by combining many Feynman diagrams into single amplitudes involving currents, it greatly simplifies the task of extracting the contributions of $O(G_F\alpha)$ [in contradistinction with those of $O(G_F{}^2)$ which are in any case negligible for our considerations]. We work in the simple renormalization framework of Ref. 2. In this approach the renormalized weak-interaction angle[7] satisfies the relation

$$\cos\theta_W = m_W/m_Z \tag{1}$$

exactly, where m_W and m_Z are the physical masses of the $W^{\pm}$ and Z bosons. Another significant feature of our calculations is that we express our final answers in terms of G_μ, the universal constant of the weak interactions, which is very accurately determined from the muon lifetime. The advantages of this procedure have been explained in Ref. 2.

Our results for the weak corrections of $O(\alpha)$ are summarized in Sec. III. Aside from certain induced currents not present in the tree approximation, we find that the weak radiative corrections give rise to a universal renormalization factor $\rho_{\rm NC}^{(\nu;h)}$ multiplying the overall ν-hadron scattering amplitude and a correction factor $\kappa^{(\nu;h)}(q^2)$ multiplying $\sin^2\theta_W$. Because the amplitude is expressed in terms of G_μ, we find that for values $10\ \text{GeV} \leq m_{\phi_1} \leq 1\ \text{TeV}$ and $m_t < m_W$ (m_{ϕ_1} and m_t are the masses of the physical Higgs scalar and the t quark, respectively), $\rho_{\rm NC}^{(\nu;h)} - 1$ is very small. For values $-q^2 \approx 20\ \text{GeV}^2$, which are characteristic of deep-inelastic scattering, we find that $1-\kappa^{(\nu;h)}(q^2) < 0.5\%$ if $m_{\phi_1} < m_Z$ and $<1.2\%$ if $m_{\phi_1} < 10 m_Z$ provided that θ_W is defined according to Eq. (1). This result shows that Eq. (1) is not only simple and appealing theoretically, but it is also very convenient in the sense that the effective Lagrangian for deep-inelastic neutral-current neutrino scattering is weakly modified by radiative corrections when expressed in terms of $\sin^2\theta_W = 1 - m_W{}^2/m_Z{}^2$ and G_μ. However, as we shall demonstrate, the radiative corrections to $\rho_{\rm NC}^{(\nu;h)}$ and $\kappa^{(\nu;h)}(-20\ \text{GeV}^2)$ are no longer small if we consider the exotic possibility that m_t is as large as a few times m_Z or that m_{ϕ_1} is extremely large,

In Sec. III C we study $\kappa^{(\nu;h)}(q^2)$ and the corresponding correction factor $\kappa^{(\nu;l)}(q^2)$ for ν-lepton scattering over the range of invariant momenta $0 \leq -q^2 \leq 100\ \text{GeV}^2$. We point out that the calculations at small values of $-q^2$ are complicated by model-dependent hadronic contributions to γZ mixing and discuss a possible strategy to overcome these difficulties. Assuming that estimates based on naive quark-model calculations remain approximately valid down to $-q^2 \approx 0$, our general conclusion is that $1-\kappa^{(\nu;h)}(q^2)$ varies slowly over the entire range $0 < -q^2 < 100\ \text{GeV}^2$ and remains quite small if $m_{\phi_1} \lesssim m_Z$.

In Sec. IV we discuss briefly the photonic corrections to hadronic vertices. These contributions form a gauge-invariant set and are model dependent in the sense that they are affected by the dynamics of strong interactions. They are strictly of $O(\alpha)$ rather than $O(g^2/4\pi) = O(\alpha/\sin^2\theta_W)$ and have been recently discussed in the leading-logarithm approximation within the framework of the quark-parton model.[8] Using the theoretical expressions of that work we explain in Sec. IV a simple way of estimating the effects of these corrections in some important integrated distributions. Fortunately, although model dependent, they turn out to have a small effect on ν-induced deep-inelastic scattering on isoscalar targets, the case of main interest to us.

Section V presents some general observations concerning the magnitude of our results and the problem of extracting the value of $\sin^2\theta_W$ by comparing theory and experiment. In particular, we point out that a calculation of the $O(\alpha)$ corrections to neutrino-induced charged-current interactions is required before a precise determination of $\sin^2\theta_W$ can be obtained from existing neutrino deep-inelastic scattering data. The appendices contain a rather detailed study of the ZZ, WW, γZ, and $\gamma\gamma$ self-energies and a discussion of possible effects associated with large values of m_t. These self-energies contribute directly to the amplitudes and, according to the renormalization procedure described in Ref. 2, they play an important role in the definition of the basic counterterms of the theory.

II. WEAK CORRECTIONS OF $O(\alpha)$ TO ν-INDUCED NEUTRAL-CURRENT PHENOMENA

In this section we discuss a number of amplitudes that contribute to ν-induced neutral-current phenomena. Our primary analysis is carried out for the case of ν-nucleon interactions and the emphasis is placed on large-momentum-transfer processes. As a by-product, in Sec. II F we extend our results to ν-charged-lepton scattering.

In this discussion we assume that $-q^2$ and $q\cdot P$, although large, satisfy $-q^2, q\cdot P \ll m_W{}^2$. More specifically, we assume that corrections of $O(G_F \alpha q^2/m_W{}^2)$ and $O(G_F\alpha q\cdot P/m_W{}^2)$ are negligible. Some of the problems involved in extending the analysis to low-momentum-transfer processes are discussed in Sec. III. We focus our attention on the contributions of $O(G_F\alpha)$ rather than $O(G_F{}^2)$ and carry out all calculations in the 't Hooft–Feynman gauge.

A. Lowest order

The lowest-order amplitude for ν-nucleon scattering [Fig. 1(a)] is given by

$$M^0 = \frac{i(g^2+g'^2)}{2(q^2-m_Z{}^2)}\ \langle f|J_Z^\mu|i\rangle \bar{u}_{\nu_f}\gamma_\mu a_- u_{\nu_i}\,, \tag{2}$$

where

$$J_Z^\mu = \tfrac{1}{2}\bar\psi C_3\gamma^\mu a_-\psi - \sin^2\theta_W \bar\psi\gamma^\mu Q\psi\,, \tag{3a}$$

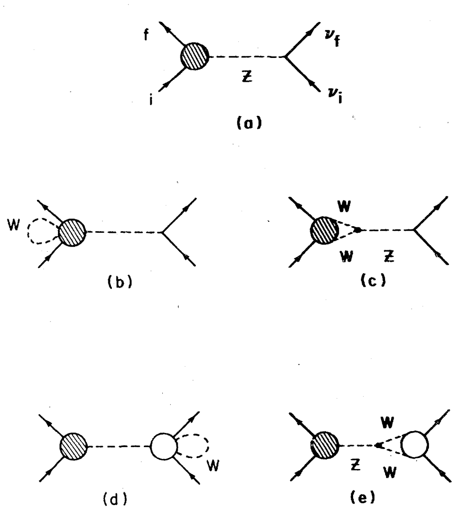

FIG. 1. Zeroth-order amplitude and $O(G_F\alpha)$ vertex corrections to neutrino neutral-current scattering. The line f symbolically represents the final hadronic state which may involve many particles. The W lines include contributions of both W^+ and W^-. An unshaded circle indicates that the virtual propagators are attached in all possible ways to the leptonic lines. In particular, Figs. 1(b) and 1(d) include field renormalizations of the external lines.

$$Q=\begin{pmatrix} \frac{2}{3}I & 0 \\ 0 & -\frac{1}{3}I \end{pmatrix}, \tag{3b}$$

$$C_3=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{3c}$$

$a_-=(1-\gamma_5)/2$, $|i\rangle$ and $|f\rangle$ represent initial and final hadronic states (f may involve many particles), I is a 3×3 unit matrix, ψ is a column vector such that $\psi^T=(uctdsb)$ and a summation over color indices is understood in Eq. (3a). Following Ref. 2 we define

$$g=e/\sin\theta_W\,, \tag{4a}$$

$$g'=g\tan\theta_W\,, \tag{4b}$$

where e is the electric charge of the proton and θ_W is given by Eq. (1). In this way, g and g' are by definition gauge-invariant and infrared-finite renormalized parameters. The relation of g and m_W to the universal constant G_μ of the weak interactions has been previously discussed in Ref. 2.

We start our analysis of the radiative corrections by considering the vertex diagrams depicted in Fig. 1.

B. Vertex diagrams

The amplitudes of Figs. 1(b), and 1(c) are best treated using the current-algebra methods of Ref. 6. We first write the amplitude of Fig. 1(b) as

$$M_{1(\mathrm{b})}=-\frac{(g^2+g'^2)}{2}\,\frac{1}{q^2-m_Z{}^2}\,L_\mu\lim_{\bar q\to q}T^\mu(\bar q)\,, \tag{5a}$$

where $L_\mu\equiv\bar u_{\nu_f}\gamma_\mu a_- u_{\nu_i}$, $q=P_i-P_f$ is the momentum transfer, and

$$T^\mu(\bar q)=\frac{g^2}{4}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m_W{}^2}\int d^4y\, e^{i\bar q\cdot y}\int d^4x\, e^{ik\cdot x}\,\langle f|\,T[J_Z^\mu(y)(J_W^{\dagger\lambda}(x)\,J_{W\lambda}(0)+\mathrm{H.c.})]\,|i\rangle-B^\mu(\bar q)\,. \tag{5b}$$

The W current is given by

$$J_W^\mu=\bar\psi\gamma^\mu a_- C_-\psi\,, \tag{6a}$$

$$C_-=\begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}, \tag{6b}$$

where U is a unitary 3×3 matrix. In Eq. (5b) $B^\mu(\bar q)$ subtracts in an appropriate manner the mass insertions of the perturbation in the external hadronic legs, so that $T^\mu(\bar q)$ is regular as $\bar q\to q$.[6] We also note that $T^\mu(\bar q)$ includes the contribution of both W^+ and W^- virtual exchanges. It is understood that the integrals in Eq. (5b) have been regularized by a method consistent with the Ward identities, such as dimensional regularization. As is well known, in applications of this method, Eq. (5b) is interpreted as an integral in n-dimensional space-time and the γ matrices are assumed to obey a generalized n-dimensional Clifford algebra. In that case it is natural to interpret also the x and y integrations in Eq. (5b) as integrals over n dimensions. In this generalized space the currents obey the same local algebra as in four-dimensional space except that the usual three-dimensional δ functions are formally replaced by $(n-1)$-dimensional δ functions. As a consequence, the Ward identities associated with the current algebra are preserved by the regularization method.

Following the discussion in Sec. III of Ref. 6, we find

$$T^{\mu}(q)=-\bar{q}_{\alpha}\frac{\partial}{\partial\bar{q}_{\mu}}T^{\alpha}+\frac{\partial}{\partial\bar{q}_{\mu}}(\mathfrak{D}-\bar{q}_{\alpha}B^{\alpha})\big|_{\bar{q}=q}+V^{\mu}\,, \tag{7a}$$

where $\mathfrak{D}$ is an expression analogous to the first term in Eq. (5b) with the replacement $J_Z^{\mu}(y)\to i\partial_{\mu}J_Z^{\mu}(y)$ and

$$V^{\mu}=\frac{ig^2c^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{k^{\mu}}{(k^2-m_W{}^2)^2}\int d^4x\, e^{i(k+q)\cdot x}\{\langle f|T[J_W^{\dagger\lambda}(x)J_{W\lambda}(0)]|i\rangle-\langle f|T[J_W^{\dagger\lambda}(0)J_{W\lambda}(x)]|i\rangle\}\,, \tag{7b}$$

where we have performed a partial integration and introduced the abbreviations

$$c^2\equiv 1-s^2\equiv\cos^2\theta_W\,. \tag{7c}$$

In Ref. 6 it was shown that, after taking into account the effects of quark-mass counterterms as well as tadpoles and tadpole counterterms, the first two terms of Eq. (7a) are convergent and indeed of $O(g^2/m_W{}^2)$. The dependence on the momentum transfer suggests that these terms are of $O(g^2q\cdot P/m_W{}^2)$, $O(g^2q^2/m_W{}^2)$, or $O(g^2m^2/m_W{}^2)$ where m is a typical hadronic mass or the mass that sets the scale in the short-distance expansions. In our approximation such terms are negligible and we are left with V^{μ}. Thus we obtain

$$M_{1(b)}=\frac{ig^4L^{\mu}}{2(q^2-m_Z{}^2)}\int\frac{d^4k}{(2\pi)^4}\frac{k^{\mu}}{(k^2-m_W{}^2)^2}\int d^4x\, e^{ik\cdot x}\langle f|T[J_W^{\lambda}(x)J_{W\lambda}^{\dagger}(0)]|i\rangle\,, \tag{8}$$

where we have set $q=0$ in the exponential of the first term of Eq. (7b). The diagram in Fig. 1(c) is given by

$$M_{1(c)}=-\frac{ig^4}{4}\frac{L^{\mu}}{q^2-m_Z{}^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m_W{}^2)[(k-q)^2-m_W{}^2]}$$
$$\times\int d^4x\, e^{ik\cdot x}\langle f|T[J_W^{\rho}(x)J_W^{\dagger\lambda}(0)]|i\rangle[g_{\rho\lambda}(2k-q)_{\mu}-(k+q)_{\lambda}g_{\rho\mu}+(2q-k)_{\rho}g_{\mu\lambda}]\,. \tag{9a}$$

Again it is easily seen that the terms involving q in Eq. (9a) are of $O(q^2/m_W{}^2, q\cdot P/m_W{}^2)$. Neglecting such contributions we obtain

$$M_{1(c)}=-\frac{ig^4}{4}\frac{L^{\mu}}{q^2-m_Z{}^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m_W{}^2)^2}\int d^4x\, e^{ik\cdot x}\langle f|T[J_W^{\rho}(x)J_W^{\dagger\lambda}(0)]|i\rangle(2k_{\mu}g_{\rho\lambda}-k_{\lambda}g_{\rho\mu}-k_{\rho}g_{\mu\lambda})\,. \tag{9b}$$

The first term of Eq. (9b) exactly cancels the expression for $M_{1(b)}$ given in Eq. (8). The last two terms in Eq. (9b) can be simplified by using the appropriate Ward identities. We get two types of terms: (a) correlation functions involving a current and a divergence which are seen to give contributions to $M_{1(c)}$ of $O(g^4/m_W{}^4)=O(G_F{}^2)$ and are therefore negligible and (b) matrix elements of single currents arising from the equal time commutators.

In this way we finally obtain the simple expression

$$M_{1(b)}+M_{1(c)}=\frac{g^4L_{\mu}}{q^2-m_Z{}^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m_W{}^2)^2}\times\langle f|J_Z^{\mu}(0)+s^2J_{\gamma}^{\mu}(0)|i\rangle\,, \tag{10}$$

where

$$J_{\gamma}^{\mu}=\bar{\psi}\gamma^{\mu}Q\psi \tag{11}$$

and $s^2\equiv\sin^2\theta_W$ as in Eq. (7c). The current-algebra approach tells us immediately that the vertex diagram in which a virtual Z meson rather than W is emitted and absorbed by the hadrons does not contribute to $O(G_F\alpha)$, although it may indeed contribute to $O(G_F\alpha q^2/m_Z{}^2)$ or $O(G_F\alpha m^2/m_Z{}^2)$, because the term corresponding to V^{μ} in the relation analogous to Eq. (7a) is absent. Thus, Eq. (10) represents all the vertex corrections of $O(G_F\alpha)$ associated with the virtual exchange of heavy bosons along the hadronic line. The situation here is in some sense simpler than in the charged current case.[6] There the vertex diagrams (some of them involving photons) give rise to two classes of terms: (a) divergent terms proportional to the matrix elements of current operators, analogous to Eq. (10), and (b) finite terms proportional to the matrix elements of two-current correlation functions which are evaluated using short-distance expansions and invoking the asymptotic freedom of the underlying theory of strong interactions. In the neutral-current case the second class of terms is absent, a welcome simplification. As we mentioned in the Introduction, the vertex diagrams in which a photon is emitted and absorbed by the hadrons form an ultraviolet-finite and gauge-invariant set, which is, however, model dependent (it depends on the strong interactions). They are briefly discussed in Sec. IV.

