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Variational principle for perfect and imperfect fluids in general relativity

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The Einstein equations and the relativistic Navier-Stokes equations for a simple fluid characterized by an arbitrary flux tensor are obtained by an action principle to which the equation for the entropy production is added as a constraint. The procedure is a generalization of the classical Herivel variational principle to relativistic and dissipative systems. The inclusion of dissipative processes requires a reformulation of the action integral to refer to a proper-time slice (τ_1,τ_2) in the limit $\tau_2\to\tau_1$, otherwise their nonconservative nature gives rise to non-Markoffian effects. Such a procedure is called a differential variational principle (DVP). The principle of least dissipation of energy can be incorporated into the DVP so that a linear form for the flux tensor can be produced as well as the above-mentioned equations.

I. INTRODUCTION

Variational principles (VP's) for hydrodynamical systems have been used for a long time. A nonrelativistic theory was given by Herivel[1,2] in 1955 for perfect fluids. It is the generalization of his method to relativistic and dissipative systems that is the subject of this paper.

Herivel's VP was of the Eulerian kind, that is, the action was formulated as an integral over the arbitrary space and time points x^i and not the initial ones X^i. The procedure was to make the action integral an extremal with the constraints that mass was conserved and that the entropy production was zero. Such a VP led to the Navier-Stokes equation (equation of motion) for a perfect fluid.

However, the Navier-Stokes equation produced this way was not completely general; it did not accommodate rotational flow. C.C. Lin[3] showed that the general equation of motion could be produced if an additional constraint was used, namely that the initial space coordinates [treated as functions of $x^1(\tau)$, $x^2(\tau)$, $x^3(\tau)$, and of τ] were independent of τ

$$dX^i/d\tau=0,\quad \delta\,dX^i/d\tau=0,\quad i=1,2,3. \tag{1.1}$$

Here τ is the proper time, and $d/d\tau=\partial/\partial t+\vec{u}\cdot\vec{\nabla}$. The combined theory, known as the Herivel-Lin VP, then produced the desired equations.

The original constraints of Herivel were conservation of mass, which we shall write as conservation of particle number,

$$(nu^i)_{;\,i}=0, \tag{1.2}$$

where n is the number of particles per volume, and u^i is the four-velocity, and zero entropy production,

$$(ns_au^i)_{;\,i}=0, \tag{1.3}$$

where s_a is the entropy per particle (the subscript a is used to denote "per particle") and ns_a is the entropy per volume.

A year before Herivel's paper, Taub[4] produced a relativistic VP for a perfect fluid. Taub's VP was stated in an Eulerian manner, but the actual computations were made for the most part in terms of the comoving (e.g., initial) coordinates. The basic dynamical variables were the world lines x^i of the fluid elements, the mass density, the temperature T, and the metric tensor g_{ik}. (g_{ik} varied both as a result of variations in the paths δx^i and variations in its own form.) The action was taken to be $\int (R-2\kappa f)\,(-g)^{1/2}d^4x$, where R is the curvature invariant, κ is Einstein's constant, and f is the free energy density $\epsilon - Tns_a$, where ϵ is the internal energy density. There was no entropy constraint. In fact, zero entropy production seemed to emerge as a consequence of the VP. There was, however, a constraint on the four-velocity:

$$g_{ik}u^iu^k/c^2-1=0. \tag{1.4}$$

This was an ingenious way to bring the kinetic energy into the principle, the action containing reference only to the internal energy. This VP produced the Einstein equations and the equations of motion independently.

There was no explicit use of a Lin condition in this VP, which preceded Lin's work. However, the use of comoving coordinates suggests that perhaps indirectly such a condition helped the argument. In fact, after Taub integrates Eq. (1.2) in comoving coordinates by Gauss's law to a three-dimensional surface integral bounding a tube oriented along u^i, he neglects the contribution from the lateral sides of the tube, i.e., the sides parallel to u^i. In comoving coordinates the integral $\int dS_i(dX^i/d\tau)n(-g)^{1/2}$ is zero since dS_i is orthogonal to $dX^i/d\tau$ on the lateral surface. However, a *variation* of this, for fixed dS_i is not zero unless $dS_i\,\delta dX^i/d\tau$ is zero. But this implies Eq. (1.1),

since the possible dS_i comprise all the local space directions. Thus implicit in the use of comoving coordinates is also the use of a form of the Lin condition.

In 1955, Fock published the *Theory of Spacetime and Gravitation*,[5] in which he includes a VP for a perfect fluid with electromagnetic effects as well. The details are somewhat similar to those of Taub. [The variation of Fock's (47.12) is essentially Taub's Eq. (3.2).] Fock did not use (1.4) above, but rather normalized the velocity vector.

A significant difference between the Taub and Herivel principles is that the entropy is contained in the Taub action integral, and conservation of entropy emerges from the VP, whereas in Herivel's approach entropy conservation is a side condition, and entropy does not appear in the action. The latter approach seems to be the natural generalization of the classical principles of thermodynamics: Maximization of entropy subject to constant energy, and minimization of energy subject to constant entropy.[6]

All this was for perfect fluids, that is, fluids without dissipative processes such as viscosity or heat conduction. When it comes to imperfect fluids variational principles run into difficulties. In the simple problem of an oscillator of mass m, force constant k, in a viscous medium with coefficient b, it is impossible to construct an action principle $\delta\int L dt = 0$ that will give the equation of motion

$$m\ddot{x} + b\dot{x} + kx = 0, \tag{1.5}$$

where an overdot denotes d/dt, *and nothing else*.[7,8] Bateman showed, however, that by choosing

$$L = m\dot{x}\dot{x}^* - \tfrac{1}{2}b(x^*\dot{x} - x\dot{x}^*) - Kxx^*, \tag{1.6}$$

and treating both x and x^* as dynamical variables, Eq. (1.5) emerged, but also

$$m^*\ddot{x}^* - b\dot{x}^* + kx^* = 0. \tag{1.7}$$

The starred functions represented a "dual system" whose nature it was to absorb heat in a frictional process (i.e., an antifriction universe). The heat produced in the physical system was in a sense balanced by the heat absorbed in the dual system. However, the VP had to contemplate *both* systems to work. Extensions of this method to the diffusion and other such equations have been made.[9] And, of course, the same formal structure exists in the quantum-mechanical VP in which the wave functions and their complex conjugates appear. By making enough constraints, one can probably formulate a VP to represent any set of differential equations.[10] However, such flexibility occurs at the expense of physical motivation.

Another VP is that of the least dissipation of energy. This was developed by Onsager in the course of his investigations into nonequilibrium thermodynamics.[11,12] It is not, however, an action principle and it was not designed to provide the Navier-Stokes type of equation of motion, but rather the linear relation in which heat flow is proportional to temperature gradient. It will turn out that inclusion of this VP into a general action principle is possible, and indeed necessary, to tie up a loose end of the latter. (See Sec. V.)

Finally, Moller[13] has reformulated the relativistic theory of thermodynamics, and has shown in this terminology that the equations of motion for an ideal fluid imply a VP in which the Lagrangian is the relativistic Gibbs function, and the independent variable is λ, where $d\lambda = T d\tau$ and where τ is the proper time following the motion. This VP has a Lagrangian not unlike that of Taub (a free energy with no entropy constraint). Moller's λ turns out to be just one of the Lagrange multipliers we use. However, Moller's approach lies outside the scope of this paper, and we shall not have occasion to refer to it again. Some other treatments of the perfect-fluid VP are listed in Refs. 14–16.

The purpose of this paper is to produce a VP which is relativistic, and can also include dissipative effects. In doing so we follow basically the Herivel approach.[17] We do not use the position coordinate x^i of the fluid element as the basic dynamical variable. Rather, the $xyzt$ coordinates are just a field over which the dynamical variables play. There are the number density $n(xyzt)$, the velocity vector $u^i(xyzt)$, the entropy per particle $s_a(xyzt)$, and the metric tensor $g_{ik}(xyzt)$. Variations with respect to these are taken in a straightforward manner, with no subtleties. The Lin constraint is not needed nor encountered. The theory for the perfect fluid is in Sec. III.