The vertex diagrams of Figs. 1(d) and 1(e) are obtained from Eq. (10) by interchanging the role of leptonic and hadronic currents. Calling l_Z^μ and l_γ^μ the leptonic counterparts of J_Z^μ and J_γ^μ and remembering $\langle \nu_f | l_Z^\mu | \nu_i \rangle = L^\mu/2$, $\langle \nu_f | l_\gamma^\mu | \nu_i \rangle = 0$, we obtain in our approximation

$$M_{1(\mathrm{d})} + M_{1(\mathrm{e})} = \frac{g^4 L_\mu}{q^2 - m_Z^2} \langle f | J_Z^\mu | i \rangle \times \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_W^2)^2} . \tag{12}$$

We now evaluate the integral in Eqs. (10) and (12) using dimensional regularization. Explicitly, $\int d^4k/(2\pi)^4$ is interpreted as

$$\int_n \equiv \int \frac{d^nk}{\mu^{n-4}(2\pi)^n} , \tag{13a}$$

where μ is an arbitrary mass parameter introduced to keep the couplings dimensionless. In this way we find[9]

$$\int \frac{d^nk}{\mu^{n-4}(2\pi)^n} \frac{1}{(k^2 - m_W^2)^2} = -\frac{i}{16\pi^2} \left[\frac{2}{n-4} + 2C + \ln\left(\frac{m_W^2}{\mu^2}\right) \right] \tag{13b}$$

with

$$C = \tfrac{1}{2} [\gamma - \ln(4\pi)] , \tag{13c}$$

where γ is the Euler constant.

C. ZZ self-energy, $J_Z^\mu Z_\mu$ and $l_Z^\mu Z_\mu$ counterterms

These diagrams are illustrated in Fig. 2. Following Ref. 2, we find the following for the sum of these graphs:

$$M_2 = M^0 \left[\frac{A_{ZZ}(q^2) - \mathrm{Re} A_{ZZ}(m_Z^2)}{q^2 - m_Z^2} - \frac{2\delta e}{e} - \left(1 - \frac{c^2}{s^2}\right) \mathrm{Re} \left(\frac{A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{A_{WW}(m_W^2)}{m_W^2} \right) \right] . \tag{14}$$

The functions $A_{ZZ}(q^2)$, $A_{WW}(q^2)$, $A_{\gamma Z}(q^2)$ (to be introduced in Sec. II D) and $A_{\gamma\gamma}(q^2)$ are the coefficients of $g_{\mu\nu}$ in the ZZ, WW, γZ, and $\gamma\gamma$ self-energies while δe is the charge-renormalization counterterm defined in Eq. (26) of Ref. 2. As explained in that paper and as we will see later, this counterterm can be eliminated by expressing the constant g^2/m_W^2 in M^0 in terms of the universal constant G_μ of the weak interactions. The self-energies are defined as $-i$ times the Feynman diagrams depicted symbolically in Fig. 2(a) and the analogous graphs for WW, γZ, and $\gamma\gamma$, with the external legs extracted. In Eq. (14) the term proportional to $A_{ZZ}(q^2)$ arises from Fig. 2(a). According to Eq. (9a) of Ref. 2, $\delta m_Z^2 = \mathrm{Re} A_{ZZ}(m_Z^2) + t_{ZZ}$ where t_{ZZ} cancels the tadpole and tadpole counterterms of Fig. 2(b) and $\mathrm{Re} A_{ZZ}(m_Z^2)$ leads to the second term in Eq. (14). Finally, the last three terms in Eq. (14) represent the effect of the $J_Z^\mu Z_\mu$ and $l_Z^\mu Z_\mu$ counterterms defined in Eqs. (16c), (28a), and (28b) of Ref. 2.

The functions $A_{ZZ}(q^2)$, $A_{WW}(q^2)$, $A_{\gamma Z}(q^2)$, and $A_{\gamma\gamma}(q^2)$ are studied in detail in the Appendices.

D. Charge-radius and γZ mixing diagrams and counterterms

These contributions are illustrated in Fig. 3. For the charge-radius diagrams of Figs. 3(a) and

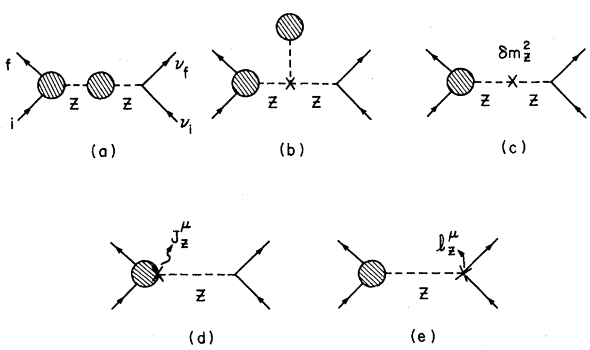

FIG. 2. ZZ self-energy, tadpoles, mass, $J_\mu^Z Z^\mu$ and $l_\mu^Z Z^\mu$ counterterms contributing to neutrino-induced neutral-current processes.

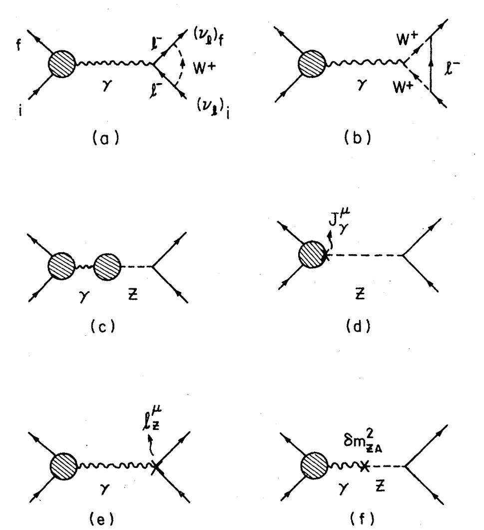

FIG. 3. Charge-radius and γZ mixing diagrams and counterterms. In Figs. 3(a) and 3(b), l^- is the charged lepton associated with ν_l ($l = e, \mu, \tau \ldots$).

3(b) we find

$$M_{3(a)}+M_{3(b)}=-\frac{ie^2g^2}{16\pi^2q^2}\langle f|J_\gamma^\mu|i\rangle L_\mu \times\left[\frac{2}{n-4}+2C+\ln\left(\frac{m_W{}^2}{\mu^2}\right)+\frac{q^2}{m_W{}^2}R_l(q^2)\right], \tag{15a}$$

where

$$R_l(q^2)=\tfrac{1}{3}+2\int_0^1 dx\,x(1-x)\ln\left(\frac{m_W{}^2}{m_l{}^2-q^2x(1-x)}\right) \tag{15b}$$

and m_l ($l=e,\mu,\tau$) is the mass of the charged lepton associated with the neutrino.

Consistent with our approximation, in the expression between brackets we have neglected terms of $O(q^4/m_W{}^4, q^2m_l{}^2/m_W{}^4)$ which are sometimes multiplied by $\ln(m_W{}^2/m_l{}^2)$ or $\ln(m_W{}^2/-q^2)$; but we have made no assumption about the relative magnitudes of $-q^2$ and $m_l{}^2$. We note the limiting cases:

$$R_l(0)=\tfrac{1}{3}[\ln(m_W{}^2/m_l{}^2)+1], \tag{15c}$$

$$R_l(q^2)\simeq\tfrac{1}{3}(\ln(m_W{}^2/-q^2)+\tfrac{8}{3}) \quad (\text{for } -q^2\gg m_l{}^2). \tag{15d}$$

For arbitrary values of $-q^2$, $R_l(q^2)$ can be calculated using Eq. (B12).

Figures 3(d)–(3f) represent the effect of the counterterms proportional to $J_\gamma^\mu Z_\mu$, $l_Z^\mu A_\mu$, and $Z^\mu A_\mu$ in Eqs. (5) and (16c) of Ref. 2. It is easy to verify that the sum of diagrams 3(e) and 3(f) is equal to 3(d). This can be readily understood on the basis of the discussion at the end of Sec. II B of Ref. 2. There it was pointed out how it is possible to rotate the fields A_μ, Z_μ in such a way that in terms of the new fields A'_μ, Z'_μ there are no counterterms of the form $J_Z^\mu A'_\mu$, $l_Z^\mu A'_\mu$ and $Z'^\mu A'_\mu$ and to $O(g^3)$ the only other relevant change is that the coefficients of the $J_\gamma^\mu Z'_\mu$ and $l_\gamma^\mu Z'_\mu$ counterterms are twice as large as in the original frame. The sum of Figs. 3(c)–3(f) gives

$$M_{3(c),(d),(e),(f)}=ie\langle f|J_\gamma^\mu|i\rangle L_\mu\frac{1}{q^2-m_Z{}^2} \times\left\{\frac{g}{2c}\frac{A_{\gamma Z}(q^2)}{q^2}+\frac{e}{2s^2}\mathrm{Re}\left[\frac{A_{ZZ}(m_Z{}^2)}{m_Z{}^2}-\frac{A_{WW}(m_W{}^2)}{m_W{}^2}\right]\right\}. \tag{16}$$

It is convenient to distinguish three classes of contributions to the self-energy functions: those arising from intermediate vector bosons, Higgs scalar particles, and ghosts which for simplicity we call the bosonic component and those arising from leptons and hadrons. We distinguish these three classes by the superscripts b (bosonic), l (leptonic), and h (hadronic). In the renormalizable gauges and to order α the leptonic and hadronic contributions to the γZ self-energy are transverse and $A_{\gamma Z}^{(l)}(q^2)$ and $A_{\gamma Z}^{(h)}(q^2)$ are proportional to q^2. However, the bosonic component is not transverse and $A_{\gamma Z}^{(b)}(0)\neq 0$. Thus, the first term in Eq. (16) contains a pole at $q^2=0$. The explicit calculation of $A_{\gamma Z}^{(b)}(q^2)$ (see Appendix A) shows that this term cancels the corresponding pole in Eq. (15a). We observe that in order to verify the cancellation of pole terms proportional to $m_l{}^2/(m_W{}^2q^2)$ in the amplitude it is necessary to consider the contributions of the unphysical Higgs scalars [specifically, we must include the contributions of vertex diagrams analogous to Fig. 3(b) with one of the two W^+ virtual lines replaced by a ϕ^+ propagator].

E. Box diagrams

The box diagrams, involving the exchange of two heavy-intermediate-boson lines are illustrated in Fig. 4. They are finite and can be studied most easily with the methods described in Sec. 2 of Ref. 10 and Sec. IV of Ref. 6. Neglecting again terms of $O(q^2/m_W{}^2)$ and very small corrections of $O(\bar{g}_s{}^2(\kappa^2)/4\pi^2)$ induced by the strong interactions in the asymptotic domain we find

$$M_{4(a)}=\frac{-ig^4}{256\pi^2m_W{}^2}L_\mu\langle f|\bar\psi\gamma^\mu a_-(10C_3-6\underline{1})\psi|i\rangle, \tag{17a}$$

$$M_{4(b)}=\frac{i3(g^2+g'^2)^2}{64\pi^2m_Z{}^2}L_\mu \times\langle f|\bar\psi\gamma^\mu a_-(\tfrac{1}{4}\underline{1}-s^2C_3Q)\psi-s^4\bar\psi Q^2\gamma^\mu\gamma_5\psi|i\rangle, \tag{17b}$$

where C_3 is given in Eq. (3c) and $\underline{1}$ is the 6×6 unit matrix. With the usual assignment of quark charges we have the relations

$$Q=\tfrac{1}{2}C_3+\tfrac{1}{6}\underline{1}, \tag{17c}$$

$$C_3Q=\tfrac{1}{6}C_3+\tfrac{1}{2}\underline{1}, \tag{17d}$$

$$Q^2=\tfrac{1}{6}C_3+\tfrac{5}{18}\underline{1}. \tag{17e}$$

In this case, Eq. (17b) can be rewritten as

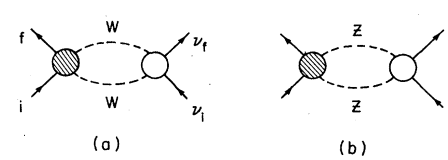

FIG. 4. Box diagrams. The unshaded circles indicate that the intermediate vector bosons are attached in all allowable ways to the ν lines.

$$M_{4(b)}=\frac{i3(g^2+g'^2)^2}{64\pi^2 m_Z{}^2}L_\mu\langle f|\overline{\psi}\gamma^\mu a_-[\tfrac{1}{4}\underline{1}(1-2s^2)-\tfrac{1}{6}C_3 s^2]\psi -\tfrac{1}{6}s^4\overline{\psi}\gamma^\mu\gamma_5[\tfrac{5}{3}\underline{1}+C_3]\psi|i\rangle\,. \tag{17f}$$

Combining Eqs. (17a) and (17f),

$$M_{4(a)}+M_{4(b)}=\frac{i(g^2+g'^2)}{m_Z{}^2}\frac{\alpha}{16\pi c^2 s^2}\times L_\mu\langle f|\overline{\psi}\gamma^\mu a_-[(\tfrac{9}{4}-3s^2)\underline{1}-(\tfrac{5}{2}-2s^2)C_3]\psi -\tfrac{1}{2}s^4\overline{\psi}\gamma^\mu\gamma_5[\tfrac{5}{3}\underline{1}+C_3]\psi|i\rangle\,, \tag{18}$$

We note that so far these are the only contributions of $O(G_F\alpha)$ which involve hadronic operators other than J_Z^μ and J_γ^μ. In particular, Eq. (18) contains a small isoscalar axial vector piece, which is absent at the tree level.[11]

As we will see in Sec. IIIC, the complete weak corrections can be expressed as contributions of $O(\alpha)$ to the conventional parameters $\epsilon_L(i),\epsilon_R(i)$ $(i=u,d,s,c,\ldots)$ in the effective Lagrangian describing the hadronic neutral current, with the understanding that some of these corrections are functions of q^2. For the box diagrams such information is fully contained in Eq. (18). However, in order to obtain a simple idea regarding the order of magnitude of the corrections, it is convenient to rewrite Eq. (18) in an alternative form which is particularly useful in the case of isoscalar targets:

$$M_{4(a)}+M_{4(b)}=\frac{-i(g^2+g'^2)}{m_Z{}^2}\frac{\alpha}{4\pi c^2 s^2}\times L_\mu\langle f|a_Z J_Z^\mu+a_\gamma s^2 J_\gamma^\mu + a_{\beta_L}J_{\beta_L}^\mu+a_{\beta_R}J_{\beta_R}^\mu|i\rangle\,, \tag{19a}$$

$$J_{\beta_L}^\mu\equiv\overline{\psi}\gamma^\mu(c^2\underline{1}+\tfrac{1}{3}s^2C_3)a_-\psi\,, \tag{19b}$$

$$J_{\beta_R}^\mu\equiv\overline{\psi}\gamma^\mu(\tfrac{1}{6}C_3-\tfrac{1}{2}\underline{1})a_+\psi\,, \tag{19c}$$

where $a_+=(1+\gamma_5)/2$ and

$$a_Z=(\tfrac{5}{2}-\tfrac{15}{4}s^2-\tfrac{1}{5}s^4+\tfrac{14}{9}s^6)/2c^2\,, \tag{19d}$$

$$a_\gamma=(\tfrac{5}{2}-\tfrac{61}{20}s^2-\tfrac{9}{10}s^4+\tfrac{14}{9}s^6)/2c^2\,, \tag{19e}$$

$$a_{\beta_L}=-(\tfrac{9}{16}-\tfrac{3}{4}s^2+\tfrac{4}{15}s^4)/c^2\,, \tag{19f}$$

$$a_{\beta_R}=-\tfrac{3}{10}s^4\,. \tag{19g}$$

Numerically, for $s^2=0.23$, these are $a_Z=1.07$, $a_\gamma=1.15$, $a_{\beta_L}=-0.525$, and $a_{\beta_R}=-0.0159$. The reason this alternative decomposition is useful is that the parton model calculations used to interpret deep-inelastic experiments the dominant contributions come from u and d quark terms in the effective Lagrangian. In the case of isoscalar targets there is no interference between the isospin-0 and isospin-1 parts of the interaction. Since the currents $J_{\beta_L}^\mu$ and $J_{\beta_R}^\mu$ have been defined to be orthogonal in the isospin of the u,d sector to the left- and right-handed zeroth-order currents of the $SU(2)_L\times U(1)$ theory, respectively, they will not contribute to $O(\alpha)$ to the terms involving u, d, $\bar{u}$, and $\bar{d}$ partons, provided the target is isoscalar. This can be readily verified as follows: In the parton model calculations for isoscalar targets and in the limit of isospin symmetry, the u, d, $\bar{u}$ and $\bar{d}$ contributions are proportional to $\epsilon_L{}^2(u)+\epsilon_L{}^2(d)$ and $\epsilon_R{}^2(u)+\epsilon_R{}^2(d)$.[12] Inserting in these combinations the tree level terms of the $SU(2)_L\times U(1)$ theory as well as the contributions of $O(\alpha)$ arising from $J_{\beta_L}^\mu$ and $J_{\beta_R}^\mu$ one readily verifies that the cross terms vanish. Thus within the framework of such calculations and approximations, and for the particular case of isoscalar targets, we may ignore the $J_{\beta_L}^\mu$ and $J_{\beta_R}^\mu$ currents, thereby obtaining once again a description to $O(\alpha)$ of the neutral-current interaction amplitude simply in terms of two operators J_Z^μ and J_γ^μ.

F. Radiative corrections to ν-lepton scattering and ν-induced processes

The analysis of the previous subsections was carried out for the case of ν-nucleon scattering. What modifications are required for other processes such as $\nu_\mu e$ scattering or the neutral-current contributions to $\nu_e e$ collisions? It is easy to see that the results of Eqs. (10), (12), (14), and (16) remain valid provided that the hadronic currents J_Z^μ and J_γ^μ are replaced by the corresponding charged lepton currents and $|i\rangle$ and $|f\rangle$ are interpreted as initial and final lepton states. This is obvious in the case of Eqs. (14) and (16). Regarding Eqs. (10) and (12) we note that these results essentially depend on the Ward identities associated with the time-time and time-space algebra of the hadronic currents. Their validity in the case of ν-lepton scattering follows from the isomorphism of the algebras defined by hadronic and leptonic currents in the $SU(2)_L\times U(1)$ theory.

The analogy is not nearly as perfect for the box diagrams. In the hadronic case there are corrections of $O(\overline{g}_s{}^2(\kappa^2)/4\pi^2)$ induced by the strong interactions in the asymptotic domain which are obviously absent in the leptonic case. However, such corrections can be estimated by invoking the asymptotic freedom of the underlying theory of strong interactions and are found to be very small.[6] Indeed, they have been neglected in Eqs. (17a) and (17b). A second more important difference is that Eqs. (17a) and (17b) depend on the charges and C_3 eigenvalues of the fundamental fields in a nontrivial way. Thus, remembering that the eigenvalues of the matrices C_3 and Q equal -1 for the charged leptons, Eq. (17b) tells us that in the case of ν-l scattering ($l=e,\mu,\tau$) the amplitude corresponding to Fig. 4(b) becomes

$$M_{4(b)}^{(l)}=\frac{i3(g^2+g'^2)^2}{64\pi^2 m_Z^2}L_\mu[\bar{u}_f\gamma^\mu a_-(\tfrac{1}{4}-s^2)u_i-s^4\bar{u}_f\gamma^\mu\gamma_5 u_i], \tag{20a}$$

where u_i and u_f are the spinors of the initial and final charged lepton. Similarly, the amplitudes corresponding to Fig. 4(a) can be read from Eq. (17a). Combining this result with Eq. (20a) we find

$$M_{4(a)}^{(l)}+M_{4(b)}^{(l)}=\frac{-i(g^2+g'^2)}{m_Z^2}\frac{\alpha}{4\pi c^2 s^2}L_\mu[c_Z l_Z^\mu+c_\gamma s^2 l_\gamma^\mu], \tag{20b}$$

where

$$l_Z^\mu=-\tfrac{1}{2}\bar{u}_f\gamma^\mu a_- u_i+s^2\bar{u}_f\gamma^\mu u_i, \tag{20c}$$

$$l_\gamma^\mu=-\bar{u}_f\gamma^\mu u_i \tag{20d}$$

are the neutral and electromagnetic currents of the charged leptons in the $\mathrm{SU}(2)_L\times\mathrm{U}(1)$ theory and

$$c_Z=\tfrac{19}{8}-\tfrac{7}{2}s^2+3s^4, \tag{20e}$$

$$c_\gamma=\tfrac{19}{8}-\tfrac{17}{4}s^2+3s^4. \tag{20f}$$

For $s^2=0.23$, $c_Z=1.73$ and $c_\gamma=1.56$.

In summary the main difference between the weak corrections to ν-hadron and ν-lepton scattering is that Eq. (19a) is replaced by Eq. (20b). On the other hand the photonic corrections associated with the hadron and lepton legs have, of course, a different theoretical status. The former depend on the dynamics of the strong interactions while the latter can be precisely evaluated. Unfortunately, present experiments involving ν-lepton scattering are very difficult and not yet very precise.

We should also point out that the amplitudes for $\bar{\nu}$-induced processes are simply obtained by replacing $L_\mu\to-\bar{v}_{\nu_i}\gamma_\mu a_- v_{\nu_f}$ in all of our previous results, where the v's denote negative energy spinors.

III. COMBINATION OF RESULTS

A. Weak corrections of $O(G_F\alpha)$ proportional to $\langle f|J_Z^\mu|i\rangle$

In this section we combine all the weak corrections of $O(G_F\alpha)$ proportional to $\langle f|J_Z^\mu|i\rangle$ obtained in Sec. II. In the case of ν-hadron scattering, using Eqs. (2) and (13b) we find for the sum of Eqs. (10), (12), (14), and (19a)

$$\delta M_{J_Z}=M^0\left\{\frac{A_{ZZ}(q^2)-\mathrm{Re}A_{ZZ}(m_Z^2)}{q^2-m_Z^2}-\frac{2\delta e}{e}-\left(1-\frac{c^2}{s^2}\right)\mathrm{Re}\left[\frac{A_{ZZ}(m_Z^2)}{m_Z^2}-\frac{A_{WW}(m_W^2)}{m_W^2}\right]-\frac{\alpha}{\pi}\frac{c^2}{s^2}\left[\frac{2}{n-4}+2C+\ln\left(\frac{m_W^2}{\mu^2}\right)\right]+\frac{\alpha}{2\pi c^2 s^2}a_Z\right\}, \tag{21a}$$

where we have used the decomposition of the box diagrams given in Eq. (19a). Using the expressions for the self-energy functions and δe given in the Appendices, one readily verifies that the sum of the terms in Eq. (21a) is convergent (as $n\to 4$) and independent of μ. Before attempting to evaluate the finite parts it is advantageous to combine Eqs. (2) and (21a):

$$M^0+\delta M_{J_Z}=M^0[1+\delta_Z(q^2)], \tag{21b}$$

where $\delta_Z(q^2)$ represents the expression between curly brackets in Eq. (21a), and to express $(g^2+g'^2)/m_Z^2$ in M^0 in terms of G_μ. To do this we remember that in the renormalization scheme being employed $(g^2+g'^2)/m_Z^2=g^2/m_W^2$ exactly and the latter is related to G_μ by Eqs. (34b), (34c), and (35b) of Ref. 2, i.e., $g^2/m_W^2=(8G_\mu/\sqrt{2})(1+\Delta r)^{-1}$ where Δr is given in Eq. (A18). In this way we find

$$M^0+\delta M_{J_Z}=\hat{M}^0[1+\delta_Z(q^2)-\Delta r], \tag{21c}$$

where

$$\hat{M}^0=\frac{im_Z^2}{q^2-m_Z^2}\frac{G_\mu}{\sqrt{2}}2\langle f|J_Z^\mu|i\rangle\bar{u}_{\nu_f}\gamma_\mu(1-\gamma_5)u_{\nu_i} \tag{22a}$$

is obtained from M^0 by the substitution $(g^2+g'^2)/m_Z^2\to 8G_\mu/\sqrt{2}$. As explained in Ref. 2, this procedure has two important advantages: The amplitude is given in terms of an accurately known constant

$$G_\mu=(1.166\,32\pm0.000\,02)\times10^{-5}\ \mathrm{GeV}^{-2} \tag{22b}$$

and the combination $\delta_Z(q^2)-\Delta r$ in (21c) is considerably simpler than either term separately (they cancel to some extent). Neglecting terms of relative order $\alpha q^2/m_Z^2$, which is equivalent to setting $q^2=0$ in $\delta_Z(q^2)$ and defining $\rho_{\mathrm{NC}}^{(\nu;h)}=1+\delta_Z(0)-\Delta r$ (NC stands for "neutral current" while $\nu;h$ refers to ν-hadron interactions) we find

$$M^0+\delta M_{J_Z}=\rho_{\mathrm{NC}}^{(\nu;h)}\hat{M}^0, \tag{23a}$$

$$\rho_{\rm NC}^{(\nu;h)} = 1 + \frac{A_{WW}(0)}{m_W^2} - \frac{A_{ZZ}(0)}{m_Z^2} + \frac{\alpha}{\pi}\left[\frac{2}{n-4} + 2C + \ln\left(\frac{m_Z^2}{\mu^2}\right)\right] + \frac{\alpha}{\pi s^2}\left[\frac{a_Z}{2c^2} - \frac{3}{2} - \frac{\ln c^2}{s^2}\left(\frac{7}{8} - \frac{1}{2}s^2 - s^4\right)\right]. \tag{23b}$$

We note that the quantities $\delta e/e$, $\mathrm{Re}A_{ZZ}(m_Z^2)$, and $\mathrm{Re}A_{WW}(m_W^2)$ which appeared in Eq. (21a) have canceled. The elimination of $\delta e/e$ is particularly welcome because, as explained in Ref. 2, the hadronic contributions to this quantity depend on the dynamics of the strong interactions. If all the leptonic and relevant hadronic masses satisfy m_l^2, $m^2 \ll m_W^2$, then the leptonic and hadronic contributions to the second and third terms of Eq. (23b) can be neglected. This is clearly the case for all the leptons and quarks of the 6 quark model with the possible exception of the t quark, which has yet to be experimentally observed. Inserting the expressions for the bosonic component $A_{WW}^{(b)}(0)/m_W^2 - A_{ZZ}^{(b)}(0)/m_Z^2$ given in Appendix A as well as the t-quark contribution to $A_{WW}^{(h)}(0)/m_W^2 - A_{ZZ}^{(h)}(0)/m_Z^2$ evaluated in the free field theory, Eq. (23b) becomes

$$\rho_{\rm NC}^{(\nu;h)} = 1 + \frac{\alpha}{4\pi}\left[\frac{3}{4s^4}\ln c^2 - \frac{7}{4s^2} + \frac{2a_Z}{c^2 s^2} + G(\xi, c^2) + \frac{3}{4s^2}\frac{m_t^2}{m_W^2}\right], \tag{24a}$$

$$G(\xi, c^2) \equiv \frac{3}{4}\frac{\xi}{s^2}\left[\frac{\ln(c^2/\xi)}{c^2 - \xi} + \frac{1}{c^2}\frac{\ln\xi}{1-\xi}\right], \tag{24b}$$

where $\xi = m_{\phi_1}^2/m_Z^2$ and m_{ϕ_1} is the mass of the physical Higgs scalar. For $s^2 = 0.23$, the first three terms inside the square brackets of Eq. (24a) add up to +0.755. For $\xi \gg 1$ we have the asymptotic formula[13]

$$G(\xi, c^2) = -\frac{3}{4}\left(\frac{\ln\xi}{c^2} + \frac{\ln c^2}{s^2}\right) + O\left(\frac{\ln\xi}{\xi}\right). \tag{24c}$$

We emphasize that $\rho_{\rm NC}^{(\nu;h)}$ in Eqs. (23a) and (24a) represents a universal renormalization of the coupling strength of the neutral-current interactions for all ν-hadron scattering processes, provided that terms of relative order $\alpha q^2/m_Z^2$ are negligible. Furthermore, in applying this correction it should be understood that $\hat{M}^0$ is defined in terms of G_μ according to Eqs. (22a) and (22b).

For values 10 GeV $\leq m_{\phi_1} \leq$ 1 TeV and $m_t < m_W$, $\rho_{\rm NC}^{(\nu;h)}$ is very close to 1. For example, for $s^2 = 0.23$ and $m_t = 18$ GeV, $\rho_{\rm NC}^{(\nu;h)} = 1.000\,54$, $1.000\,28$, $1.000\,23$, and 0.9984 for $\xi = 0$, c^2, 1, and 100, respectively. A very large value of m_{ϕ_1}, of the order of $1.7 \times 10^4 m_Z$, is required to make $\rho_{\rm NC}^{(\nu;h)} = 0.99$. Conversely, one would need $m_t \simeq 2.3 m_W$ in order to make $\rho_{\rm NC}^{(\nu;h)} = 1.01$.

According to the discussion in Sec. II F, the only change in the case of ν-lepton scattering is the substitution of c_Z for a_Z in Eq. (24a). In that way we obtain the appropriate renormalization factor $\rho_{\rm NC}^{(\nu;l)}$. Comparing the values of a_Z and c_Z given after Eqs. (19g) and (20f), respectively, we see that $\rho_{\rm NC}^{(\nu;l)}$ is larger than $\rho_{\rm NC}^{(\nu;h)}$ by 0.0043 for any given values of m_{ϕ_1} and m_t. Thus, for $s^2 = 0.23$ and $m_t = 18$ GeV, $\rho_{\rm NC}^{(\nu;l)} = 1.0048$, 1.0046, 1.0045, and 1.0027 for $\xi = 0$, c^2, 1, and 100, respectively.

The numerical results of this subsection are summarized in Table I.

B. Weak corrections of $O(G_Z\alpha)$ proportional to $\langle f | J_\gamma^\mu | i \rangle$

From Eqs. (10), (13b), (15a), (16), and (19a) we find the following for the sum of the contributions proportional to J_γ^μ in the case of ν-hadron interactions:

$$\delta M_{J_\gamma} = \frac{i(g^2 + g'^2)}{2(q^2 - m_Z^2)} s^2 \langle f | J_\gamma^\mu | i \rangle L_\mu \left\{ \frac{c}{s}\frac{A_{\gamma Z}(q^2)}{q^2} + \frac{c^2}{s^2}\mathrm{Re}\left[\frac{A_{ZZ}(m_Z^2)}{m_Z^2} - \frac{A_{WW}(m_W^2)}{m_W^2}\right] + \frac{\alpha}{2\pi s^2}\left[c^2\left(\frac{m_Z^2}{q^2} - 2\right)\left(\frac{2}{n-4} + 2C + \ln(m_W^2/\mu^2)\right) + R_l(q^2) + \frac{a_\gamma}{c^2}\right]\right\}, \tag{25a}$$

where we have again used the decomposition of the box diagrams given in Eq. (19a) and $R_l(q^2)$ is defined in Eq. (15b). At this stage it is convenient to separate the bosonic, leptonic, and hadronic contributions to self-energies. That is, we write $A_{\gamma Z}(q^2) = A_{\gamma Z}^{(b)}(q^2) + A_{\gamma Z}^{(l)}(q^2) + A_{\gamma Z}^{(h)}(q^2)$ and similarly decompose $\mathrm{Re}A_{ZZ}(m_Z^2)$ and $\mathrm{Re}A_{WW}(m_W^2)$. The bosonic contributions to the self-energies combine with the other terms between the curly brackets in Eq. (25a) to provide a finite gauge-invariant contribution which, for want of a better name, we call

TABLE I. Values of $\rho_{\rm NC}^{(\nu;h)} - 1$ corresponding to ν-hadron scattering as a function of m_{ϕ_1} for $\sin^2\theta_W = 0.23$ and $m_t = 18$ GeV. The values of $\rho_{\rm NC}^{(\nu;l)} - 1$ corresponding to ν-lepton scattering are obtained by adding a positive contribution 4.3×10^{-3} to all the entries in the table.

m_{ϕ_1}	$\rho_{\rm NC}^{(\nu;h)} - 1$
0	5.4×10^{-4}
m_W	2.8×10^{-4}
m_Z	2.3×10^{-4}
$10 m_Z$	-1.6×10^{-3}

the bosonic contribution $\delta_\theta^{(b)}(q^2)$. The leptonic and hadronic contributions arising from the self-energies in Eq. (25a) are separately finite and gauge invariant: We call them $\delta_\theta^{(l)}(q^2)$ and $\delta_\theta^{(h)}(q^2)$. Thus,

$$\delta M_{J_\gamma} = \frac{i(g^2+g'^2)}{2(q^2-m_Z{}^2)} s^2\langle f|J_\gamma^\mu|i\rangle L_\mu \Delta^{(\nu;h)}(q^2)\,, \quad (25b)$$

where

$$\Delta^{(\nu;h)}(q^2) = \delta_\theta^{(b)}(q^2)+\delta_\theta^{(l)}(q^2)+\delta_\theta^{(h)}(q^2) \quad (25c)$$

stands for the complete expression between the curly brackets in Eq. (25a). Comparing Eq. (25b) with Eqs. (2) and (3a) we see that the correction $\Delta^{(\nu;h)}(q^2)$ can be absorbed into an effective parameter $\sin^2\theta_W(q^2)_{\text{eff}}$

$$\sin^2\theta_W(q^2)_{\text{eff}} = \kappa^{(\nu;h)}(q^2)\sin^2\theta_W\,, \quad (25d)$$

where $\kappa^{(\nu;h)}(q^2)$ represents a momentum-dependent renormalization factor given by

$$\kappa^{(\nu;h)}(q^2) = 1-\Delta^{(\nu;h)}(q^2)\,. \quad (25e)$$

Using the results of Appendix A we find for the bosonic contributions

$$\begin{aligned}\delta_\theta^{(b)}(q^2) = \frac{\alpha}{2\pi s^2}\Big\{ & R_l(q^2) - \tfrac{1}{9}[26c^2+\tfrac{35}{4}-\tfrac{19}{2}s^2] \\ & -\frac{1}{2s^2}\ln c^2[\tfrac{5}{6}-3c^2] + \frac{a_\gamma}{c^2} \\ & +\frac{c^2}{2s^2}[I_1(c^2)-I_2(c^2)] \\ & +\frac{1}{2s^2}\left[H(\xi)-c^2H\left(\frac{\xi}{c^2}\right)\right]\Big\}\,, \end{aligned}\quad (26a)$$

where

$$\begin{aligned}I_1(c^2) = -\int_0^1 dx & \left[8c^4+7c^2-\tfrac{3}{2}-x(1-x)\left(12c^2+\frac{1}{2c^2}-2\right)\right] \\ & \times \ln\left(1-\frac{x(1-x)}{c^2}\right)\,, \end{aligned}\quad (26b)$$

$$\begin{aligned}I_2(c^2) = -\int_0^1 dx & \left[10+4c^2-\frac{1}{2c^2}-x\left(8+\frac{1}{2c^2}+2c^2\right)\right. \\ & \left.+x^2(10c^2+\tfrac{1}{2})\right]\ln\left[x^2+\frac{1-x}{c^2}\right]\,, \end{aligned}\quad (26c)$$

and $H(\xi)$ is the function introduced in Ref. 2:

$$\begin{aligned}H(\xi) = \int_0^1 dx & \left[1-\frac{x^2}{2}-\frac{\xi}{2}(1-x)\right]\ln[x^2+\xi(1-x)] \\ & +\frac{\xi}{4}(\ln\xi-\tfrac{1}{2})\,. \end{aligned}\quad (26d)$$