In Sec. IV we show that if the entropy production is not zero, but of the form

$$(ns_a u^i + u^i Q_i{}^k/T)_{;k} = (u^i/T)_{;k} Q_i{}^k, \tag{1.8}$$

where $Q_i{}^k$ is unspecified and treated as a fixed function, not a dynamical variable, then the VP yields the Einstein equations and the Navier-Stokes equations, with $Q_i{}^k$ entering just as the viscosity and heat conduction parts of the energy-momentum tensor should. The procedure leads to a number of unwanted terms also, and these must be argued away. In the latter part of Sec. IV a reformulation of the VP is discussed which accomplishes this.

The VP so far does not lead to expressions for $Q_i{}^k$ in terms of gradients of velocity or temperature. To get such equations one must consider the Onsager type of minimum principle. It is shown in Sec. V how such a principle can be absorbed into that of Sec. IV. Electromagnetism is

added on in the usual way in Sec. VI. Section II contains a review of Eckart's formulation of irreversible thermodynamics,[18] the basic ideas and notation of which are needed in the subsequent sections.

II. IRREVERSIBLE THERMODYNAMICS

We shall review here the basic ideas of irreversible thermodynamics so as to assemble the notation and basic equations needed in the VP. The basic paper is that of Eckart,[18] in which the laws of thermodynamics were correlated with the energy-momentum tensor relativistically.

Eckart's analysis relied on the projection of tensors parallel and orthogonal to a given direction, say u^i. The operator

$$\Delta_k^{\ a} = g_k^{\ a} - u_k u^a/c^2 \tag{2.1}$$

projects out the part orthogonal to u^a when operating on a vector V^k. Any vector V^k can then be expanded, i.e.,

$$V^i = (u_k V^k/c^2)u^i + \Delta^i_{\ k} V^k . \tag{2.2}$$

Similarly, any second-rank tensor can be expanded, i.e.,

$$T_k^{\ i} = \epsilon u^i u_k/c^2 + q_k u^i + u_k q^i + \overline{\Pi}_k^{\ i} + (\Pi - p)\Delta_k^{\ i}, \tag{2.3}$$

where each of the two indices i and k is projected parallel and orthogonal to u^m, with the "parallel-parallel" part being

$$u^i u_k \epsilon/c^2 = u^i u_k u^a u_b c^{-4} T_a^{\ b} \equiv P_0 \left(\begin{smallmatrix} ia \\ kb \end{smallmatrix}\right) T_a^{\ b}, \tag{2.4}$$

which *defines* the projection operator P_0 onto the parallel-parallel domain. The other terms in (2.3) have corresponding definitions and corresponding projection operators:

$$u_k q^i = u_k \Delta^i_{\ b} u^a c^{-2} T_a^{\ b} = P_1 \left(\begin{smallmatrix} ia \\ kb \end{smallmatrix}\right) T_a^{\ b}, \tag{2.5}$$

$$u^i q_k = u^i \Delta_k^{\ a} u_b c^{-2} T_a^{\ b} = P_{1'} \left(\begin{smallmatrix} ia \\ kb \end{smallmatrix}\right) T_a^{\ b}, \tag{2.6}$$

$$\overline{\Pi}_k^{\ i} = (\Delta^i_{\ b}\Delta_k^{\ a} - \tfrac{1}{3}\Delta_k^{\ i}\Delta_b^{\ a}) T_a^{\ b} = P_2 \left(\begin{smallmatrix} ia \\ kb \end{smallmatrix}\right) T_a^{\ b}, \tag{2.7}$$

$$(\Pi - p)\Delta_k^{\ i} = \tfrac{1}{3}\Delta_k^{\ i}\Delta_b^{\ a} T_a^{\ b} = P_3 \left(\begin{smallmatrix} ia \\ kb \end{smallmatrix}\right) T_a^{\ b}. \tag{2.8}$$

Here both (2.7) and (2.8) refer to the "perpendicular-perpendicular" part, but they are separated into a traceless term (2.7) and a term with trace (2.8).

The operators P_α ($\alpha = 0, 1, 1', 2, 3$) satisfy

$$P_\alpha \left(\begin{smallmatrix} ia \\ kb \end{smallmatrix}\right) P_{\alpha'} \left(\begin{smallmatrix} bs \\ ar \end{smallmatrix}\right) = P_\alpha \left(\begin{smallmatrix} is \\ kr \end{smallmatrix}\right) \delta_{\alpha\alpha'}, \tag{2.9}$$

and form the completeness relation

$$\delta_r^i \delta_k^s = \sum_{\alpha, ab} P_\alpha \left(\begin{smallmatrix} ia \\ kb \end{smallmatrix}\right) P_\alpha \left(\begin{smallmatrix} bs \\ ar \end{smallmatrix}\right) = \sum_\alpha P_\alpha \left(\begin{smallmatrix} is \\ kr \end{smallmatrix}\right). \tag{2.10}$$

The quantities in (2.7) and (2.8) form the parts of a general tensor $\Pi_k^{\ i}$,

$$\Pi^i_{\ k} = \overline{\Pi}^i_{\ k} + \Pi \Delta^i_{\ k}, \tag{2.11}$$

$$\overline{\Pi}^i_{\ k} = \Pi^i_{\ k} - \Pi^m_{\ m}\Delta_k^{\ i}/3, \quad \Pi = \Pi^m_{\ m}/3 . \tag{2.12}$$

The expansion in Eq. (2.3) is general, but if T^{ik} represents the energy-momentum tensor, it satisfies the Einstein equation

$$G^{ik} = \kappa T^{ik}, \tag{2.13}$$

where G^{ik} is the Einstein tensor. From this it follows that

$$T^{ik}_{\ \ ;k} = 0, \tag{2.14}$$

which is the relativistic equation of motion.

Eckart noticed that the projection of (2.14) parallel to u_i,

$$u_i T^{ik}_{\ \ ;k} = 0, \tag{2.15}$$

resembles the first law of thermodynamics, with ϵ being the internal energy density, q^i the heat flux vector, p the hydrostatic pressure, and Π^{ik} the viscosity tensor. The formal mathematical expansion in (2.3) corresponds to a physically meaningful separation when applied to the energy-momentum tensor. The perfect-fluid part of T^{ik} is

$$T^{ik}(\text{perf}) = nh_a u^i u^k/c^2 - p g^{ik}, \tag{2.16}$$

where h_a is the enthalpy per particle (nh_a the enthalpy per volume)

$$nh_a = \epsilon + p . \tag{2.17}$$

Thus, the total T^{ik} may be written as

$$T^{ik} = T^{ik}(\text{perf}) + Q^{ik}, \tag{2.18}$$

where

$$Q_i^{\ k} = u_i q^k + u^k q_i + \Pi_i^{\ k} \tag{2.19}$$

contains the dissipative processes, heat conduction and viscosity.

Eckart went further and showed that the functions ϵ and p imply the existence of two other functions, entropy per volume s, and temperature T according to

$$d\epsilon = T ds + \mu_a dn, \tag{2.20}$$

$$\epsilon + p = n(\mu_a + T s_a) = n h_a . \tag{2.21}$$

Eckart's proof did not include the terms in μ_a, the chemical potential per particle, nor did he write down (2.21). We shall need both of these additions, which go beyond Eckart's discussion. For their inclusion, see below Eq. (2.44). For reference we include an expression for dp which can be derived from Eqs. (2.20), (2.21), and (2.17):

$$dp = n(dh_a - T ds_a) . \tag{2.22}$$

Next, Eckart went on to the entropy-production

equation. This can be derived by rewriting (2.15), using (2.20) and (2.21). The μ_a terms drop out, using Eq. (1.2). Then

$$S^k{}_{;k}=(u^i/T)_{;k}Q_i{}^k\,, \tag{2.23}$$

where

$$S^k=ns_a u^k+u^i\bar{Q}_i{}^k/T \tag{2.24}$$

is the *entropy flux vector*. We have placed a bar on $\bar{Q}_i{}^k$ when Q appears in S^k to distinguish its presence here from its contribution to the *entropy-production rate*, which is the right-hand side of (2.23). It is this right-hand side *only* that is taken to be necessarily positive, not all the terms containing heat conduction and viscosity. This distinction becomes important in Sec. V.