For large ξ we have the asymptotic formula

$$H(\xi) = \tfrac{5}{6}\ln\xi - \tfrac{31}{36} + O\left(\frac{\ln\xi}{\xi}\right)\,, \quad (26e)$$

while for small ξ

$$H(\xi) = -\tfrac{17}{9}+\pi\sqrt{\xi}+O(\xi\ln\xi)\,. \quad (26f)$$

For completeness the functions $I_1(c^2)$ and $I_2(c^2)$ are calculated in Appendix A. For $s^2=0.23$, their values are $I_1(0.77)=1.768$ and $I_2(0.77)=0.147$. Thus, recalling the asymptotic form of $R_l(q^2)$ given in Eq. (15d) and the numerical value of a_γ quoted in Sec. II B, Eq. (26a) becomes for $s^2=0.23$ and $-q^2 \gg m_l{}^2$

$$\begin{aligned}\delta_\theta^{(b)}(q^2) = \frac{\alpha}{2\pi s^2}\Big\{ & \tfrac{1}{3}\ln(m_W{}^2/-q^2)+1.30 \\ & +\frac{1}{2s^2}\left[H(\xi)-c^2H\left(\frac{\xi}{c^2}\right)\right]\Big\}\,. \end{aligned}\quad (26g)$$

The third term between the curly brackets of Eq. (26g) equals -0.944, -0.410, -0.355, and $+1.15$ for $\xi=0$, 0.77, 1, and 100 respectively. For $-q^2=20$ GeV^2, which is a typical invariant momentum transfer for deep-inelastic ν scattering, we find $\delta_\theta^{(b)}(-20\text{ GeV}^2)=0.0115$, 0.0142, 0.0145, and 0.0221 for $\xi=0.$ 0.77, 1, and 100, respectively.

For large values of $-q^2$, theoretical arguments based on the asymptotic freedom of the underlying strong interaction theory suggest that the hadronic contributions to the self-energies in Eq. (25a) are given to a good approximation by the quark loops of the free field theory.[14] Using the results of Appendix B this leads to

$$\delta_\theta^{(h)}(q^2) = \frac{\alpha}{2\pi s^2}\left\{\sum_{f=1}^{2N} 3(C_{3f}Q_f-4s^2Q_f{}^2)\int_0^1 dx\, x(1-x)\ln\left(\frac{m_f{}^2-q^2x(1-x)}{m_Z{}^2x(1-x)}\right)-\frac{Nc^2\ln c^2}{2s^2}\right\}\,, \quad (27)$$

where the sum is over flavors, N is the number of left-handed quark doublets, Q_f and C_{3f} are eigenvalues of the matrices Q and C_3 [see Eqs. (3b) and (3c)] associated with flavor f, m_f is the corresponding mass, a summation over color indices has been carried out explicitly, and we have neglected terms of $O(\alpha m_f{}^2/m_W{}^2)$.

The leptonic contributions coming from the self-energies in Eq. (25a) can be read off Eq. (27) by choosing the eigenvalues of Q and C_3 appropriate to the leptons and dividing the overall result by 3 to eliminate the color degrees of freedom. This leads to

$$\delta_\theta^{(l)}(q^2) = \frac{\alpha}{2\pi s^2}\left\{(1-4s^2)\sum_{l=1}^{N}\int_0^1 dx\, x(1-x)\ln\left(\frac{m_l{}^2-q^2x(1-x)}{m_Z{}^2x(1-x)}\right)-\frac{Nc^2\ln c^2}{6s^2}\right]\,, \quad (28)$$

where the sum is over the charged leptons ($e^-, \mu^-, \tau^- \ldots$).

For large values of $-q^2$ we can neglect the masses of the lightest leptons and quarks in the evaluation of the integrals of Eqs. (27) and (28). Thus, for example, in the six-flavor model it is a good approximation to set $m_e, m_\mu, m_u, m_d, m_s = 0$. For the τ lepton and heavy quarks c, b, t, it is better to use the accurate formulas given in Appendix B.

Setting $-q^2 = 20$ GeV^2, $N = 3$, $s^2 = 0.23$, $m_Z = 91.5$ GeV,[15] $m_\tau = 1.807$ GeV, $m_c = m_b/3 = 1.5$ GeV, $m_t = 18$ GeV, and adopting the usual charge assignments for the quarks we obtain $\delta_\theta^{(l)}(-20\text{ GeV}^2) = 0.00103$ and $\delta_\theta^{(h)}(-20\text{ GeV}^2) = -0.0110$.[16] We note that $\delta_\theta^{(l)}(-20\text{ GeV}^2)$ is quite small: This is largely due to the factor $1-4s^2$ which suppresses the contribution of the first term of Eq. (28). The hadronic (or quark) contribution is of $O(1\%)$ but has opposite sign than the bosonic contribution. Adding the bosonic, leptonic, and hadronic contributions at $-q^2 = 20$ GeV^2 and $s^2 = 0.23$ we obtain $\Delta^{(\nu;h)}(-20\text{ GeV}^2) = 0.0015$, 0.0042, 0.0045, and 0.012 for $\xi = 0$, 0.77, 1, and 100, respectively. From Eqs. (25d) and (25e) we see that the effect of $\Delta^{(\nu;h)}(-20\text{ GeV}^2)$ is to shift the value of $\sin^2\theta_W$ by a small amount $\sim 1\%$ or less.

In the case of ν-lepton scattering, according to the discussion in Sec. II F, the only change is the substitution of c_γ for a_γ in Eq. (25a). Using the values of these constants given after Eqs. (19g) and (20f), we see that the value of $\Delta^{(\nu;l)}(q^2)$ appropriate to ν-lepton scattering is larger than the corresponding correction in ν-hadron interactions by 0.0027. Thus, for example, for $-q^2 = 20$ GeV^2 and $s^2 = 0.23$ we find $\Delta^{(\nu;l)}(-20\text{ GeV}^2) = 0.0042$, 0.0069, 0.0072, and 0.015 for $\xi = 0$, 0.77, 1, and 100, respectively.

So far we have considered the case $m_t^2 \ll m_W^2$. In Appendix B we discuss the corrections to Eq. (27) in the free field theory if terms of $O(m_t^2/m_Z^2)$ are not negligible. If $m_t = m_W$, for instance, we find a departure from the previous values of $\Delta^{(\nu;h)}$, but it is in the direction of making $\Delta^{(\nu;h)}$ even smaller. For values of $m_t \simeq 3m_Z$, however, the corrections to $\kappa^{(\nu;h)}$ and $\rho_{\text{NC}}^{(\nu;h)}$ become fairly large, of the order of several percent. However, the two corrections affect the determination of $\sin^2\theta_W$ from the experimental ratio $R^{(\nu)} = \sigma(\nu + N \to \nu + X)/\sigma(\nu + N \to \mu + X)$ in opposite directions. A simple argument in the Appendix shows that the overall effect is in the direction of increasing the value of $\sin^2\theta_W$ extracted from experiment. The effect of contributions proportional to m_t^2/m_Z^2 on the corrections to the relationship between m_W and $\sin\theta_W$ is also briefly discussed in Appendix B.

The numerical results of this subsection are summarized in Table II.

TABLE II. Values of $\Delta^{(\nu;h)}(q^2) \equiv 1 - \kappa^{(\nu;h)}(q^2)$ at $-q^2 = 20$ GeV^2 and $q^2 = 0$ as a function of m_{ϕ_1} for $\sin^2\theta_W = 0.23$. To evaluate $\Delta^{(\nu;h)}(-20\text{ GeV}^2)$ we have assumed $m_c = m_b/3 = 1.5$ GeV and $m_t = 18$ GeV. The values of $\Delta^{(\nu;h)}(0)$ are for ν_μ beams; they depend on the light-quark masses and are therefore less reliable. The quoted values of $\Delta^{(\nu;h)}(0)$ in the table correspond to $m_u = m_d = m_s = 300$ MeV. The values of $\Delta^{(\nu;l)}$ for ν-lepton scattering are obtained by adding a positive contribution 2.7×10^{-3} to all entries in the table. For ν_e beams a contribution of 1.79×10^{-2} must be added to the values of $\Delta^{(\nu;h)}(0)$ and $\Delta^{(\nu;l)}(0)$.

m_{ϕ_1}	$\Delta^{(\nu;h)}(-20\text{ GeV}^2)$	$\Delta^{(\nu;h)}(0)$
0	1.5×10^{-3}	2.2×10^{-3}
m_W	4.2×10^{-3}	4.9×10^{-3}
m_Z	4.5×10^{-3}	5.2×10^{-3}
$10m_Z$	1.2×10^{-2}	1.3×10^{-2}

C. The case of small momentum transfers

Provided that we neglect corrections of $O(\alpha q^2/m_W^2)$, we have seen that $\rho_{\text{NC}}^{(\nu;h)}$ and $\rho_{\text{NC}}^{(\nu;l)}$ are constants. Therefore, the results obtained in Sec. III A remain valid for low values of $-q^2$. However, the corrections $\Delta^{(\nu;h)}(q^2)$ and $\Delta^{(\nu;l)}(q^2)$ and the corresponding renormalization factors $\kappa^{(\nu;h)}(q^2)$ and $\kappa^{(\nu;l)}(q^2)$ do depend on q^2 as illustrated in Eqs. (26a), (27), and (28). This dependence arises from charge-radius diagrams as well as hadronic and leptonic contributions to the γZ self-energy. In the considerations of Sec. III B we focused our attention on large momentum transfers, characteristic of deep-inelastic scattering.

In attempting to discuss $\Delta^{(\nu;h)}(q^2)$ and $\Delta^{(\nu;l)}(q^2)$ for low values of $-q^2$, say $-q^2 < 1$ GeV^2, we face the problem that the hadronic contribution $\delta_\theta^{(h)}(q^2)$ to the γZ self-energy becomes sensitive to the light-quark as well as the heavy-quark masses [see Eq. (27)]. This is a clear signal of dependence on the dynamics of the strong interactions. Indeed, the theoretical arguments alluded to in Sec. III B to justify the application of free field theory in the evaluation of these contributions are no longer valid. Thus, in a very definite sense the radiative corrections represented by $\kappa^{(\nu;h)}(q^2)$ and $\kappa^{(\nu;l)}(q^2)$ are theoretically better defined at large than at low momentum transfers. Nevertheless, we may attempt to estimate these corrections in the free-quark model using constituent masses $m_u \simeq m_d \simeq 100$ to 300 MeV and $m_s \simeq 300$ MeV. As an example, let us consider the case $q^2 = 0$. Using Eqs. (15c) and (26a) we find

$$\delta_\theta^{(b)}(0) = \frac{\alpha}{2\pi s^2}\left\{\tfrac{2}{3}\ln(m_W/m_l) + 0.744 + \frac{1}{2s^2}\left[H(\xi) - c^2 H\left(\frac{\xi}{c^2}\right)\right]\right\} \tag{29}$$

instead of Eq. (26g). For an incident ν_μ beam, $m_l = m_\mu$ and we obtain $\delta_\theta^{(b)}(0)=0.0213$, 0.0240, 0.0243, and 0.0317 for $\xi=0$, 0.77, 1, and 100, respectively. For an incident ν_e beam there is an additional contribution $[\alpha/(3\pi s^2)]\ln(m_\mu/m_e)=0.0179$ arising from the charge-radius contribution in Eq. (29). We mention in passing that, for a given target, the distinction between the corrections for the cases of ν_μ and ν_e beams exists only at low values of q^2. As soon as $-q^3 \gg m_\mu{}^2$, the corrections become essentially identical.

Using the values of m_Z, m_c, m_b, and m_t given in Sec. III B and $m_s=300$ MeV we find from Eq. (27) $\delta_\theta^{(h)}(0)=-0.0186$ if $m_u=m_d=300$ MeV, $\delta_\theta^{(h)}(0) = -0.0213$ if $m_u=m_d=100$ MeV.[16] Finally, Eq. (28) leads to $\delta_\theta^{(l)}(0)=-0.00052$. Adding up the values of $\delta_\theta^{(b)}(0)$, $\delta_\theta^{(h)}(0)\simeq-0.0186$, and $\delta_\theta^{(l)}(0)$ we find for an incident ν_μ beam $\Delta^{(\nu;h)}(0)=0.0022$, 0.0049, 0.0052, and 0.013 for $\xi=0$, 0.77, 1, and 100, respectively. Comparison with the results of Sec. III B show that $\Delta^{(\nu;l)}(0)$ and $\Delta^{(\nu;l)}(-20\text{ GeV}^2)$ are numerically quite similar. Of course, any comparison depends on the particular values assumed for the masses of the light quarks. However, if the above estimates based on the free field theory are approximately valid, the main conclusion would be that the departure of $\kappa^{(\nu;h)}(0)$ from 1 remains quite small for ν_μ beams provided that $\sin^2\theta_W$ is defined according to Eq. (1). It is interesting to note that the smallness of $\Delta^{(\nu;h)}(0)$ can be traced, to a large extent, to a cancellation between bosonic and hadronic contributions. Indeed, both $\delta_\theta^{(b)}(0)$ and $\delta_\theta^{(h)}(0)$ are significantly larger in magnitude than $\delta_\theta^{(b)}(-20\text{ GeV}^2)$ and $\delta_\theta^{(h)}(-20\text{ GeV}^2)$, respectively, but they have opposite signs and their combined effect remains quite small.

The above numerical results were obtained for ν_μ hadron scattering. According to our discussion in Sec. III B, for the case of ν_μ-charged lepton scattering there is a further additive contribution of 0.0027 to $\Delta^{(\nu;l)}(0)$. This, however, is a constant contribution and does not affect our discussion about the slowness of the q^2 dependence.

Finally, for ν_e beams, there is a more significant q^2 variation: In that case $\Delta^{(\nu;h)}(0)$ and $\Delta^{(\nu;l)}(0)$ are larger than their corresponding values at $q^2=-20\text{ GeV}^2$ by about 1.8%, an effect that can be traced to the m_l dependence of Eq. (29). The numerical results described above are summarized in Table II.

The $-q^2$ dependence of $\delta_\theta^{(b)}(q^2)$, $\delta_\theta^{(h)}(q^2)$, $\delta_\theta^{(l)}(q^2)$ and $\Delta^{(\nu;h)}(q^2)$ is studied in Table III for $\xi=1$ over a wide range of momentum transfers. We note that $\delta_\theta^{(l)}(q^2)$ remains small while $\delta_\theta^{(b)}(q^2)$ and $\delta_\theta^{(h)}(q^2)$ are considerably larger and vary significantly from $-q^2\simeq 0$ to $-q^2=100\text{ GeV}^2$. However, they are of opposite sign so the overall correction $\Delta^{(\nu;h)}(q^2)$ is quite small and varies slowly with $-q^2$.

TABLE III. Dependence of $\delta_\theta^{(b)}(q^2)$, $\delta_\theta^{(h)}(q^2)$, $\delta_\theta^{(l)}(q^2)$, and $\Delta^{(\nu;h)}(q^2)\equiv 1-\kappa^{(\nu;h)}(q^2)$ on $-q^2$. The calculation of $\delta_\theta^{(b)}(q^2)$ has been done for $\xi=1$, i.e., $m_{\phi_1}=m_Z$ and $\delta_\theta^{(b)}(0)$ is for a ν_μ beam. In evaluating $\delta_\theta^{(h)}(q^2)$ we have assumed the same quark masses as in Table II. To obtain the values corresponding to ν_μ-lepton scattering a contribution of 0.27 must be added to all entries in the second and fifth columns. For ν_e beams a contribution of 1.79% must be added to $\delta_\theta^{(b)}(0)$, $\Delta^{(\nu;h)}(0)$, and $\Delta^{(\nu;l)}(0)$.

$-q^2$ (GeV^2)	$\delta_\theta^{(b)}(q^2)$, $\xi=1$ (%)	$\delta_\theta^{(h)}(q^2)$ (%)	$\delta_\theta^{(l)}(q^2)$ (%)	$\Delta^{(\nu;h)}(q^2)$, $\xi=1$ (%)
0	2.43	−1.86	−0.052	0.52
1	1.94	−1.64	0.059	0.36
10	1.57	−1.25	0.092	0.41
20	1.45	−1.10	0.10	0.45
50	1.30	−0.88	0.12	0.54
100	1.18	−0.69	0.13	0.62

One may wonder whether it is possible to circumvent the difficulties associated with the calculation at $q^2=0$ by means of a dispersive approach, similar for example to the method followed in Ref. 2 to discuss the hadronic contributions to the relation between m_W and $\sin\theta_W$. A detailed study of this problem lies beyond the scope of the present paper, so we will limit ourselves to a brief discussion of a possible strategy. The hadronic contributions to the γZ self-energy involve the amplitude

$$\frac{ige}{c}\int d^4y\, e^{iq\cdot y}\langle 0|\,T[J_\gamma^\mu(y)J_Z^\nu(0)]\,|\,0\rangle = \Pi_{\gamma Z}^{(h)}(q^2)(q^\mu q^\nu - g^{\mu\nu}q^2)\,, \quad (30)$$

where $\Pi_{\gamma Z}^{(h)}(q^2)=-A_{\gamma Z}^{(h)}(q^2)/q^2$. Only the vector part of J_Z^ν contributes to (30). For that part of J_Z^ν proportional to J_γ^ν [i.e., the second term of Eq. (3a)], the contribution to $\Pi_{\gamma Z}^{(h)}(q^2)$ is proportional to $\Pi_{\gamma\gamma}^{(h)}(q^2)$ which can be related by a dispersion relation to $\sigma(e^+e^-\to\text{hadrons})$. For the part of J_Z^ν proportional to $\bar\psi C_3\gamma^\mu\psi$ that is not immediately possible. However, we may proceed as follows: There are theoretical arguments which suggest that the heavy-quark contributions can be reliably computed by perturbative methods, even at $q^2=0$.[14] The terms involving light quarks in $\bar\psi C_3\gamma^\mu\psi$ are of the form $\bar u\gamma^\mu u-\bar d\gamma^\mu d-\bar s\gamma^\mu s$. The first two terms are proportional to the isovector part of the electromagnetic current. Therefore their contribution to $\Pi_{\gamma Z}^{(h)}(q^2)$ can be related by a dispersion relation to $\sigma_{I=1}(e^+e^-\to\text{hadrons})$. To evaluate the contribution of $\bar s\gamma^\mu s$ we may invoke SU(3) symmetry. Decomposing $\bar s\gamma^\mu s$ into unitary singlet and octet parts, only the latter contributes to $\Pi_{\gamma Z}^{(h)}(q^2)$ in the limit of SU(3) symmetry, provided we neglect "interference terms" between heavy and light quarks in the two-current correlation function of Eq. (30). Moreover, this octet term is proportional to the isoscalar part of J_γ^ν in the light-quark sector and can be related, therefore, to $\sigma_{I=0}(e^+e^- \to \text{hadrons})$ by a dispersion relation.

It is worthwhile to point out that the theoretical ambiguities in the radiative corrections at $-q^2=0$ arising from the above-mentioned hadronic complications as well as uncertainties in the values of m_{ϕ_1} and m_t would be automatically eliminated in the ratio of two different neutral-current processes. As an example, aside from calculable QED corrections, the ratio $\sigma(\nu_\mu+e^- \to \nu_\mu+e^-)/\sigma(\nu_e+\mu^- \to \nu_e+\mu^-)$ at $q^2=0$ is reduced by 1.8% for $s^2=0.23$ relative to the Born approximation on account of the difference between ν_μ and ν_e charge-radius diagrams.

D. Summary of results for the weak corrections of $O(G_F\alpha)$

The total weak correction of $O(\alpha)$ to ν-hadron neutral-current processes consists of the terms proportional to J_Z^μ and J_γ^μ given in Secs. III A and III B, respectively, and contributions proportional to $a_{\beta_L}J_{\beta_L}^\mu$ and $a_{\beta_R}J_{\beta_R}^\mu$ which were obtained in Sec. II E. The first two give rise to an overall renormalization factor and a correction to $\sin^2\theta_W$.

Thus, our entire answer can be written as an effective amplitude

$$M_{\text{eff}} = M^0 + \delta M = M^0_{\text{eff}} + \frac{i(g^2+g'^2)}{q^2-m_Z{}^2}\,\frac{\alpha}{8\pi c^2 s^2}\,\bar u_{\nu_f}\gamma_\mu(1-\gamma_5)u_{\nu_i}\langle f|\,a_{\beta_L}J_{\beta_L}^\mu + a_{\beta_R}J_{\beta_R}^\mu\,|i\rangle\,, \tag{31a}$$

where the terms proportional to $J_{\beta_L}^\mu$ and $J_{\beta_R}^\mu$ are the contributions from box diagrams not proportional to J_Z^μ or J_γ^μ, M^0 is the zeroth-order amplitude, and

$$M^0_{\text{eff}} = \frac{im_Z{}^2}{q^2-m_Z{}^2}\,\frac{G_\mu}{\sqrt2}\,\rho_{\text{NC}}^{(\nu;h)}\,\bar u_{\nu_f}\gamma_\mu(1-\gamma_5)u_{\nu_i}\langle f|\,\bar\psi C_3\gamma^\mu a_-\psi - 2\kappa^{(\nu;h)}(q^2)\sin^2\theta_W\,\bar\psi\gamma^\mu Q\psi\,|i\rangle\,. \tag{31b}$$

In Eq. (31b) $\rho_{\text{NC}}^{(\nu;h)}$ is the universal renormalization factor discussed in Sec. III A and $\kappa^{(\nu;h)}(q^2)$ is the $\sin^2\theta_W$ correction factor studied in Secs. III B and III C. We recall that the induced currents $J_{\beta_L}^\mu$ and $J_{\beta_R}^\mu$ were defined to be orthogonal in the isospin space of the u,d quark sector to the corresponding currents in the tree approximation. As explained in Sec. II E, this implies that $J_{\beta_L}^\mu$ and $J_{\beta_R}^\mu$ give very small order α corrections to deep-inelastic scattering on isosclar targets. Thus, for such reactions Eq. (31b) is an excellent approximation. The fact that the correction factor $\kappa^{(\nu;h)}(q^2)$ for typical values of $-q^2$, ξ, and $m_t{}^2/m_W{}^2$ is very close to 1 indicates that the definition of $\sin^2\theta_W$ used in this paper is not only theoretically simple but it is also convenient in the sense that it is very weakly renormalized by radiative corrections.

An alternative frequently used parametrization of the effective neutral-current interaction is

$$M_{\text{eff}} = \frac{im_Z{}^2}{(q^2-m_Z{}^2)}\,\frac{G_\mu}{\sqrt2}\,\bar u_{\nu_f}\gamma_\mu(1-\gamma_5)u_{\nu_i} \times\sum_f\left[\epsilon_L(f)\,\bar q_f\gamma^\mu(1-\gamma_5)q_f + \epsilon_R(f)\,\bar q_f\gamma^\mu(1+\gamma_5)q_f\right], \tag{32}$$

where the sum is over all quark flavors $f=u,c,t,d,s,b\ldots$. Comparing Eqs. (32), (31a), and (31b) and remembering the structure of $J_{\beta_L}^\mu$ and $J_{\beta_R}^\mu$ we find

$$\epsilon_L(u) = \rho_{\text{NC}}^{(\nu;h)}\left(\tfrac12 - \tfrac23\kappa^{(\nu;h)}(q^2)s^2\right) + \frac{\alpha}{2\pi c^2 s^2}a_{\beta_L}(c^2+\tfrac13 s^2)\,, \tag{33a}$$

$$\epsilon_L(d) = \rho_{\text{NC}}^{(\nu;h)}\left(-\tfrac12 + \tfrac13\kappa^{(\nu;h)}(q^2)s^2\right) + \frac{\alpha}{2\pi c^2 s^2}a_{\beta_L}(c^2-\tfrac13 s^2)\,, \tag{33b}$$

$$\epsilon_R(u) = -\rho_{\text{NC}}^{(\nu;h)}\kappa^{(\nu;h)}(q^2)\tfrac23 s^2 + \frac{\alpha}{\pi c^2}\,\frac{s^2}{20}\,, \tag{33c}$$

$$\epsilon_R(d) = \rho_{\text{NC}}^{(\nu;h)}\kappa^{(\nu;h)}(q^2)\,\tfrac13 s^2 + \frac{\alpha}{\pi c^2}\,\frac{s^2}{10}\,, \tag{33d}$$

$$\epsilon_{L,R}(c) = \epsilon_{L,R}(t) = \epsilon_{L,R}(u)\,, \tag{33e}$$

$$\epsilon_{L,R}(s) = \epsilon_{L,R}(b) = \epsilon_{L,R}(d)\,, \tag{33f}$$

where a_{β_L} is defined in Eq. (19f). The tree level results of the $\text{SU(2)}_L\times\text{U(1)}$ theory are obtained by setting $\rho_{\text{NC}}^{(\nu;h)} = \kappa^{(\nu;h)}(q^2)=1$ and neglecting the explicit terms of $O(\alpha)$ in Eq. (33). Thus, the deviations of $\rho_{\text{NC}}^{(\nu;h)}$ and $\kappa^{(\nu;h)}(q^2)$ from unity given in Secs. III A–III C and the explicit terms of $O(\alpha)$ in Eq. (33) represent the effect of the $O(\alpha)$ weak corrections in ν-hadron neutral-current interactions.

As explained in Secs. III A–III C, in the case of ν-lepton interaction we should use $\rho^{(\nu;l)}$ and $\kappa^{(\nu;l)}(q^2) = 1-\Delta^{(\nu;l)}(q^2)$, which are slightly different from $\rho^{(\nu;h)}$ and $\kappa^{(\nu;h)}$; there are no further contributions. In that case the answer is given by an expression analogous to Eq. (31b) with the factors $\rho_{\text{NC}}^{(\nu;h)}$ and $\kappa^{(\nu;h)}(q^2)$ and hadronic operators replaced by their leptonic counterparts.

IV. PHOTONIC CORRECTIONS

In this section we discuss briefly the photonic corrections associated with the hadronic legs in ν-hadron interactions. As mentioned in the Introduction, these corrections are model dependent in the sense that they are affected by the dynamics of strong interactions. Following the analysis of De Rújula, Petronzio, and Savoy-Navarro[8] we discuss them in the leading-logarithmic approximation of the quark-parton model. According to these authors, in this approximation the effect of such photonic corrections is to change the "bare" parton distribution $q_i^{(0)}(x)$ into a radiatively corrected distribution $q_i^{(r)}(x)$:

$$q_i^{(r)}(x) = q_i^{(0)} + e_i^2 \frac{\alpha}{2\pi} \ln(-q^2/m_i^2) \times \int_0^1 dz\, \frac{1+z^2}{1-z}\left[\frac{1}{z} q_i^{(0)}\left(\frac{x}{z}\right)\theta(z-x) - q_i^{(0)}(x)\right], \qquad (34a)$$

where e_i and m_i are the charge and effective mass of quark i. The first term in the square brackets represents the effect of real photon emission while the second is associated with virtual photons.

We now observe that Eq. (34a) simplifies greatly if we consider the integrated distribution $\int_0^1 dy \int_0^1 x q_i^{(r)}(x) dx$ and at the same time ignore the detailed dependence of the slowly varying function $\ln(-q^2/m_i^2)$ on x and y. To a good approximation we replace $-q^2 = 2m_p E_\nu xy$ in the argument of the logarithm (x and y are the usual scaling variables) by an average value $\langle -q^2\rangle \equiv 2m_p E_\nu \langle xy\rangle$. Interchanging the order of x and z integrations, introducing the new variable $u = x/z$ instead of x in the first term in the square brackets of Eq. (34a) and combining the result with the second term leads to

$$Q_i^{(r)} = Q_i^{(0)}\left[1 - \tfrac{4}{3} e_i^2 \frac{\alpha}{2\pi} \ln(\langle -q^2\rangle/m_i^2)\right], \qquad (34b)$$

where

$$Q_i^{(a)} \equiv \int_0^1 dy \int_0^1 dx\, x q_i^{(a)}(x), \quad a = r, 0 . \qquad (34c)$$

Thus the effect of the photonic corrections in the leading-logarithm approximation is simply to multiply each integrated parton distribution by the correction factor $[1 - \frac{4}{3} e_i^2 (\alpha/2\pi) \ln(\langle -q^2\rangle/m_i^2)]$. A more detailed analysis shows that no significant change is obtained if the exact dependence of $\ln(-q^2/m_i^2)$ on x and y is taken into account.[17]

What is the effect of these corrections on the ratio $R^{(\nu)} = \sigma(\nu + N \to \nu + X)/\sigma(\nu + N \to \mu + X)$ and the determination of $\sin^2\theta_W$? If we neglect the contribution of s and c quarks, $R^{(\nu)}$ in the case of isoscalar targets is given by

$$R^{(\nu)} = \frac{\sum_{i=u,d}\{[\epsilon_L^2(i) + \frac{1}{3}\epsilon_R^2(i)][Q_{i;p}^{(r)} + Q_{i;n}^{(r)}] + (\epsilon_L \longleftrightarrow \epsilon_R, Q_i \to \overline{Q}_i)\}}{Q_{d;p}^{(r)} + Q_{d;n}^{(r)} + \frac{1}{3}(\overline{Q}_{u;p}^{(r)} + \overline{Q}_{u;n}^{(r)})}, \qquad (35a)$$

where $\overline{Q}_i$ is the integrated antiparton distribution, the subscripts p and n label the distributions corresponding to protons and neutrons and the last expression in the numerator is obtained from the first two by interchanging ϵ_L and ϵ_R and substituting $\overline{Q}_i$ for Q_i. The correction factor in Eq. (34b) also holds for integrated antiparton distributions, if we approximate $\ln(-q^2/m_i^2)$ by $\ln(\langle -q^2\rangle/m_i^2)$ in integrals of the type $\int_0^1 (1-y)^2 dy \int_0^1 dx\, x\, q_i^{(r)}(x)$.

The bare integrated distributions satisfy the usual charge-symmetry relations: $Q_{u;n}^{(0)} = Q_{d;p}^{(0)}$, $Q_{d;n}^{(0)} = Q_{u;p}^{(0)}$ and similarly for the $\overline{Q}_i^{(0)}$. However, Eq. (34b) tells us that this is no longer true for the $Q_i^{(r)}$ and $\overline{Q}_i^{(r)}$. Taking into account the charge symmetry relations obeyed by the $Q_i^{(0)}$ and $\overline{Q}_i^{(0)}$, assuming that m_u and m_d are constituent masses $m_u \simeq m_d = m$ and using Eq. (34b), $R^{(\nu)}$ reduces to

$$R^{(\nu)} = \frac{[\epsilon_L^2(u) + \frac{1}{3}\epsilon_R^2(u)][1 - \frac{1}{3}\gamma] + \epsilon_L^2(d) + \frac{1}{3}\epsilon_R^2(d) + \{[\epsilon_R^2(u) + \frac{1}{3}\epsilon_L^2(u)](1 - \frac{1}{3}\gamma) + [\epsilon_R^2(d) + \frac{1}{3}\epsilon_L^2(d)]\}\overline{Q}^{(0)}/Q^{(0)}}{1 + \frac{1}{3}(\overline{Q}^{(0)}/Q^{(0)})(1 - \frac{1}{3}\gamma)}, \qquad (35b)$$

where $\gamma = \frac{4}{3}(\alpha/2\pi)\ln(\langle -q^2\rangle/m^2)$, $Q^{(0)} = Q_{u;p}^{(0)} + Q_{d;p}^{(0)}$, and $\overline{Q}^{(0)} = \overline{Q}_{u;p}^{(0)} + \overline{Q}_{d;p}^{(0)}$. Taking $\overline{Q}^{(0)}/Q^{(0)} \simeq 0.14$ and $s^2 = 0.23$, an elementary analysis using the explicit expressions for the ϵ's [see Eqs. (33a)–(33f)] shows that the effect of the corrections proportional to γ is to change the value of s^2 obtained from $R^{(\nu)}$ by

$$ds^2 = -0.062\gamma . \qquad (35c)$$

For $\langle -q^2\rangle = 20$ GeV^2 and $m = 0.3$ GeV this leads to a decrease of 5.2×10^{-4} for s^2, i.e., a 0.23% effect. Even if we consider $m = 100$ MeV the magnitude of the decrease would be only 7.3×10^{-4}. For $R^{(\bar\nu)}$ the roles of Q and $\overline{Q}$ are interchanged in Eq. (35b) and one expects a somewhat larger correction.