It is convenient to define

$$X^i{}_k=(u^i/T)_{;k}\,. \tag{2.25}$$

Then (2.23), the entropy-production equation, becomes

$$S^k{}_{;k}=X^i{}_k Q_i{}^k\,. \tag{2.26}$$

It is customary to break up the right-hand side into the terms for heat conduction, viscosity, etc. This can be done easily with the help of the completeness relation (2.10):

$$S^k{}_{;k}=X^i{}_k\delta^s_i\delta^k_r Q_s{}^r=\sum_\alpha X^a{}_b(\alpha)Q_a{}^b(\alpha)\,, \tag{2.27}$$

where the so-called "fluxes" are

$$Q_a{}^b(\alpha)=P_\alpha\binom{bs}{ar}Q_s{}^r\,, \tag{2.28}$$

and the "forces" are

$$X^a{}_b(\alpha)=X^i{}_k P_\alpha\binom{ka}{ib}\,. \tag{2.29}$$

The fluxes of (2.28) have already been listed in Eqs. (2.5)–(2.8), and the forces, using (2.25), turn out to be

$$X^a{}_b(1)=\tfrac{1}{2}u^a\Delta_b{}^m(-T_{,m}/T^2+u_{m;r}u^r/c^2T)\,, \tag{2.30}$$

$$X^a{}_b(1')=\tfrac{1}{2}u_b\Delta^{am}(-T_{,m}/T^2+u_{m;r}u^r/c^2T)\,, \tag{2.31}$$

$$X^a{}_b(2)=(2T)^{-1}\Delta^{ar}\Delta_b{}^m(u_{r;m}+u_{m;r}-\tfrac{2}{3}u^s{}_{;s}\Delta_{rm})\,, \tag{2.32}$$

$$X^a{}_b(3)=(3T)^{-1}u^r{}_{;r}\Delta^a{}_b\,. \tag{2.33}$$

In getting these equations we used the symmetrized form of $X^i{}_k$, that is, $\frac{1}{2}(X^i{}_k+X_k{}^i)$.

All this is general. If we now require that the components of $Q_i{}^k$ be linear in the gradients of velocity and temperature and, for arbitrary values of these, that the right side of (2.27) be positive, then the Q's must have the form

$$Q_i{}^k(\alpha)=L(\alpha)X_i{}^k(\alpha)\,,\quad \alpha=1,1',2,3 \tag{2.34}$$

or, inversely

$$X_i{}^k(\alpha)=R(\alpha)Q_i{}^k(\alpha)\,, \tag{2.35}$$

where

$$R(\alpha)=1/L(\alpha)\,. \tag{2.36}$$

By comparing (2.34) with the usual expressions for the fluxes,

$$q^i=\chi T\Delta^{ik}(-T_{,k}/T+u_{k;m}u^m/c^2)\,, \tag{2.37}$$

$$\bar{\Pi}^{ik}=\eta\Delta^i{}_p\Delta^k{}_q[u^{p;q}+u^{q;p}-(\tfrac{2}{3})\Delta^{pq}u^m{}_{;m}]\,, \tag{2.38}$$

$$\Pi=(\zeta/3)u^m{}_{;m}\,, \tag{2.39}$$

where χ is the coefficient of heat conductivity, η the coefficient of shear viscosity, and ζ the coefficient of bulk viscosity (all positive), we can identify the L's as follows:

$$L(1')=L(1)=-2\chi T^2\leqslant 0\,, \tag{2.40}$$

$$L(2)=2T\eta\geqslant 0\,, \tag{2.41}$$

$$L(3)=T\zeta\geqslant 0\,. \tag{2.42}$$

Equations (2.34)–(2.42) constitute the linear approximation.

The results so far have been obtained from projecting $T^{ik}{}_{;k}$ along u_i as in (2.15). One can project along Δ_{mi} also, and this will lead to the Navier-Stokes equation. Eckart discussed the heat terms of this equation. The full equation comes from

$$\Delta^i{}_m T^{mk}{}_{;k}=0 \tag{2.43}$$

and is

$$(\epsilon+p)u^k u^i{}_{;k}/c^2-\Delta^{ik}p_{,k}+\Delta^{ik}Q_k{}^m{}_{;m}=0\,, \tag{2.44}$$

which is the desired equation.

All this was formulated or implied in Eckart's article. Another point of view was developed by Prigogine and co-workers.[19] Whereas Eckart purported to have derived Eq. (2.20) (without the μ_a term) from an analysis of (2.15), this other view regarded Eqs. (2.20) and (2.21), and similar thermodynamic equations, as the fundamental assumptions of nonequilibrium thermodynamics, namely that the equilibrium equations of Gibbs are valid also when systems are out of equilibrium in a first-order approximation. This was argued in terms of a kinetic-theory derivation that showed that the Gibbs equation is valid to within first-order terms in the deviation of the distribution function from the equilibrium distribution.

In the subsequent discussion in this paper we shall adopt this point of view, treating Eqs. (2.20) and (2.21), and related equations such as

$$T=n^{-1}\partial\epsilon/\partial s_a\,, \tag{2.45}$$

$$h_a=\partial\epsilon/\partial n\,, \tag{2.46}$$

as consequences of a fundamental and independent

postulate that is valid to first order in the gradients of velocity, temperature, and density.

Now how does a variational principle relate to this basic description? The discussion above (a) *assumed* the Einstein equations (2.13) and (2.14), (b) *assumed* the nonequilibrium postulates (2.20) and (2.21), then (c) *derived* the entropy-production equation (2.23), and (d) *derived* the generalized Navier-Stokes equation (2.44).

The variational principle we use will (a) *assume* the action principle in Eq. (3.1) below (and others in subsequent sections), (b) *assume* the nonequilibrium postulates (2.20) and (2.21), then will (c) *derive* the Einstein equations (2.13) and (2.14), and (d) *derive* independently the generalized Navier-Stokes equation (2.44). Further, in Sec. V shall show that a modified VP will also yield the linearized equations (2.34).

III. THE VARIATIONAL PRINCIPLE FOR A SIMPLE PERFECT FLUID

This is the prototype of all the VP's to follow. For a system near equilibrium whose thermodynamic variables satisfy (2.20) and (2.21), and also satisfy an equation of state $p = p(n, s_a)$, the action integral over an arbitrary volume Ω,

$$I = \int_\Omega (R - 2\kappa\epsilon)(-g)^{1/2} d^4x , \tag{3.1}$$

is to be an extremal

$$\delta I = 0 , \tag{3.2}$$

with respect to variations in the dynamical variables n, u^i, s_a, and g_{ik} that vanish on the boundary of Ω, subject to the constraints (1.2)–(1.4), which are repeated here with their Lagrange multipliers:

$$(nu^i)_{;\,i} = 0, \quad -\phi , \tag{3.3}$$

$$g_{ik} u^i u^k / c^2 - 1 = 0, \quad -\kappa n\psi , \tag{3.4}$$

$$(nu^k s_a)_{;\,k} = 0, \quad -2\kappa\lambda . \tag{3.5}$$

(In the VP, u^k is defined as the direction along which there is zero entropy production.) Thus, we have

$$I' = \int \Lambda (-g)^{1/2} d^4x, \quad \delta I' = 0 , \tag{3.6}$$

where

$$\Lambda = R - 2\kappa\epsilon - \phi (nu^i)_{;\,i} - \kappa n\psi (g_{ij} u^i u^j / c^2 - 1) - 2\kappa\lambda n u^k s_{a,\,k} . \tag{3.7}$$

This is the basic VP. To implement this we have (G^{ik} is the Einstein tensor)

$$\delta \int R(-g)^{1/2} d^4x = \int G^{ik} \delta g_{ik} (-g)^{1/2} d^4x , \tag{3.8}$$

$$\delta \int (-2\kappa\epsilon)(-g)^{1/2} d^4x = -2\kappa \int [(\partial\epsilon/\partial n)\delta n + (\partial\epsilon/\partial s_a)\delta s_a + \tfrac{1}{2}\epsilon g^{ik} \delta g_{ik}](-g)^{1/2} d^4x , \tag{3.9}$$

$$\delta \int (-)\phi (nu^i)_{;\,i} (-g)^{1/2} d^4x = \delta \int \phi_{,\,i} n u^i (-g)^{1/2} d^4x = \int (\phi_{,\,i} u^i \delta n + \phi_{,\,i} n \delta u^i + \phi_{,\,m} n u^m \tfrac{1}{2} g^{ik} \delta g_{ik})(-g)^{1/2} d^4x , \tag{3.10}$$

$$\delta \int (-)\kappa n\psi (g_{ik} u^i u^k / c^2 - 1)(-g)^{1/2} d^4x = -\kappa \int n\psi (2u_i c^{-2} \delta u^i + u^i u^k c^{-2} \delta g_{ik})(-g)^{1/2} d^4x , \tag{3.11}$$

$$-\delta 2\kappa \int \lambda (nu^k s_a)_{;\,k} (-g)^{1/2} d^4x = 2\kappa\delta \int \lambda_{,\,k} n u^k s_a (-g)^{1/2} d^4x$$

$$= 2\kappa \int [\delta n (\lambda_{,\,k} u^k s_a) + \delta u^k (\lambda_{,\,k} n s_a) + \delta s_a (n \lambda_{,\,k} u^k) + \tfrac{1}{2} \delta g_{ik} g^{ik} n u^m s_a \lambda_{,\,m}](-g)^{1/2} d^4x . \tag{3.12}$$

In working out these equations, use of the four-dimensional Gauss's theorem and the assumption that all variations vanish on the (arbitrary) surface of the four-volume integrated over were made. Also, *after* the variations were taken, the constraints in Eqs. (3.3)–(3.5) were used to simplify the expressions.