In summary, we find that if the leading-logarithm approximation in the framework of the parton model[8] is a valid description of the photonic corrections associated with the hadronic legs, their effect is rather small and can be approximately taken into account by analyses analogous to the discussion in this section.

V. DISCUSSION

The general results of this paper are mainly contained in Sec. III. We have seen that the effect of the order-α weak corrections on ν-induced neutral-current phenomena can be described by means of an effective amplitude involving two renormalization factors ρ_{NC} and $\kappa(q^2)$ and, in the case of ν-hadron interactions, two induced currents $J^{\mu}_{\beta_L}$ and $J^{\mu}_{\beta_R}$ not present at the tree level. The values of $\rho_{NC}^{(\nu;h)}$ and $\kappa^{(\nu;h)}(q^2)$ appropriate for ν-hadron interactions differ by rather small amounts from $\rho_{NC}^{(\nu;l)}$ and $\kappa^{(\nu;l)}(q^2)$ which correspond to ν–charged-lepton interactions, the difference being due to the nontrivial dependence of box diagrams on the Q and C_3 eigenvalues of the fundamental fields of the theory. Alternately, the results for ν-hadron interactions can be expressed in terms of radiatively corrected expressions for the parameters $\epsilon_L(i)$, $\epsilon_R(i)$, $(i=u,d,s,c\dots)$ frequently used to discuss the effective Lagrangian [see Eqs. (33a)–(33f)].

We have also seen that for nonexotic values of m_{ϕ_1} and m_t, the parameters ρ_{NC} and $\kappa(q^2)$ differ from unity by rather small corrections provided we use the renormalization framework of Ref. 2 in which $\cos\theta_W$ obeys Eq. (1) exactly and the results are expressed in terms of G_μ. Indeed, a glance at Table I tells us that $\rho_{NC}^{(\nu;l)}-1<0.5\%$ for ν-lepton interactions and considerably less for ν-hadron processes. Looking at Table II we see that for ν-hadron interactions at $-q^2=20$ GeV^2, typical momentum transfer in deep-inelastic scattering, $1-\kappa^{(\nu;h)}(q^2)<0.5\%$ for $m_{\phi_1}<m_Z$ reaching 1.2% only for very large values $m_{\phi_1}\approx 10m_Z$ of the physical Higgs mass. For ν-lepton scattering the corresponding corrections are only slightly larger. Regarding very small momentum transfers, $-q^2\simeq 0$, we pointed out in Sec. III C that the evaluation of the correction $1-\kappa(0)$ is complicated by model-dependent hadronic contributions to γZ mixing and we outlined there a strategy to partially overcome the associated difficulties by a judicious use of dispersion relations. Nevertheless, if estimates based on the naive quark-model calculations are approximately valid, we have seen that $1-\kappa(0)$ is still very small for ν_μ beams. (The corrections at $-q^2=0$ for ν_e beams are larger by about 1.8% due to the well-known "neutrino charge-radius" effect).

The smallness of the radiative corrections we have encountered is by no means *a priori* obvious: Indeed, one does find corrections of order $\alpha/(2\pi s^2)$, sometimes multiplied by potentially large logarithms such as $\ln(m_W{}^2/-q^2)$ if $-q^2$ is large or $\ln(m_Z{}^2/m_f{}^2)$ and $\ln(m_Z{}^2/m_l{}^2)$ if $-q^2$ is small. However, a close inspection of the analysis in Secs. III B and III C and Table III indicates that the smallness of the corrections to $\kappa(q^2)$ over a wide range of momentum transfers can be partially traced to a delicate cancellation between "bosonic" and "hadronic" contributions.

We have also repeatedly emphasized that the smallness of the radiative corrections is closely related to the chosen renormalization framework, namely, our decision to express the results in terms of G_μ and $\cos\theta_W$ defined according to Eq. (1). The reason is that, by so doing, all the counterterms contributing to ν-induced neutral-current phenomena are effectively defined at large invariant momenta $\sim m_W{}^2$ and, as a consequence, they do not introduce spurious lepton and quark mass singularities in the explicit expressions for the radiative corrections. Perhaps this point can be best illustrated by considering an alternative scheme in which the effective NC Lagrangian is expressed in terms of $\sin^2\theta_W(0)$ rather than the $\sin^2\theta_W$ we have employed. This frequency used parameter is defined by $\sin^2\theta_W(0)\equiv e^2/\hat{g}^2$ where e is still the electric charge of the proton and $\hat{g}^2/8m_W{}^2\equiv G_\mu/\sqrt{2}$. We know from the work of Refs. 1 and 2 that $\sin^2\theta_W(0)$ and $\sin^2\theta_W$, defined according to Eq. (1), differ by about 6.7%. It follows that $1-\kappa(q^2)$ would be of $O(6.7\%)$ had we expressed our results in terms of $\sin^2\theta_W(0)$ rather than $\sin^2\theta_W$, a sizable correction. It is worthwhile recalling that the difference between $\sin^2\theta_W$ and $\sin^2\theta_W(0)$ is due, to a considerable extent, to large logarithmic terms involving fermion mass singularities.

At this point, two natural questions emerge: (a) What is the effect of radiative corrections on the value of $\sin^2\theta_W$ extracted from experiment? (b) What is the relation between the quantity $\sin^2\theta_W(m_W)$ predicted by grand unified theories[1] to lie near 0.21 and $\sin^2\theta_W$ defined according to Eq. (1)?

If $\sin^2\theta_W$ were extracted from measurements restricted to neutrino-induced neutral-current phenomena, at reasonably large momentum transfers, the results of this paper would provide an adequate theoretical answer to the first question. Indeed, we have given explicit forms for the effective ν-hadron and ν-lepton amplitudes including the effect of weak $O(\alpha)$ corrections. This result should be complemented by the photonic corrections which can be evaluated completely for ν-lepton interactions[18] or by the approximation methods outlined in Sec. IV for ν-hadron interactions. As is well known, the precise form in which these photonic corrections are to be implemented depends on the details of the experimental detection procedures. At present, however, $\sin^2\theta_W$ is precisely extracted mainly from the ratio of neutral- to charged-current ν-induced deep-inelastic scattering and from asymmetry measurements in deep-inelastic polarized e-D scattering. Although the charged-current neutrino cross section does not explicitly involve $\sin^2\theta_W$, loop effects will induce a renormalization factor $\rho_{CC}^{(\nu;h)}$ (CC means "charged current") multiplying the charged-current amplitude. The ratio $R^{(\nu)} = \sigma(\nu+N\to\nu+X)/\sigma(\nu+N\to\mu+X)$ would then be proportional to $(\rho_{NC}^{(\nu;h)}/\rho_{CC}^{(\nu;h)})^2$ and a significant deviation of $\rho_{CC}^{(\nu;h)}$ from unity could indirectly affect the determination of $\sin^2\theta_W$ from $R^{(\nu)}$. Thus, a complete theoretical analysis requires an investigation of the $O(\alpha)$ radiative corrections to charged-current ν-induced deep-inelastic scattering. A discussion of these effects and the connection between $\sin^2\theta_W(m_W)$ in grand unified theories and $\sin^2\theta_W$ will be presented in separate communications.

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APPENDIX A: BOSONIC CONTRIBUTIONS TO THE ZZ, WW, γZ, and $\gamma\gamma$ SELF-ENERGIES

The bosonic contributions to the ZZ, WW, γZ, and $\gamma\gamma$ self-energies are depicted in Figs. 5 and 6. The precise definitions of these quantities are given in Sec. II C. Explicit evaluation in the 't Hooft–Feynman gauge leads to

$$\begin{aligned}A_{ZZ}^{(b)}(q^2) = \frac{g^2 m_Z^2}{16\pi^2}\Bigg\{&\left[\frac{1}{n-4}+C\right]\left[\frac{2}{c^2}-4(1-2s^2)-\frac{q^2}{3m_Z^2}\left(19c^2-\frac{s^4}{c^2}\right)\right]-\frac{q^2}{2m_Z^2}\left(c^2+\frac{s^4}{3c^2}\right)\\&-\left[\frac{17c^2}{2}+\frac{s^4}{2c^2}-s^2\right]\frac{K_1(m_W^2,m_W^2,q^2)}{m_Z^2}-\frac{1}{2c^2}\frac{K_1(m_Z^2,m_{\phi_1}^2,q^2)}{m_Z^2}+\frac{K_2(m_Z^2,m_{\phi_1}^2,q^2)}{c^2}\\&+2s^4K_2(m_W^2,m_W^2,q^2)-\frac{q^2c^2}{m_Z^2}\int_0^1 dx[5-2x(1-x)]\ln\left(\frac{m_W^2-q^2x(1-x)}{\mu^2}\right)\\&+c^2\ln\left(\frac{m_W^2}{\mu^2}\right)\left[\frac{13c^2}{2}+\frac{s^4}{2c^2}-s^2\right]+\frac{1}{4c^2}\left[\ln\left(\frac{m_Z^2}{\mu^2}\right)+\frac{m_{\phi_1}^2}{m_Z^2}\ln\left(\frac{m_{\phi_1}^2}{\mu^2}\right)\right]\Bigg\},\end{aligned}\tag{A1}$$

$$\begin{aligned}A_{WW}^{(b)}(q^2) = \frac{g^2 m_W^2}{16\pi^2}\Bigg\{&\left[\frac{1}{n-4}+C\right]\left[\frac{-2(1-2s^2)}{c^2}-\frac{19q^2}{3m_W^2}\right]-\frac{q^2}{2m_W^2}-[8c^2+\tfrac{1}{2}]\frac{K_1(m_W^2,m_Z^2,q^2)}{m_W^2}\\&-\frac{8s^2}{m_W^2}K_1(m_W^2,0,q^2)-\frac{1}{2}\frac{K_1(m_W^2,m_{\phi_1}^2,q^2)}{m_W^2}+K_2(m_W^2,m_{\phi_1}^2,q^2)+K_2(m_W^2,0,q^2)s^2\\&+\frac{s^4}{c^2}K_2(m_W^2,m_Z^2,q^2)-\frac{q^2s^2}{m_W^2}\int_0^1 dx[5-2x(1-x)]\ln\left(\frac{m_W^2x-q^2x(1-x)}{\mu^2}\right)\\&-\frac{q^2c^2}{m_W^2}\int_0^1 dx[5-2x(1-x)]\ln\left(\frac{m_W^2x+m_Z^2(1-x)-q^2x(1-x)}{\mu^2}\right)\\&+\frac{7}{2}\ln\left(\frac{m_W^2}{\mu^2}\right)+3\ln\left(\frac{m_Z^2}{\mu^2}\right)+\frac{1}{4c^2}\left[\ln\left(\frac{m_Z^2}{\mu^2}\right)+\frac{m_{\phi_1}^2}{m_Z^2}\ln\left(\frac{m_{\phi_1}^2}{\mu^2}\right)\right]\Bigg\},\end{aligned}\tag{A2}$$

where μ and C were introduced in Eqs. (13a) and (13c) and

$$K_1(m_1^2,m_2^2,q^2)=\int_0^1 dx[m_1^2x+m_2^2(1-x)-q^2x(1-x)]\ln\left[\frac{m_1^2x+m_2^2(1-x)-q^2x(1-x)}{\mu^2}\right],\tag{A3}$$

$$K_2(m_1{}^2, m_2{}^2, q^2) = \int_0^1 dx \ln\left[\frac{m_1{}^2 x + m_2{}^2(1-x) - q^2 x(1-x)}{\mu^2}\right]. \tag{A4}$$

We note that Eqs. (A1) and (A2) become identical in the limit $s^2 \to 0$ (keeping g^2 fixed). In this limit there is no mixing between the $SU(2)_L$ and $U(1)$ gauge bosons so that Z_μ and A_μ are identical to the neutral fields $W_\mu{}^3$ and B_μ associated with those groups. It is then clear the the diagrams involving γ and C_A in Fig. 6 vanish while W, Z, C^+, C^-, C_Z, ϕ^+, and ϕ_2^0 become mass degenerate. The equality of Eqs. (A1) and (A2) in the limit $s^2 \to 0$ can then be easily seen by a cursory comparison of Figs. 5 and 6.

From Eqs. (A1) and (A2) we find

$$\frac{A_{ZZ}^{(b)}(m_Z{}^2)}{m_Z{}^2} = \frac{g^2}{16\pi^2}\left\{\left[\frac{1}{n-4} + C + \ln\left(\frac{m_Z}{\mu}\right)\right]\left[\frac{7}{3c^2} + \frac{10}{3} - 14c^2\right] - \frac{1}{6}\left(\frac{1}{c^2} + 4c^2 - 2\right)\right.$$
$$\left. + \ln c^2\left(\frac{1}{12c^2} + \frac{5}{3} - 7c^2\right) + I_1(c^2) + \frac{1}{c^2}H(\xi) + \frac{\xi}{8c^2}\right\}, \tag{A5}$$

$$\frac{A_{WW}^{(b)}(m_W{}^2)}{m_W{}^2} = \frac{g^2}{16\pi^2}\left\{\left[\frac{1}{n-4} + C + \ln\left(\frac{m_Z}{\mu}\right)\right]\left[\frac{2}{c^2} - \frac{31}{3}\right] + \frac{157 - 166c^2}{18} + \ln c^2\left[\frac{3}{4c^2} - \frac{49}{6}\right]\right.$$
$$\left. + I_2(c^2) + H\left(\frac{\xi}{c^2}\right) + \frac{\xi}{8c^2}\right\}, \tag{A6}$$

where $I_1(c^2)$, $I_2(c^2)$, and $H(\xi)$ are defined in Eqs. (26b), (26c), and (26d). They can be expressed in terms of elementary functions:

$$I_1(c^2) = 4c^2(3+2c^2) - \frac{5}{9}\left(5 + \frac{1}{4c^2}\right) - (4c^2-1)^{1/2}\left[8c^4 + \frac{34c^2}{3} - \frac{8}{3} - \frac{1}{6c^2}\right]\tan^{-1}\left(\frac{1}{(4c^2-1)^{1/2}}\right), \tag{A7}$$

$$I_2(c^2) = \frac{83c^2}{9} + \frac{91}{9} - \frac{13}{6c^2} - \frac{1}{12c^4} - \{\tan^{-1}[(4c^2-1)^{1/2}]\}\frac{(4c^2-1)^{1/2}}{c^4}\left[4c^4 + \frac{17c^2}{3} - \frac{4}{3} - \frac{1}{12c^2}\right]$$
$$+ \left[3c^4 + \frac{7c^2}{2} - \frac{7}{12} - \frac{1}{24c^2}\right]\frac{\ln c^2}{c^4}, \tag{A8}$$

$$H(\xi) = \frac{\xi\ln\xi}{4}[3 - \xi + \tfrac{1}{6}\xi^2] + \tfrac{3}{8}\xi - \tfrac{1}{12}\xi^2 - \tfrac{17}{9}$$
$$+ [\xi(1 - \tfrac{1}{4}\xi)]^{1/2}[2 - \tfrac{2}{3}\xi + \tfrac{1}{6}\xi^2]\{\tan^{-1}[((4/\xi) - 1)^{1/2}]\} \tag{A9}$$

for $\xi < 4$ and

$$H(\xi) = \left[\left(1 + \frac{a^2}{2}\right)\left(\frac{\xi}{2} - a\right) + \frac{1}{6}\left(a^3 - \frac{\xi^3}{8} + \frac{3\xi}{2}\right)\right]\ln\xi + \frac{3\xi}{8} - \frac{\xi^2}{12} - \frac{17}{9} + 2a\left(1 + \frac{a^2}{3}\right)\ln\left(\frac{a + \xi/2}{a - 1 + \xi/2}\right), \tag{A10}$$

where $a^2 = (\xi^2/4) - \xi$, for $\xi > 4$. In writing down Eqs. (A7) and (A8) we have considered only the phenomenologically realistic case $c^2 > \frac{1}{4}$.

As pointed out in Ref. 2, the following combination of self-energies is gauge invariant and plays a crucial role in the definition of the basic counterterms:

$$\frac{A_{ZZ}^{(b)}(m_Z{}^2)}{m_Z{}^2} - \frac{A_{WW}^{(b)}(m_W{}^2)}{m_W{}^2} = \frac{\alpha}{2\pi}\left\{\left[\frac{1}{n-4} + C + \ln\left(\frac{m_Z}{\mu}\right)\right]\frac{1}{2c^2}[\tfrac{43}{3} - 14s^2] - \frac{1}{36c^2}[157 - 154s^2]\right.$$
$$\left. + \frac{\ln c^2}{2s^2}\left(\frac{59}{6} - 7c^2 - \frac{2}{3c^2}\right) + \frac{1}{2s^2}[I_1(c^2) - I_2(c^2)] + \frac{1}{2s^2}\left[\frac{H(\xi)}{c^2} - H\left(\frac{\xi}{c^2}\right)\right]\right\}. \tag{A11}$$

The values of the self-energies at $q^2 = 0$ are also important quantities; they can be readily obtained from Eqs. (A1) and (A2):

$$\frac{A_{ZZ}^{(b)}(0)}{m_Z^2}=\frac{g^2}{16\pi^2}\left\{-\left[\frac{1}{n-4}+C\right]\frac{2}{c^2}[1-6s^2+4s^4]-\frac{\ln c^2}{4c^2}-\frac{1}{c^2}\ln\left(\frac{m_W^2}{\mu^2}\right)[\tfrac{7}{4}-6s^2+4s^4]-\frac{7}{8c^2}\right.$$
$$\left.+\frac{\xi}{8c^2}+\frac{3}{4c^2}\left[\frac{\ln(m_Z^2/\mu^2)-\xi\ln(m_{\phi_1}^2/\mu^2)}{1-\xi}\right]\right\}, \tag{A12}$$

$$\frac{A_{WW}^{(b)}(0)}{m_W^2}=\frac{g^2}{16\pi^2}\left\{-\left[\frac{1}{n-4}+C\right]\frac{2}{c^2}(1-2s^2)+\frac{1}{c^2}\ln c^2\left[\frac{17}{4s^2}-\frac{29}{4}+2s^2\right]+\frac{1}{c^2}\ln\left(\frac{m_W^2}{\mu^2}\right)\left[-\frac{7}{4}+\frac{11s^2}{4}\right]\right.$$
$$\left.+\frac{1}{8c^2}[27-34s^2+\xi]+\frac{3}{4}\frac{[c^2\ln(m_W^2/\mu^2)-\xi\ln(m_{\phi_1}^2/\mu^2)]}{(c^2-\xi)}\right\}. \tag{A13}$$

The combination

$$\frac{A_{WW}^{(b)}(0)}{m_W^2}-\frac{A_{ZZ}^{(b)}(0)}{m_Z^2}=\frac{\alpha}{4\pi s^2}\left\{-8s^2\left[\frac{1}{n-4}+C\right]+\frac{1}{c^2}\ln c^2\left[\frac{17}{4s^2}-7+2s^2\right]-\frac{s^2}{c^2}[\tfrac{13}{4}-4s^2]\ln\left(\frac{m_W^2}{\mu^2}\right)\right.$$
$$+\frac{17}{4}+\frac{3}{4}\frac{[c^2\ln(m_W^2/\mu^2)-\xi\ln(m_{\phi_1}^2/\mu^2)]}{(c^2-\xi)}$$
$$\left.-\frac{3}{4c^2}\frac{[\ln(m_Z^2/\mu^2)-\xi\ln(m_{\phi_1}^2/\mu^2)]}{(1-\xi)}\right\} \tag{A14}$$

contributes to the renormalization factor $\rho_{NC}^{(\nu;h)}$ [see Eq. (23b)].

Because the γZ self-energy is inserted between photon and Z propagators [see Fig. 3(c)], we expand $A_{\gamma Z}^{(b)}(q^2)$ up to $O(q^2)$. However, consistent with our approximation, we may neglect terms of $O(q^4/m_W^2)$ since these would give rise to contributions of $O(G_f\alpha q^2/m_W^2)$ in the amplitude. Consideration of the γZ diagrams illustrated in Fig. 5 leads to

$$A_{\gamma Z}^{(b)}(q^2)=\frac{\alpha}{4\pi sc}\left\{-\left[\frac{1}{n-4}+C+\ln\left(\frac{m_W}{\mu}\right)\right][4m_W^2+q^2(6c^2+\tfrac{1}{3})]+q^2\left(c^2+\frac{s^2}{3}\right)+O\left(\frac{q^4}{m_W^2}\right)\right\}. \tag{A15}$$

The analogous calculation for the $\gamma\gamma$ self-energy gives

$$A_{\gamma\gamma}^{(b)}(q^2)=-\frac{\alpha q^2}{2\pi}\left[\frac{3}{n-4}+3C+3\ln\left(\frac{m_W}{\mu}\right)-\frac{1}{3}+O\left(\frac{q^2}{m_W^2}\right)\right], \tag{A16}$$

or, introducing $A_{\gamma\gamma}^{(b)}(q^2)=-q^2\,\Pi_{\gamma\gamma}^{(b)}(q^2)$,

$$\Pi_{\gamma\gamma}^{(b)}(0)=\frac{\alpha}{2\pi}\left[\frac{3}{n-4}+3C+3\ln\left(\frac{m_W}{\mu}\right)-\frac{1}{3}\right] \tag{A17}$$

which coincides with Eq. (25) of Ref. 2. As shown in that paper, $\Pi_{\gamma\gamma}^{(b)}(0)$ is part of the charge renormalization counterterm and contributes to the relation between m_W and $\sin\theta_W$.

More explicitly, according to Eq. (34b) of Ref. 2, the corrections of $O(\alpha)$ to the relation between m_W and $\sin\theta_W$ are given in the present framework by

$$m_W=\left(\frac{\alpha\pi}{\sqrt{2}G_\mu}\right)^{1/2}\frac{(1+\Delta r/2)}{\sin\theta_W}=\frac{37.281\text{ GeV}}{\sin\theta_W}(1+\Delta r/2), \tag{A18a}$$

$$\frac{\Delta r}{2}=\frac{\mathrm{Re}A_{WW}(m_W^2)-A_{WW}(0)}{2m_W^2}-\frac{\delta e}{e}+\frac{c^2}{2s^2}\mathrm{Re}\left[\frac{A_{ZZ}(m_Z^2)}{m_Z^2}-\frac{A_{WW}(m_W^2)}{m_W^2}\right]$$
$$+\frac{g^2}{16\pi^2}\left\{-4\left[\frac{1}{n-4}+C+\ln\left(\frac{m_Z}{\mu}\right)+\frac{\ln c^2}{s^2}\left(\frac{7}{4}-3s^2\right)+3\right\}. \tag{A18b}$$

Inserting in this expression the bosonic contributions to $A_{WW}(m_W^2)$, $A_{WW}(0)$, and $A_{ZZ}(m_Z^2)/m_Z^2-A_{WW}(m_W^2)/m_W^2$ given in Eqs. (A6), (A13), and (A11) of this paper, $\delta e^{(b)}/e=-(7e^2/16\pi^2)[(1/(n-4)+C+\ln(m_W/\mu)-1/21]$ obtained in Eq. (26) of Ref. 2, and adding the explicit terms given in Eq. (A18b) lead to a correction[2]

$$\frac{\Delta r^{(1)}}{2}=\frac{\alpha}{4\pi s^2}\left\{\frac{F(s^2)}{2}+\frac{1}{2s^2}\left[H(\xi)-(1-2s^2)H\left(\frac{\xi}{c^2}\right)\right]-\frac{3}{8}\left(\frac{\xi\ln\xi-c^2\ln c^2}{\xi-c^2}\right)\right\}, \tag{A19}$$

where

$$F(s^2)=\frac{\ln c^2}{12c^2}\left(\frac{17}{s^2}-71+63s^2\right)+\frac{1}{c^2}\left(-\frac{475}{72}+\frac{295s^2}{12}-\frac{154s^4}{9}\right)+\frac{1}{s^2}[c^2I_1(c^2)-(1-2s^2)I_2(c^2)]\ . \tag{A20}$$

For $s^2=0.23$, $F(0.23)=2.68$, in agreement with the numerical result quoted in Ref. 2. The leptonic and hadronic contributions to Δr arise from self-energies and the charge renormalization counterterm δe. They have been discussed in the leading-logarithm approximation in Ref. 1 and in complete detail to $O(\alpha)$ in Ref. 2.

APPENDIX B: HADRONIC CONTRIBUTIONS TO THE SELF-ENERGIES IN THE FREE FIELD THEORY

In the free-field-theory (FFT) approximation (i.e., neglecting strong interaction effects) the hadronic contributions to the self-energies are given by the usual quark-loop diagrams. A straightforward calculation gives

$$\begin{aligned}[\Pi_{ZZ}^{\mu\nu(h)}(q^2)]_{\rm FFT}=\frac{3(g^2+g'^2)}{24\pi^2}\sum_{f=1}^{2N}\Bigg\{&(q^2g^{\mu\nu}-q^\mu q^\nu)[4s^4Q_f^2-2s^2C_{3f}Q_f+\tfrac{1}{2}]\\ &\times\left[\frac{1}{n-4}+C+3\int_0^1 dx\,x(1-x)\ln\left(\frac{m_f^2-q^2x(1-x)}{\mu^2}\right)\right]\\ &-\frac{3m_f^2g^{\mu\nu}}{2}\left[\frac{1}{n-4}+C+\tfrac{1}{2}\int_0^1 dx\ln\left(\frac{m_f^2-q^2x(1-x)}{\mu^2}\right)\right]\Bigg\}\ ,\end{aligned} \tag{B1}$$

$$\begin{aligned}[\Pi_{WW}^{\mu\nu(h)}(q^2)]_{\rm FFT}=\frac{3g^2}{24\pi^2}\sum_{i,j=1}^{N}|U_{ji}|^2\Bigg\{&(q^2g^{\mu\nu}-q^\mu q^\nu)\left[\frac{1}{n-4}+C\right.\\ &\left.+3\int_0^1 dx\,x(1-x)\ln\left(\frac{m_i^2x+m_j^2(1-x)-q^2x(1-x)}{\mu^2}\right)\right]\\ &-\frac{3}{2}g^{\mu\nu}\left[(m_i^2+m_j^2)\left(\frac{1}{n-4}+C\right)+\int_0^1 dx[m_i^2x+m_j^2(1-x)]\right.\\ &\left.\times\ln\left(\frac{m_i^2x+m_j^2(1-x)-q^2x(1-x)}{\mu^2}\right)\right]\Bigg\}\ ,\end{aligned} \tag{B2}$$

$$[\Pi_{\gamma\gamma}^{\mu\nu(h)}(q^2)]_{\rm FFT}=\frac{3e^2}{6\pi^2}(q^2g^{\mu\nu}-q^\mu q^\nu)\sum_f Q_f^2\left[\frac{1}{n-4}+C+3\int_0^1 dx\,x(1-x)\ln\left(\frac{m_f^2-q^2x(1-x)}{\mu^2}\right)\right]\ , \tag{B3}$$

$$\begin{aligned}[\Pi_{\gamma Z}^{\mu\nu(h)}(q^2)]_{\rm FFT}=&\frac{3e^2}{12\pi^2cs}(q^2g^{\mu\nu}-q^\mu q^\nu)\\ &\times\sum_f\left[\frac{Q_fC_{3f}}{2}-2s^2Q_f^2\right]\left[\frac{1}{n-4}+C+3\int_0^1 dx\,x(1-x)\ln\left(\frac{m_f^2-q^2x(1-x)}{\mu^2}\right)\right]\ ,\end{aligned} \tag{B4}$$

where $f=1, 2, \ldots, 2N$ is the flavor index, $i=1, 2, \ldots, N$ labels the charge $+2/3$ quarks, $j=1, 2, \ldots, N$ labels the charge $-\frac{1}{3}$ quarks, U is the unitary matrix introduced in Eqs. (6a) and (6b), Q_f and C_{3f} are the eigenvalues of the matrices Q and C_3 defined in Eqs. (3b) and (3c), the m's stand for quark masses, the explicit factors of 3 multiplying all the expressions take into account the color degrees of freedom, and we have used the convention $\mathrm{Tr}[I]_{\rm Dirac}=4$ in evaluating the dimensionally regularized integrals. The subscript FFT reminds us that these expressions are valid in the free field approximation.

Equation (23b) tells us that in order to evaluate the renormalization factor $\rho_{\rm NC}^{(\nu;h)}$ we need the combination $A_{WW}^{(h)}(0)/m_W^2-A_{ZZ}^{(h)}(0)/m_Z^2$. Using $\sum_{i,j}|U_{ji}|^2=N$, $\sum_{i,j}|U_{ji}|^2(m_i^2+m_j^2)=\sum_f m_f^2$ which follow from the unitary property of the matrix U_{ji}, Eqs. (B1) and (B2) lead to

$$\begin{aligned}\left[\frac{A_{WW}^{(h)}(0)}{m_W^2}-\frac{A_{ZZ}^{(h)}(0)}{m_Z^2}\right]_{\rm FFT}=&\frac{3g^2}{32\pi^2m_W^2}\left[\sum_f m_f^2\ln\left(\frac{m_f^2}{m_W^2}\right)\right.\\ &\left.-2\sum_{i,j}|U_{ji}|^2\int_0^1 dx[m_i^2x+m_j^2(1-x)]\ln\left(\frac{m_i^2x+m_j^2(1-x)}{m_W^2}\right)\right]\ .\end{aligned} \tag{B5}$$

If all the quark masses $m_f^2\ll m_W^2$, the contribution of Eq. (B5) to Eq. (23b) can be neglected. This is clearly the case for all of the known quarks with the possible exception of m_t. Setting all the quark masses equal to zero with the exception of the t-quark mass leads to

$$\left[\frac{A_{WW}^{(h)}(0)}{m_W^2}-\frac{A_{ZZ}^{(h)}(0)}{m_Z^2}\right]_{\mathrm{FFT}}=\frac{3g^2}{64\pi^2}\frac{m_t^2}{m_W^2} \tag{B6}$$

which is the contribution included in Eq. (24a). The possible effect of very heavy fermions through loop contributions has been previously discussed in the literature.[19]

Assuming that all quark masses satisfy $m_f^2 \ll m_W^2$, Eqs. (B1) and (B2) lead to

$$\left[\frac{\mathrm{Re}A_{ZZ}^{(h)}(m_Z^2)}{m_Z^2}\right]_{\mathrm{FFT}}=\frac{g^2}{8\pi^2c^2}\sum_{f=1}^{2N}\left\{\left[\tfrac{1}{2}-2s^2C_{3f}Q_f+4s^4Q_f^2\right]\left[\frac{1}{n-4}+C+\ln\left(\frac{m_Z}{\mu}\right)-\frac{5}{6}\right]-\frac{3}{2}\frac{m_f^2}{m_Z^2}\frac{1}{n-4}\right\}, \tag{B7}$$

$$\left[\frac{\mathrm{Re}A_{WW}^{(h)}(m_W^2)}{m_W^2}\right]_{\mathrm{FFT}}=\frac{g^2}{8\pi^2}\left\{N\left[\frac{1}{n-4}+C+\ln\left(\frac{m_W}{\mu}\right)-\frac{5}{6}\right]-\frac{3}{2}\sum_f\frac{m_f^2}{m_W^2}\frac{1}{n-4}\right\}, \tag{B8}$$

where we have neglected finite terms of $O(g^2m_f^2/m_W^2)$. Combining Eqs. (B7) and (B8),

$$\mathrm{Re}\left[\frac{A_{ZZ}^{(h)}(m_Z^2)}{m_Z^2}-\frac{A_{WW}^{(h)}(m_W^2)}{m_W^2}\right]_{\mathrm{FFT}}=-\frac{\alpha}{2\pi c^2}\sum_{f=1}^{2N}\left\{\left[2C_{3f}Q_f-\tfrac{1}{2}-4s^2Q_f^2\right]\left[\frac{1}{n-4}+C+\ln\left(\frac{m_Z}{\mu}\right)-\frac{5}{6}\right]+\frac{c^2\ln c^2}{4s^2}\right\}. \tag{B9}$$

The function $[A_{\gamma Z}^{(h)}(q^2)]_{\mathrm{FFT}}$ is obtained from Eq. (B4):

$$\left[\frac{A_{\gamma Z}^{(h)}(q^2)}{q^2}\right]_{\mathrm{FFT}}=\frac{\alpha}{2\pi cs}\sum_f\left[C_{3f}Q_f-4s^2Q_f^2\right]\left[\frac{1}{n-4}+C+3\int_0^1 dx\,x(1-x)\ln\left(\frac{m_f^2-q^2x(1-x)}{\mu^2}\right)\right]. \tag{B10}$$

Inserting Eq. (B9) and (B10) in the expression between curly brackets in Eq. (25a) and recalling the relation $Q_f=(C_{3f}+Y_f)/2$ where Y_f is the weak hypercharge assignment corresponding to flavor f, we see that the divergences and mass parameter μ cancel for arbitrary charge assignments and we are led to Eq. (27). For the usual assignment of quark charges,

$$\sum_{f=1}^{2N}\left[C_{3f}Q_f-4s^2Q_f^2\right]=N\left(1-\frac{20s^2}{9}\right) \tag{B11}$$

a relation that can be used to simplify Eqs. (B9) and (B10).