We now put Eqs. (3.8)–(3.12) into (3.6). We get

$$\delta I' = \int (A^{ik} \delta g_{ik} + B\delta n + C_i \delta u^i + D\delta s_a)(-g)^{1/2} d^4x = 0 , \tag{3.13}$$

where for arbitrary variations of the dynamical variables at every point x^i of the integrand,

$$A^{ik} \equiv G^{ik} - \kappa\epsilon g^{ik} - \tfrac{1}{2} n g^{ki} \phi_{,\,m} u^m - \kappa n\psi u^i u^k / c^2 + \kappa g^{ik} n u^m \lambda_{,\,m} s_a = 0 , \tag{3.14}$$

$$B \equiv -u^k [\phi_{,\,k} - 2\kappa(\partial\epsilon/\partial n) u_k / c^2 + 2\kappa\lambda_{,\,k} s_a] = 0 , \tag{3.15}$$

$$C_i \equiv n[\phi_{,\,i} - 2\kappa\psi u_i / c^2 + 2\kappa\lambda_{,\,i} s_a] = 0 , \tag{3.16}$$

$$D \equiv -2\kappa\partial\epsilon/\partial s_a + 2\kappa n \lambda_{,\,k} u^k = 0 . \tag{3.17}$$

Equation (3.14) will give us the Einstein equa-

tions, and Eqs. (3.15) and (3.16) will give us the Navier-Stokes equation. The mathematical problem is to eliminate the three Lagrange multipliers ψ, λ, and ϕ. The usual way to do this is to substitute the solutions back into the constraints. Equation (3.17) gives us, however, directly using Eq. (2.44),

$$\lambda_{,k}u^k = T\,, \tag{3.18}$$

which turns out to be all we shall ever know about λ.

As for ψ, we notice that the quantities in square brackets in Eqs. (3.15) and (3.16) are almost identical. By forming $C_i u^i/n$ we bring (3.16) into the same form as (3.15). Thus from

$$C_i u^i/n - B = 0\,, \tag{3.19}$$

we find

$$\psi = h_a \tag{3.20}$$

using (2.46). This much information enables us to write (3.14) as

$$G^{ik} = \kappa(nh_a u^i u^k/c^2 - pg^{ik})\,, \tag{3.21}$$

where the ϕ terms were eliminated by (3.15). Equation (3.21) is the desired Einstein equation for a perfect fluid. [See Eqs. (2.13) and (2.16).]

Equation (3.19) may be regarded as an indirect use of the constraint (3.4) to evaluate a multiplier. The other constraints, (3.3) and (3.5), involve *derivatives* of u^i or s_a. To substitute the variational equations into these constraints requires, therefore, that derivatives be taken of the variational equations. It turns out that by use of [20]

$$B_{,i} - u^m(C_i/n)_{;m} = 0\,, \tag{3.22}$$

all reference to ϕ and λ disappears. We shall interpret (3.22) as the appropriate procedure for substituting the variational equations into the constraints to eliminate reference to the multipliers.

Substitution of (3.15) and (3.16) into (3.22), and using (3.16) again to eliminate $\phi_{,i}$, leads after a tedious calculation to

$$nh_a u^m u_{i;m}/c^2 - \Delta_i{}^m(nh_{a,m} - nTs_{a,m}) + nTu_i u^m s_{a,m}/c^2 = 0\,. \tag{3.23}$$

The last term is zero by (3.5) with (3.3), and the quantity in parentheses is just $p_{,m}$ by (2.22). Thus (3.23) reduces to the perfect-fluid part of (2.44), i.e., the part not containing $Q_k{}^m$. Thus, we have arrived at the Navier-Stokes equation for a perfect fluid. Note that the derivations of the Einstein equations and equations of motion were independent, since the former came from (3.14) and the latter from (3.15) and (3.16). Note also that although (3.5) defined u^k, the results in (3.21) and (3.23) tell us that u^k is also the direction of motion.

IV. THE VP FOR A FLUID WITH VISCOSITY AND HEAT CONDUCTION. REFORMULATION OF THE VP

To take into account dissipative processes in the VP, we modify the entropy-production equation (3.5) to the form in (2.23):

$$(ns_a u^k + u^i \bar{Q}_i{}^k/T)_{;k} - (u^i/T)_{;k}Q_i{}^k = 0, \quad -2\kappa\lambda \tag{4.1}$$

where the barred $\bar{Q}$ contributes to the entropy flux vector S^k [see Eq. (2.24)], and the unbarred one contributes to the entropy production. They are the same quantity, of course. In this Section we are interested in the VP for the Einstein and Navier-Stokes equations and have no need of the Q's, so we set their variations to zero:

$$\delta Q_i{}^k = 0\,, \tag{4.2}$$

$$\delta \bar{Q}_i{}^k = 0\,. \tag{4.3}$$

In Sec. V we shall relax (4.2). It should be remarked that in order to get the desired equations, it must be $Q_i{}^k$ that has zero variation and not, say, Q^{ik}, Q_{ik}, or $Q^i{}_k$. With any of these other choices the variational equations do not contain the correct terms. There seems to be no obvious reason why it turns out this way, unless it is that $Q_i{}^k$ is what naturally enters the expressions in Eq. (4.1).

We now return to the plan of Sec. III. After Eqs. (3.8)–(3.11), which remain unaltered, Eq. (3.12) must be replaced by

$$\begin{aligned}-2\kappa\delta\int \lambda[(nu^k s_a + u^i \bar{Q}_i{}^k/T)_{;k} - (u^i/T)_{;k}Q_i{}^k](-g)^{1/2}d^4x \\ = 2\kappa\int\{\delta n(\lambda_{,k}u^k s_a) + \delta s_a(\lambda_{,k}nu^k) + \delta u^i(\lambda_{,i}ns_a - \lambda Q_i{}^k{}_{;k}/T) \\ + \delta g_{ik}[\tfrac{1}{2}g^{ik}(\lambda u^m/T)_{;r}Q_m{}^r - \tfrac{1}{2}(\lambda u^m Q^{ik}/T)_{;m} + \tfrac{1}{2}g^{ik}\lambda_{,m}u^m ns_a] + \lambda(u^i/T)_{;k}\delta Q_i{}^k + \lambda_{,k}(u^i/T)\delta\bar{Q}_i{}^k\}(-g)^{1/2}d^4x\,.\end{aligned} \tag{4.4}$$

In order to get this result we had to use

$$\delta(u^i{}_{;k}) = (\delta u^i)_{;k} + \tfrac{1}{2}u^r g^{im}[(\delta g_{mr})_{;k} + (\delta g_{mk})_{;r} - (\delta g_{rk})_{;m}], \tag{4.5}$$

which follows from writing $u^i{}_{;k}$ in terms of Christoffel symbols $\Gamma^i{}_{jk}$ and using

$$\delta\Gamma^i{}_{kj} = \tfrac{1}{2}g^{im}[(\delta g_{mk})_{;j} + (\delta g_{mj})_{;k} - (\delta g_{kj})_{;m}], \tag{4.6}$$

along with the symmetry of Q^{ik}. Many of the terms had to be integrated by parts.

Instead of (3.13) we now have

$$\delta I' = \int (A^{ik}\delta g_{ik} + B\delta n + C_i\delta u^i + D\delta s_a + E^i{}_k\delta Q_i{}^k + F^i{}_k\delta\overline{Q}_i{}^k)(-g)^{1/2}d^4x\,. \tag{4.7}$$

Here

$$A^{ik} = G^{ik} - \kappa\epsilon g^{ik} + \tfrac{1}{2}\phi_{,m}nu^m g^{ik} - \kappa n\psi u^i u^k/c^2 + \kappa\lambda_{,r}u^m g^{ik}Q_m{}^r/T - \kappa\lambda_{,m}u^m Q^{ik}/T + \kappa g^{ik}\lambda_{,m}u^m n s_a$$
$$+ \kappa\lambda(u^m Q_m{}^r/T)_{;r}g^{ik} - \kappa\lambda(u^m Q^{ik}/T)_{;m} - \kappa\lambda g^{ik}u^r Q_r{}^m{}_{;m}/T = 0\,, \tag{4.8}$$

where we separated off all terms containing derivatives of λ from those simply multiplied by λ. Also,

$$B = -2\kappa\partial\epsilon/\partial n + \phi_{,m}u^m + 2\kappa\lambda_{,k}u^k s_a = 0\,, \tag{4.9}$$

$$C_i = n(\phi_{,i} - 2\kappa\psi u_i/c^2 + 2\kappa\lambda_{,i}s_a - 2\kappa\lambda Q_i{}^k{}_{;k}/nT) = 0\,, \tag{4.10}$$

$$D = -2\kappa\partial\epsilon/\partial s_a + 2\kappa n\lambda_{,k}u^k = 0\,, \tag{4.11}$$

$$E^i{}_k = \lambda(u^i/T)_{;k}\,, \tag{4.12}$$

$$F^i{}_k = \lambda_{,k}(u^i/T)\,. \tag{4.13}$$

Equation (4.11) is the same as (3.17) whence

$$\lambda_{,k}u^k = T\,. \tag{4.14}$$

We can therefore expand $\lambda_{,k}$ in the Eckart manner in a term parallel and a term orthogonal to u_k:

$$\lambda_{,k} = Tu_k/c^2 + \lambda_{,m}\Delta^m{}_k\,. \tag{4.15}$$

Following the argument of Sec. III we now form (3.19), obtaining with (4.9) and (4.10),

$$\psi = h_a - \lambda u^i Q_i{}^k{}_{;k}/nT\,. \tag{4.16}$$

We substitute (4.15) and (4.16) into (4.8) to get the Einstein equation

$$G^{ik} = \kappa(nh_a u^i u^k/c^2 - pg^{ik} + Q^{ik} + Z^{ik})\,, \tag{4.17}$$

where all the unwanted terms are lumped in Z^{ik}, i.e.,

$$Z^{ik} = -\lambda[u^r Q_r{}^m{}_{;m}u^i u^k/c^2 T + g^{ik}(u^m Q_m{}^r/T)_{;r} - (u^m Q^{ik}/T)_{;m} - g^{ik}u^r Q_r{}^m{}_{;m}/T]$$
$$- \lambda_{,s}\Delta^s{}_r g^{ik}u^m Q_m{}^r/T\,. \tag{4.18}$$

In (4.17), p is not as in Eq. (2.17) but rather

$$p = nh_a - (\epsilon - u^m u_r Q_m{}^r/T)\,. \tag{4.19}$$

In fact, we could define a new energy density ϵ^0 as

$$\epsilon^0 = \epsilon - u^m u_r Q_m{}^r/T\,, \tag{4.20}$$

since only this combination appears directly, and $\partial\epsilon^0/\partial n = \partial\epsilon/\partial n$, since the second term of Eq. (4.20) contains only independent dynamical variables or functions of position.

We expect the second term of (4.20) to be zero, since $Q_m{}^r$ is defined in Eq. (2.18) not to have a parallel-parallel part. However, the VP does not know this; in the entropy-production constraint equation (4.1), $Q_i{}^k$ appears as a function with no restrictions. The VP is supposed, in fact, to lead to the expression for T^{ik}, as it does in (4.17), and it cannot utilize the answer beforehand. Thus, the argument of the present section in no way can suggest what $Q_i{}^k$ is, and cannot get rid of the second term in Eq. (4.20). In the next section, however, we consider varying the Q's also. When that is done, we get expressions for them and in particular it will be possible to argue that there is no parallel-parallel component so that ϵ^0 reduces to ϵ. In anticipation of this result we shall use the notation ϵ rather than ϵ^0 in our discussion.

To get the equations of motion we again use Eq. (3.22), but now with Eqs. (4.9) and (4.10). A very tedious but straightforward calculation [in which we eliminate $\phi_{,m}$ by (4.10) and utilize (4.15)] leads to

$$nh_a u^m u_{i;m}/c^2 - \Delta_i{}^m(nh_{a,m} - nTs_{a,m}) + \Delta_i{}^m Q_m{}^k{}_{;k} + Z_i = 0\,, \tag{4.21}$$

where the quantity in parentheses is $p_{,m}$ by (2.22), and where all the unwanted terms [those that deviate from (2.43)] are lumped in Z_i:

$$Z_i = \Delta_i{}^r\{\lambda_{,r}u^m Q_m{}^k{}_{;k}/T - \lambda[u_{r;m}u^m u^b Q_b{}^k{}_{;k}/c^2 T - u^m{}_{;r}Q_m{}^k{}_{;k}/T - nu^m(Q_r{}^k{}_{;k}/nT)_{;m}]\}\,. \tag{4.22}$$

Two consistency conditions must be satisfied. Since $G^{ik}{}_{;k}=0$ identically, the divergence of (4.17) times $\Delta^p{}_i$ will be zero. However, the first three terms in this result are the first three terms of (4.21). Thus,

$$\Delta_{ir}Z^{rk}{}_{;k}-Z_i=0. \tag{4.23}$$

Secondly, if $G^{ik}{}_{;k}u_i$ is subtracted from (4.1), we get

$$u_iZ^{ik}{}_{;k}=0. \tag{4.24}$$

Any approximation involving Z^{ik} and Z_i must satisfy Eqs. (4.23) and (4.24).

We now turn to a discussion of the basic results of this section, Eqs. (4.17) and (4.21). For these to reduce to the standard equations, Z^{ik} and Z_i must drop out. Each term in the Z's depends linearly on Q and is proportional either to λ itself or its space gradient. We cannot of course set Q equal to zero and still contemplate the dissipative processes. However, Q is linear at least in the gradients of u^i, n, or T, and the first question we might ask is whether or not the Z's are of higher order in these gradients than the desired terms in Eqs. (4.17) and (4.21). Secondly, we could ask whether λ and its space gradient could be zero. It is, in fact, easy to see that if λ is regarded as of zeroth order in the gradients, then Z^{ik} is of second order in (4.17)—one order higher than Q^{ik}—and Z_i is of third order in (4.21)—one order higher than $Q^k{}_{m;k}$. (In this argument a gradient is regarded as increasing the order by 1 so that $Q^k{}_{m;k}$ is of second order, $\lambda_{,k}\Delta^{km}$ is of first order, etc.) Thus, one could conclude that the Z terms can be neglected on the basis of being of higher order.

This argument rests on the assumption that λ is of zeroth order. However, λ from (4.14) is an integral of T, and hence might be considered to be of order minus 1. The general solution to (4.14) may be written as

$$\lambda=\int_{\tau_0}^{\tau}T\,d\tau'=\overline{T}(\tau-\tau_0), \tag{4.25}$$

where $\overline{T}$ is an average temperature defined by (4.25), and τ_0 must be independent of τ:

$$\tau_{0,k}u^k=0, \tag{4.26}$$

but is otherwise arbitrary. The integral in (4.25) follows the motion from the "initial" time τ_0 to τ. (τ_0 has nothing to do with solving the equations of motion.)

The right-hand side of (4.25) can be made as small as desired by making $\tau-\tau_0$ small. Thus, an argument based on order could be upheld if the domain of integration is such and τ_0 is such that $\tau-\tau_0$ is small everywhere in the domain. If the domain is over all four-space, then clearly the argument fails. One must choose a narrow proper-time slice (τ_1,τ_2) for the domain, and place τ_0 between τ_1 and τ_2. There is nothing preventing us from choosing such a domain and such a τ_0. Since the results are then independent of what the time slice and τ_0 actually are, the results are generally valid for any time τ.