Even if we consider the large momentum transfers characteristic of deep-inelastic scattering, say $-q^2\simeq 20\ \mathrm{GeV}^2$, we note that the masses of the heavy quarks are not negligible in Eq. (27). To evaluate their contributions we have used the following relation, valid for $q^2<0$:

$$\int_0^1 dx\,x(1-x)\ln\left(\frac{m_f^2-q^2x(1-x)}{m_Z^2}\right)=\frac{1}{6}\left\{\ln\left(\frac{m_f^2}{m_Z^2}\right)+\frac{1}{3}+(3-X^2)\left[\frac{X}{2}\ln\left(\frac{X+1}{X-1}\right)-1\right]\right\}, \tag{B12}$$

where $X=(1-4m_f^2/q^2)^{1/2}$. For $q^2>4m_f^2$ the integral in Eq. (B12) becomes complex. Its real part is obtained from the right-hand side of Eq. (B12) by changing $\ln[(X+1)/(X-1)]\to\ln[(1+X)/(1-X)]$. Finally, for $0<q^2<4m_f^2$ the integral is again real and equals

$$\int_0^1 dx\,x(1-x)\ln\left(\frac{m_f^2-q^2x(1-x)}{m_Z^2}\right)=\tfrac{1}{6}\ln\left(\frac{m_f^2}{m_Z^2}\right)-\frac{5}{18}-\frac{2m_f^2}{3q^2}+a\tan^{-1}\left(\frac{1}{2a}\right)\left[1+\tfrac{4}{3}a^2\right], \tag{B13}$$

where $a=[(m_f^2/q^2)-(1/4)]^{1/2}$.

We emphasize that in obtaining Eq. (27) we have neglected finite terms of $O(m_f^2/m_W^2)$. If the ratio m_t^2/m_W^2 turns out to be nonnegligible, one can go back to the exact expressions given in Eqs. (B1) and (B2) and calculate the t-quark contribution more accurately. We list the corrections that should be added to Eqs. (B7), (B8), and (B9) if terms of $O(m_t^2/m_W^2)$ are retained but all other quark masses are kept equal to zero:

$$\Delta\left[\frac{\mathrm{Re}A_{ZZ}^{(h)}(m_Z^2)}{m_Z^2}\right]_{\mathrm{FFT}}=\frac{g^2}{8\pi^2c^2}\left\{3\left(\frac{1}{2}-\frac{4s^2}{3}+\frac{16s^4}{9}\right)\mathrm{Re}\int_0^1 dx\,x(1-x)\ln\left(\frac{m_t^2-m_Z^2x(1-x)}{m_Z^2x(1-x)}\right)\right.$$
$$\left.-\frac{3m_t^2}{4m_Z^2}\mathrm{Re}\int_0^1 dx\ln\left(\frac{m_t^2-m_Z^2x(1-x)}{\mu^2}\right)\right\}\ , \tag{B14}$$

$$\Delta\left[\frac{\mathrm{Re}A_{WW}^{(h)}(m_W^2)}{m_W^2}\right]_{\mathrm{FFT}}=\frac{g^2}{8\pi^2}\left\{3\,\mathrm{Re}\int_0^1 dx\,x(1-x)\ln\left(\frac{m_t^2x-m_W^2x(1-x)}{m_W^2x(1-x)}\right)\right.$$
$$\left.-\frac{3}{2}\frac{m_t^2}{m_W^2}\mathrm{Re}\int_0^1 dx\,x\ln\left(\frac{m_t^2x-m_W^2x(1-x)}{\mu^2}\right)\right\}\ , \tag{B15}$$

$$\Delta\left[\frac{\mathrm{Re}A_{ZZ}^{(h)}(m_Z^2)}{m_Z^2}-\frac{\mathrm{Re}A_{WW}^{(h)}(m_W^2)}{m_W^2}\right]_{\mathrm{FFT}}=\frac{3\alpha}{2\pi c^2s^2}\left\{\left(\frac{1}{2}-\frac{4s^2}{3}+\frac{16s^4}{9}\right)\mathrm{Re}\left[\int_0^1 dx\,x(1-x)\ln\left(\frac{m_t^2-m_Z^2x(1-x)}{m_Z^2x(1-x)}\right)\right.\right.$$
$$\left.-c^2\int_0^1 dx\,x(1-x)\ln\left(\frac{m_t^2x-m_W^2x(1-x)}{m_W^2x(1-x)}\right)\right]$$
$$-\frac{1}{2}\frac{m_t^2}{m_Z^2}\mathrm{Re}\left[\frac{1}{2}\int_0^1 dx\ln\left(\frac{m_t^2-m_Z^2x(1-x)}{m_Z^2}\right)\right.$$
$$\left.\left.-\int_0^1 dx\,x\ln\frac{m_t^2x-m_W^2x(1-x)}{m_Z^2}\right]\right\}\ . \tag{B16}$$

As an example, if $m_t=m_W$ and $s^2=0.23$ this leads to a further contribution of $-0.001\,80$ to Eq. (B9). Recalling Eq. (25a) and taking into account the variation in $A_{\gamma Z}$ (-20 GeV^2) we find that changing $m_t=18$ GeV to $m_t=m_W=80.3$ GeV adds an additional correction of $-0.004\,11$ to $\Delta^{(\nu;h)}(q^2)$ which is of opposite sign to the overall result quoted in Sec. III B. In particular, we note that if both m_{ϕ_1} and m_t equal m_W, the overall $\Delta^{(\nu;h)}(q^2)$ would be extremely small, of order 10^{-4}. For completeness we give the asymptotic form of Eq. (B16) when $m_t^2/m_Z^2\gg 1$:

$$\Delta\left[\frac{\mathrm{Re}A_{ZZ}^{(h)}(m_Z^2)}{m_Z^2}-\frac{\mathrm{Re}A_{WW}^{(h)}(m_W^2)}{m_W^2}\right]_{\mathrm{FFT}}=-\frac{\alpha}{2\pi c^2s^2}\left\{\frac{3m_t^2}{8m_Z^2}+\frac{1}{2}\ln\left(\frac{m_t^2}{m_Z^2}\right)\left[\frac{1}{2}+\frac{s^2}{3}-\frac{16s^4}{9}\right]-\frac{c^2}{2}\ln c^2+\frac{1}{8}\right.$$
$$\left.+\frac{4s^2}{9}\left(1-\frac{10s^2}{3}\right)+O\left(\frac{m_Z^2}{m_t^2}\right)\right\}\ . \tag{B17}$$

The quadratic dependence on m_t and the fact that the contribution of Eq. (B17) to $\delta_\theta^{(h)}(q^2)$ is enhanced by a factor c^2/s^2 [see Eq. (25a)] indicate that the values of $\delta_\theta^{(h)}(q^2)$ and $\kappa(q^2)$ obtain large corrections if we consider the possibility of t-quark masses equal to a few times m_Z. For example, if $m_t=3m_Z$ an estimate using Eq. (B17) shows that $\kappa(q^2)$ would exceed 1 by roughly 9%. Remembering Eq. (24a) we see, therefore, that the effect of a t mass of the order of a few times m_Z is to increase both ρ_{NC} and $\kappa(q^2)$ above unity. Because of the enhancement factor c^2/s^2 alluded to before, the effect is more important for $\kappa(q^2)$. In the extraction of $\sin^2\theta_W$ from the ratio $R^{(\nu)}=\sigma(\nu+N\to\nu+X)/\sigma(\nu+N\to\mu+X)$ for isoscalar targets, these two effects work in opposite directions. Values of $\rho_{\mathrm{NC}}^{(\nu;h)}$ and $\kappa^{(\nu;h)}(q^2)$ larger than one tend to increase and decrease $\sin^2\theta_W$, respectively. Although the corrections to $\kappa^{(\nu;h)}(q^2)$ are larger, it turns out that the value of $\sin^2\theta_W$ extracted from $R^{(\nu)}$ measurements depends more sensitively on $\rho_{\mathrm{NC}}^{(\nu;h)}$ and the two contributions tend to cancel. A rough error analysis can be made by considering a parton model calculation in which only the contribution of u and d quarks is included. $R^{(\nu)}$ is then proportional to $g_L^2+g_R^2/3+(\bar{Q}/Q)[g_R^2+g_L^2/3]$ where $g_L^2=\epsilon_L^2(u)+\epsilon_L^2(d)$, $g_R^2=\epsilon_R^2(u)+\epsilon_R^2(d)$ and $\bar{Q}/Q$ is the ratio of the integrated antiquark and quark distributions. To simplify the argument we may consider the dominant contribution g_L^2, which according to Eqs. (31a) and (31b) is given by $g_L^2=(\rho_{\mathrm{NC}}^{(\nu;h)})^2[\frac{1}{2}-s^2\kappa^{(\nu;h)}+\frac{5}{9}s^4(\kappa^{(\nu;h)})^2]$. Varying $\rho_{\mathrm{NC}}^{(\nu;h)}$ and $\kappa^{(\nu;h)}$ we find

$$[1-\tfrac{10}{9}s^4]ds^2=2d\rho[\tfrac{1}{2}-s^2+\tfrac{5}{9}s^4]-s^2d\kappa[1-\tfrac{10}{9}s^2]\ . \tag{B18}$$

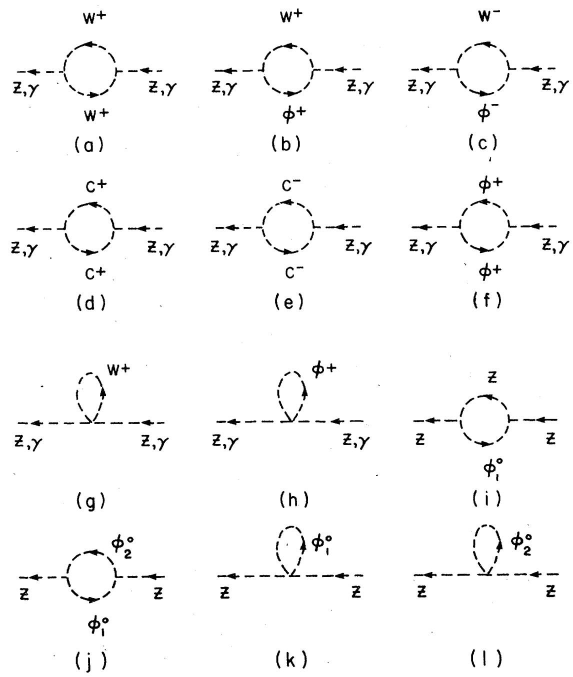

FIG. 5. Bosonic contributions to ZZ, γZ, and $\gamma\gamma$ self-energies. The symbol ϕ_1^0 represents the physical Higgs scalar while ϕ_2^0, ϕ^+, and ϕ^- denote the unphysical Higgs bosons associated with Z, W^+, and W^-, respectively. Similarly, C_Z, C^+, C^-, and C_A are the ghosts. Diagrams (a) through (h) contribute to ZZ, γZ, and $\gamma\gamma$ self-energies with the proper identification of the vertices of the incoming and outgoing lines. Diagrams (i) through (l) involve neutral particles and hence contribute only to the ZZ self-energy.

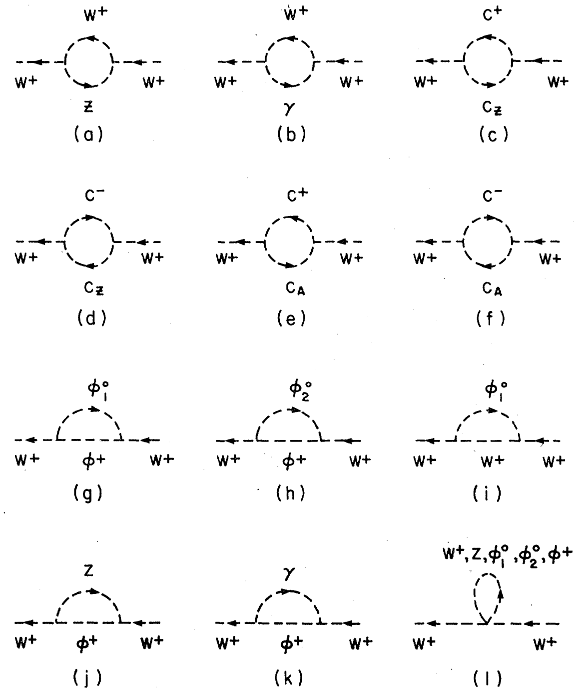

FIG. 6. Bosonic contributuions to the WW self-energy. Diagram (l) represents the five distinct amplitudes involving W^+, Z, ϕ_1^0, ϕ_2^0, and ϕ^+ in the virtual line, respectively.

If, for simplicity, we consider only the effect of the terms proportional to m_t^2/m_Z^2 in Eqs. (24a) and (B17) we have the relation $d\kappa = (c^2/s^2)d\rho$ and the right-hand side of Eq. (B18) becomes $+(s^2/9)d\rho$. A more careful analysis, keeping the terms proportional to g_R^2 and $\bar{Q}/Q \simeq 0.2$ leads to $ds^2/s^2 \simeq d\rho \simeq (3\alpha/16\pi s^2)\, m_t^2/m_W^2$. Thus, the sign of the effect is to increase the value of $\sin^2\theta_W$ extracted from experiment, but the large enhancement factor c^2/s^2 has been effectively eliminated.

On the other hand, this suppression of the factor c^2/s^2 does not occur in the corrections to the relation between m_W and $\sin\theta_W$. Comparing Eqs. (A18) and (B17) we see that there is a contribution $-3\alpha/(32\pi s^4)(m_t^2/m_Z^2) \simeq -0.0041\ (m_t^2/m_Z^2)$ which becomes quite large for $m_t = 3m_Z$ and is in the direction opposite to the corrections found in Refs. 1 and 2 (it tends to reduce m_W).

APPENDIX C: LEPTONIC CONTRIBUTIONS TO THE SELF-ENERGIES

The leptonic contributions to the self-energy tensors can be immediately obtained from Eqs. (B1), (B2), (B3), and (B4) by inserting the eigenvalues of Q^2 and C_3Q appropriate for leptons and dividing out the factor 3 associated with the color degree of freedom. Morever, if all the neutrino masses are negligible, the matrix U can be effectively replaced by an $N\times N$ unit matrix. The contribution of the known leptons to $\rho_{NC}^{(\nu;h)}$, i.e., $A_{WW}^{(l)}(0)/m_W^2 - A_{ZZ}^{(l)}(0)/m_Z^2$, is of $O(g^2 m_l^2/m_W^2)$ and therefore clearly negligible.

We list a number of useful expressions:

$$\frac{\mathrm{Re}A_{ZZ}^{(l)}(m_Z^2)}{m_Z^2} = \frac{g^2}{24\pi^2 c^2}\left\{N[1-2s^2+4s^4]\left[\frac{1}{n-4}+C+\ln\left(\frac{m_Z}{\mu}\right)-\frac{5}{6}\right]-\tfrac{3}{2}\sum_l \frac{m_l^2}{m_Z^2}\frac{1}{n-4}\right\}, \tag{C1}$$

$$\frac{\mathrm{Re}A_{WW}^{(l)}(m_W^2)}{m_W^2} = \frac{g^2}{24\pi^2}\left\{N\left[\frac{1}{n-4}+C+\ln\left(\frac{m_W}{\mu}\right)-\frac{5}{6}\right]-\tfrac{3}{2}\sum_l \frac{m_l^2}{m_W^2}\frac{1}{n-4}\right\}, \tag{C2}$$

where we have again terms of $O(g^2 m_l^2/m_W^2)$ and

$$\frac{A_{\gamma Z}^{(l)}(q^2)}{q^2}=\frac{\alpha(1-4s^2)}{6\pi cs}\sum_{l=1}^{N}\left[\frac{1}{n-4}+C+3\int_0^1 dx\,x(1-x)\ln\left(\frac{m_l^2-q^2x(1-x)}{\mu^2}\right)\right], \tag{C3}$$

where the sum is over the charged leptons. One can readily verify that inserting Eqs. (C1), (C2), and (C3) in the expression between the curly brackets in Eq. (25a) leads to Eq. (28).

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*Permanent address: Department of Physics, New York University, New York, N.Y. 10003.

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[16]In the results of $\delta_\theta^{(h)}(-q^2)$ we have included a small contribution $+4\times10^{-4}$ arising from terms of $O(m_t^2/m_W^2)$ not exhibited in Eq. (27). They can be obtained from Eq. (B16) and are given by $[3\alpha/(2\pi s^2)][m_t^2/m_Z^2][\ln c^2/(4s^2)+4(1-4s^2/3)/3]$.

[17]If the detailed x dependence of $-q^2$ in Eq. (34a) is taken into account, after interchanging the order of the x and z integrations and introducing the new variable $u=x/z$, we encounter integrals proportional to $\int_0^1 uq_i^{(0)}(u)du$ and $\int_0^1 uq_i^{(0)}(u)\ln u\,du$. The latter can be estimated by using the approximate analytic expressions for $q_i^{(0)}(u)$ frequently employed (see, for example, Ref. 8).

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