This discussion has led us to the point of considering the second possibility mentioned in the paragraph below Eq. (4.24). That is, since each term in the Z's is proportional either to λ or its space gradient, could we make λ equal to zero? From Eq. (4.25), this would require setting $\tau_0=\tau$, which contradicts Eq. (4.26). And further, if λ were zero in any finite time slice (as described in the previous paragraph), its time derivative would be zero, not T as (4.14) requires.

Both these objections could be overcome, however, if we consider the time slice (τ_1,τ_2), mentioned two paragraphs above, *in the limit* $\tau_2\to\tau_1$, with of course both τ and τ_0 required to be trapped between τ_1 and τ_2. The VP would then be designed to give information only about a particular (but arbitrary) constant τ surface. Since the results would be independent of whichever surface was involved, the results would be general. It should be kept in mind that taking the limit is the last thing that is done; integration by parts timewise must precede this limit taking, otherwise we would not have obtained the normal terms in Eq. (4.17) and (4.21). We shall describe this procedure as a *differential VP*, DVP, since the limit taking here is reminiscent of taking a derivative.

The motivation so far for such a DVP has been mathematical; this is what is required to get the desired equations, independent of order. However, there is a compelling physical reason for setting λ to zero also. The terms in λ, if allowed to appear in the equations, bring with them a dependence on the history of the system from τ_0 to τ (in the equations themselves, not just the solutions) where τ_0 is arbitrary. The equations would not just be relations among the dynamical variables and their derivatives, but would depend on the time at which the system is considered, and on its historical development. Such a system would be non-Markoffian in the language of irreversible thermodynamics.[21] Thus, to eliminate any hint of non-Markoffian effects, we *must* set λ and its space gradients equal to zero.

As an illustration, consider Eq. (4.16). Using (2.21) we get

$$n\psi=n\mu_a+[ns_a+(\overline{T}/T)(ns_au^k)_{;k}(\tau-\tau_0)]. \tag{4.27}$$

The square brackets contain the normal expression ns_a for the entropy density in the first term and, in the second term an average increase of

entropy density caused by the dissipative processes from τ_0 to τ. It is this second term that exhibits (in the equations themselves) an explicit development of the system from τ_0 to τ. In fact, it is possible to identify all the terms in Z^{ik} as history-dependent supplements to the normal terms in (4.17). We shall call such supplements *accumulation terms*. In the present case we would wish that the normal term ns_a would contain the entire entropy density instantaneously as the system evolves.

The problem with λ has an analogy with ϕ in Eq. (3.3). From (3.15) and (3.26),

$$\phi = \int_{\tau_0'}^{\tau} 2\kappa\mu_a d\tau' , \tag{4.28}$$

which is analogous to (4.25), with τ_0' independent of τ but otherwise arbitrary. Equation (3.3) concerns the *particle* production. No trouble arises in this paper with ϕ since the particle production is zero. (Just as no trouble arose with λ in Sec. III.) If the particle production were not zero, terms proportional to ϕ and its space gradient would appear in the equations of motion, representing the effects of the accumulation of particles from τ_0' to τ. However, use of a time slice (τ_1, τ_2), with $\tau_2 \to \tau_1$, sweeps away all such accumulation terms along with those proportional to λ and its space gradient.

Getting rid of accumulation terms has its analog in the Bateman VP discussed briefly in Sec. I. There, two systems had to be considered simultaneously: The physical system, in which friction emits heat, and a dual system, in which friction absorbs heat. The VP is designed so that no accumulation of heat (in the physical system) or vacancy of heat (in the dual system) occurs. In effect, the heat that would have accumulated is delivered from the physical to the dual system. The scheme is quite different from ours, but the issue is the same.

The crux of the matter, as emphasized by Gyarmati,[22] is that action principles involving an integral over time are *global VP*'s, whereas problems involving dissipative processes lead to *constant time VP*'s. When both effects are present we cannot on the one hand keep the time interval finite, for then the dissipative processes give rise to non-Markoffian effects, and we cannot on the other hand go at once to a constant τ surface, for then the kinetic terms cannot be integrated by parts over time intervals and develop time derivatives.

The *differential VP* takes an intermediate route: It works the problem first over a small proper-time slice (τ_1, τ_2) and then takes the limit $\tau_2 \to \tau_1$. We then suggest: For a system whose thermodynamic variables satisfy Eqs. (2.20), and (2.21), whence (2.45) and (2.46), and an equation of state $p = p(n, s_a)$, the action integral over the time slice (τ_1, τ_2),

$$I(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} (R - 2\kappa\epsilon)(-g)^{1/2} d^4x , \tag{4.29}$$

is to be an extremal in the sense that

$$\delta_L I \equiv \lim_{\tau_2 \to \tau_1} \delta I(\tau_1, \tau_2) = 0 , \tag{4.30}$$

with respect to variations in the dynamical variables n, u^i, s_a, and g_{ik} that vanish on the boundary surface of the slice, subject to the constraints (3.3), (3.4), and (4.1). The limit sign means that after the variational equations are obtained and substituted back into the constraints, as in Eqs. (3.19) and (3.22), the limit is to be taken. It is implied here that all parameters appearing directly in the equations refer to the time slice (τ_1, τ_2) in question.

The principle here might be called a differential VP of first order, in the sense that first derivatives of the variational equations are taken before the limit is taken. This is related to the fact that the constraint equations involve only first derivatives. It might happen in other circumstances that the constraint equations contain higher-order derivatives, and that higher-order derivatives of the variational equations would have to be taken before the limit was taken. In that case, it would be a differential VP of higher order.

The VP was worked out in a general-relativistic formalism. The special-relativistic version can be obtained directly by setting $g_{ik} = (1, -1, -1, -1)$, $\delta g_{ik} = 0$, and $R = 0$. As such it differs from the usual particle VP by having the kinetic term $m_0 c^2 (1 - v^2/c^2)^{1/2}$ replaced by the constraint (3.4). Nevertheless, the argument goes through with no alterations. The inclusion of dissipative processes in a VP is, of course, a classical problem as well. The formalism here, however, does not reduce directly to a classical limit since the constraint (3.4) has no classical analog. Presumably, one must return to the Herivel-Lin VP, modify the entropy constraint appropriately, and then invoke the same limit procedure. However, we have not worked this out.

V. THE PRINCIPLE OF LEAST DISSIPATION OF ENERGY

The pioneering work of Onsager in irreversible thermodynamics[11] found expression in the VP known by the title of this section. A rather full

development can be found in Gyarmati[12] and in Lavenda.[23] In this section we shall formulate the principle in relativistic terms, and discuss what it takes to incorporate it into the action principle of the previous section. With it, the form of the $Q_i{}^k$ can be determined.

The entropy production in Eq. (2.23) or (2.26) can be written, using Eq. (2.10), as

$$\sigma(Q,X)\equiv(u^i/T)_{;k}\left[\sum_\alpha P_\alpha\left({}^{ka}_{ib}\right)P_\alpha\left({}^{bs}_{ar}\right)\right]Q_s{}^r$$

$$=\sum_\alpha Q_i{}^k(\alpha)X^i{}_k(\alpha)\,. \tag{5.1}$$

We shall, in what follows, use σ', $\sum'_\alpha$ (i.e., primes) to mean σ and $\sum$ without the $\alpha=0$ term. In addition, we need to define the dissipation functions

$$\Psi'(X,X)=\tfrac{1}{2}X^i{}_k\left[\sum_\alpha{}' P_\alpha\left({}^{ka}_{ib}\right)L(\alpha)P_\alpha\left({}^{bs}_{ar}\right)\right]X^r{}_s$$

$$=\tfrac{1}{2}\sum_\alpha{}' X^a{}_b(\alpha)L(\alpha)X^b{}_a(\alpha)\,, \tag{5.2}$$

$$\Phi'(Q,Q)=\tfrac{1}{2}Q_k{}^i\left[\sum_\alpha{}' P_\alpha\left({}^{ka}_{ib}\right)R(\alpha)P_\alpha\left({}^{bs}_{ar}\right)\right]Q_s{}^r$$

$$=\tfrac{1}{2}\sum_\alpha{}' Q_b{}^a(\alpha)R(\alpha)Q_a{}^b(\alpha)\,, \tag{5.3}$$

using the notation of Eqs. (2.28)–(2.33). The quantity in square brackets might be called resistance dyadics, or transport dyadics.

The local form of the VP can be stated[12] as

$$\delta_Q[\sigma'(Q,X)-\Phi'(Q,Q)]=0\,, \tag{5.4}$$

where variations are taken with respect to the Q's but not the X's or

$$\delta_X[\sigma'(Q,X)-\Psi'(X,X)]=0\,, \tag{5.5}$$

where variations are taken with respect to the X's but not the Q's. This VP yields the linear equations (2.34) and (2.35). This can be seen by varying the $Q_a{}^b(\alpha)$ directly [or the $X^a{}_b(\alpha)$'s]. Thus, using the last forms in Eqs. (5.1) and (5.3), we get from (5.4)

$$\sum{}' [X^a{}_b(\alpha)-R(\alpha)Q_b{}^a(\alpha)]\delta Q_a{}^b(\alpha)=0\,. \tag{5.6}$$

Since the δQ are arbitrary, the quantity in square brackets must be zero for each α and we obtain Eq. (2.35).

Lavenda[24] has shown that Eq. (5.4) can be regarded as the result of a VP in which

$$\Phi'(Q,Q)=\min \tag{5.7}$$

is the minimum principle, but subject to the constraint

$$\sigma'(Q,X)-2\Phi'(Q,Q)=0\,, \tag{5.8}$$

which is in fact obtained by multiplying (2.34) by $R(\alpha)Q^i{}_k(\alpha)$ and summing over the α. If Eq. (5.8) is handled with a Lagrange multiplier, then this principle leads to the unconstrained principle (5.4).

Although the above discussion constitutes a relativistic version of the VP, it would be preferable if it could be incorporated into the action principle of the previous sections to produce a unified VP. With this in mind we notice that the Lavenda form is fairly close to that of our action principle, but it has in (5.7) a different quantity that is made an extremal, and the side condition is not the same as (4.1), although it is closely related.

In order to obtain the unification, we go back to Eq. (5.3) and define a *transport operator* $\hat{R}$ (the caret is used to distinguish it from the curvature scalar R) whose projections onto the P_α are the $R(\alpha)$. We define the tensor components of $\hat{R}$ to be

$$\hat{R}\left({}^{ks}_{ir}\right)=\sum_\alpha P_\alpha\left({}^{ka}_{ib}\right)R(\alpha)P_\alpha\left({}^{bs}_{ar}\right) \tag{5.9a}$$

and the inverse is

$$R(\alpha)=d(\alpha)^{-1}P_\alpha\left({}^{ia}_{kb}\right)\hat{R}\left({}^{bc}_{ad}\right)P_\alpha\left({}^{dk}_{ci}\right), \tag{5.9b}$$

where the $d(\alpha)$ are numbers needed because of normalization:

$$d(0)=1=d(3),\quad d(2)=8,\quad d(1)=3\,. \tag{5.9c}$$

It should be noted that the $\alpha\neq 0$ terms in Eq. (5.9a) are what appear in Eq. (5.3), but that there is also in general an $\alpha=0$ term. The form in Eq. (5.9a) is similar to that of the completeness relation in Eq. (2.10), and in fact is typical of the expansion of any operator in terms of projectors.[25] The diagonality of Eq. (5.9) is not essential to the theory. In terms of $\hat{R}$ we define a generalized dissipation function

$$\Phi=\tfrac{1}{2}Q^i{}_k\hat{R}\left({}^{ks}_{ir}\right)Q_s{}^r\,, \tag{5.10a}$$

which can be symbolized in a quantum-mechanical notation

$$\Phi=\tfrac{1}{2}\langle Q\,|\,\hat{R}\,|\,Q\rangle\,. \tag{5.10b}$$

In sum, there are three objects in the theory: X, Q, and R. The projections of X onto the P_α give the forces $X(\alpha)$, the projections of Q onto the P_α give the fluxes $Q(\alpha)$, and the projections of $\hat{R}$ onto the P_α give the transport coefficients $R(\alpha)$. The dissipation function, Eq. (5.10), has a form in which the Q's appear as wave functions and the $\hat{R}$ as an operator.

Now, just as in the quantum-mechanical VP, when we come to take variations of Φ we shall vary the wave functions Q, but not the operator $\hat{R}$:

$$\delta\Phi=\tfrac{1}{2}(\delta Q)\hat{R}Q+\tfrac{1}{2}Q\hat{R}\delta Q\,. \tag{5.11}$$

Equation (5.11) is the fundamental postulate that allows the unification of the VP's to take place, and establishes $\hat{R}$ as an independent geometrical operator in the theory.

We now add Φ to the action integrand[26] in Eq. (3.1) to obtain

$$\delta_L J = \lim_{\tau_2 \to \tau_1} \delta \int_{\tau_1}^{\tau_2} [(R - 2\kappa\epsilon) + 2\kappa\lambda f \tfrac{1}{2} Q^i{}_k R(^{ks}_{ir}) Q_s{}^r](-g)^{1/2} d^4x , \tag{5.12}$$

where $2\kappa\lambda f$ is a multiplier put in for generality, f being an arbitrary function. Then the constraint in (4.1) is written

$$(n s_a u^k + u^i \overline{Q}_i{}^k/T)_{;k} - X^i{}_k \left[\sum_\alpha P_\alpha(^{ka}_{ib}) P_\alpha(^{bs}_{ar})\right] Q_s{}^r = 0 . \tag{5.13}$$

In the VP we vary the Q's but not the $\overline{Q}$'s. This is necessary if we are to generate a VP analogous to that of Onsager, for in that, what enters is the entropy production [which is the second term of (5.13)] containing the Q's. The entropy flux vector, representing flux through the surface, does not enter the Onsager VP. In adapting our side condition (5.13) to this problem, we must therefore distinguish between the physical significance of the two terms and ask for the response of the entropy volume production term to variations in the fluxes, but not for the response of the entropy surface flux term. In principle, there is nothing difficult or inconsistent about keeping the $\overline{Q}$'s fixed as we vary the Q's.

The VP is then (5.12), with (5.13) as a constraint. To proceed, we form from the J of (5.12) a J' analogous to the I' of (3.6), and then set $\delta J' = 0$. There are two new effects. First of all, the new Φ term in (5.12) makes a contribution to the A^{ik}:

$$\Delta A^{ik} = \tfrac{1}{2} g^{ik}(2\kappa\lambda f)\Phi(Q, Q) . \tag{5.14}$$

And second, by having $\delta Q \neq 0$, we get an addition to $\delta J'$ we shall call $\delta_Q J'$:

$$\delta_Q J' = \lim \sum \int 2\kappa\lambda \{f R(\alpha) Q_b{}^a(\alpha) - X^a{}_b(\alpha)\} \times [P_\alpha(^{bs}_{ar}) \delta Q_s{}^r](-g)^{1/2} d^4x . \tag{5.15}$$

If now the $\delta Q_s{}^r$ are completely arbitrary, then so are the projections $P_\alpha \delta Q$. Then the quantity in curly brackets of (5.15) must be zero for each α. For $\alpha = 1, 1', 2, 3$ these equations will bring back Eqs. (2.34) and (2.35) provided

$$f = 1 , \tag{5.16}$$

or equivalently, provided we absorb f into the $R(\alpha)$ and allow $L(\alpha) = [fR(\alpha)]^{-1}$ to be identified with the experimental quantities χ, η, and ζ as in Eqs. (2.40)–(2.42). The VP does not determine χ, η, and ζ of course, and from the point of view of the VP we might as well absorb f into these undetermined quantities.

There is, however, by our procedure an equation for $\alpha = 0$ as well, since the variations $\delta Q_s{}^r$, if perfectly arbitrary, must have arbitrary contributions projected along the parallel-parallel domain. This leads to an $\alpha = 0$ equation

$$R(0) Q_b{}^a(0) = X^a{}_b(0) . \tag{5.17}$$

This equation would seem to have no physical significance. The right-hand side is not, however, zero since if $X^a{}_b$ is defined as in Eq. (2.25), there is a parallel-parallel component

$$X^a{}_b(0) = -T^{-2} T_{,k} u^k u^a u_b / c^2 . \tag{5.18}$$

We recall that in Sec. IV there was also a problem with the parallel-parallel part of $Q_i{}^k$. In Eq. (4.19) what appeared was ϵ^0, which contained the Q contribution written in Eq. (4.20). To regain the usual equations we would have to force this part of Q to be zero. However, in Sec. IV there was no procedure for determining the Q's. In the present section we have such a procedure and in fact, the $\alpha = 1, 1', 2$, and 3 equations resulting from (5.15) give the usual forms for the $Q_a{}^b(\alpha)$ in terms of the $L(\alpha)$ or $R(\alpha)$. These parameters are determined either by experiment or by a theory of the microscopic structure of the material.

In the same vein we expect Eq. (5.17), or its inverse

$$Q_b{}^a(0) = L(0) X^a{}_b(0) , \tag{5.19}$$

to give us an expression for $Q_b{}^a(0)$ in terms of the parameter $L(0)$, obtained from another theory. The VP does not tell us the various $L(\alpha)$. The resolution of the problem is therefore that $L(0)$ must be chosen equal to zero

$$L(0) = 0 , \tag{5.20}$$

when u^i is the local four-velocity of the fluid under consideration. We must go beyond the VP to make this statement, just as we must go beyond the VP to determine the $L(\alpha)$, $\alpha \neq 0$. With it, however, we get back what we know must be the case: The Q must refer to dissipative processes only. And, of course, it will eliminate the second term of Eq. (4.20).

With $Q_i{}^k$ and f determined, we can return to Eq. (5.14) and argue that since the right-hand side is proportional to λ, it will eventually drop out of the equations by the procedure described at the end of Sec. IV.[27] It is interesting to note that the dissipation term in (5.12) is, in fact, just another accumulation term. However, it has an important

effect in determining the Q's. In this way, with the postulate in Eq. (5.11) and the "experimental" determination of $L(0)$ in Eq. (5.20), the Onsager type of VP that produces the linear equations connecting fluxes and forces is incorporated into (at the same time tying together a loose end of) the action principle of the previous section.

VI. INCLUSION OF ELECTROMAGNETISM

The VP of the preceding sections can be generalized to include the equations for an em field, and for the interaction between this field and the fluid if it is charged. The manner for doing this is standard. One adds to the I of (3.1) $I_{\rm em}$, where

$$I_{\rm em} = 2\kappa \int (-F_{ik}F^{ik}/16\pi + J^i A_i/c)(-g)^{1/2} d^4x , \tag{6.1}$$

where A_i is the electromagnetic four-potential, F_{ik} the field tensor,

$$F_{ik} = A_{k,i} - A_{i,k} , \tag{6.2}$$

and J^i is the charge flow of the particles:

$$J^i = qnu^i , \tag{6.3}$$

where q is the charge on the individual particle.

The VP is then that $I + I_{\rm em}$ is to be an extremal, subject to the constraints (3.3), (3.4), and (4.1). There is the new dynamical variable A_k which gets varied, so (3.13) becomes

$$\delta(I' + I_{\rm em}) = \int (\overline{A}^{ik}\delta g_{ik} + \overline{B}\delta n + \overline{C}_i \delta u^i + D\delta s_a + E^k \delta A_k)(-g)^{1/2} d^4x = 0 , \tag{6.4}$$

where

$$\overline{A}^{ik} \equiv (4.8) - (\kappa/4\pi)(F^i{}_m F^{km} - \tfrac{1}{4} g^{ik} F_{rm} F^{rm}) + (\kappa/c) g^{ik} J^m A_m = 0 , \tag{6.5}$$

$$\overline{B} \equiv (4.9) + (2\kappa/c) q A_k u^k = 0 , \tag{6.6}$$

$$\overline{C}_i \equiv (4.10) + (2\kappa/c) nq A_i = 0 , \tag{6.7}$$

$$D \equiv (4.11) = 0 , \tag{6.8}$$

$$E^k \equiv -(\kappa/2\pi)[F^{km}{}_{;m} - (4\pi/c) J^k] = 0 . \tag{6.9}$$

Equation (6.8) yields again Eqs. (4.14) and (4.15), and Eq. (3.19) yields again (4.16), there being no electromagnetic effect in this quantity. When (6.5) is used to get the Einstein equations, it turns out that the last term of (6.5) is canceled by a term coming from the $\phi_{,m} u^m g^{ik}$ term of (4.8) [which of course appears in (6.5)] when $\phi_{,m} u^m$ is obtained from Eq. (6.6). Thus, G^{ik} ends up as what (4.17) has plus the usual em term:

$$G^{ik} = \kappa\left[nh_a u^i u^k/c^2 - p g^{ik} + Q^{ik} + \frac{1}{4\pi}(F^i{}_m F^{km} - \tfrac{1}{4} g^{ik} F_{mr} F^{mr}) \right] . \tag{6.10}$$

The equations of motion are obtained from (3.22) using (6.6) and (6.7). The result after a tedious calculation is to give the usual em correction:

$$nh_a u^m u_{i;m} / {}^2 - \Delta_i{}^m p_{,m} + \Delta_i{}^m Q_m{}^k{}_{;k} + J^k F^i{}_k/c = 0 . \tag{6.11}$$

The last term may be written in the same form as the previous two $\Delta^i{}_m J^k F^m{}_k/c$, since its projection onto u_i is zero.

Finally Eq. (6.9) gives the Maxwell equation

$$F^{km}{}_{;m} - (4\pi/c) J^k = 0 . \tag{6.12}$$

Thus, all the electromagnetic effects are absorbed into the VP in the usual way.

VII. REVIEW

In this paper, the variational principle for a perfect fluid in general relativity is reformulated in a way that constitutes the natural relativistic generalization of the Herivel classical hydrodynamical principle. The explicit use of the Lin auxiliary condition proves to be unnecessary in this formulation. There is an entropy constraint which is the generalization of the classical one that energy is to be a minimum for no change in entropy. The statement is that entropy production is to be zero.

This VP for a perfect fluid is then extended to include imperfect fluids exhibiting dissipative processes by allowing the entropy production to have the form in Eq. (4.1). When this is done, the Einstein and Navier-Stokes equations containing these processes are produced independently, as was the case for a perfect fluid, but there are a number of unwanted terms (Z^{ik} and Z_i) in both equations. These unwanted terms may first of all be of higher order in the gradients of velocity, density, and temperature, and can perhaps be neglected for this reason.

However, these terms have the physical significance of representing directly the accumulation of entropy, heat, etc., via the dissipative processes between an arbitrary initial time τ_0 and the time τ of observation of the system. In this sense they then introduce a non-Markoffian character into the equations of motion and the Einstein equations. If such a character is to be eliminated, then all such "accumulation terms" must be eliminated.

The mechanism for doing this is to limit the time interval during which the accumulation takes place, in fact, to set it equal to zero. Thus, the

action integral is reformulated to refer to a narrow proper-time slice τ_1, τ_2 and then, after the variational equations have been substituted back into the constraints to obtain all the desired equations, the two times are allowed to collapse to some arbitrary time τ'. The resulting equations refer then to an arbitrary time, but the accumulation terms have collapsed to zero by this mechanism. Thus by modifying the time range of the action integral to be of differential size, we can introduce dissipative processes into a modified action principle.

By the same token, the Onsager type of VP which produces the linear flow equations can be incorporated into the principle by defining a transport operator or dyadic [Eq. (5.9)] which generalizes all the transport coefficients into a single geometrical object which does not undergo variations. Then a generalized dissipation function [Eq. (5.10)] is added to the action integrand with an appropriate multiplier, and variations are taken of the flux tensor Q_i^k appearing in the entropy production and dissipation terms. This is described in Sec. V. The parameters in the transport dyadic must be obtained from experiment or from a microscopic theory outside of the VP. One of them must be chosen to be zero in order to get the correct equations of motion. Electromagnetism can be taken into account in the standard way as described in Sec. VI.

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[1]J. W. Herivel, Proc. Cambridge Philos. Soc. 51, 344 (1955).

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[3]Lin's result was known by private communication soon after Herivel's work. It is referred to, for example, in Ref. 17 below.

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[5]V. Fock, *The Theory of Spacetime and Gravitation*, English translation (Pergamon, London, 1959). The first Russian edition appeared in 1955.

[6]H. Callen, *Thermodynamics* (Wiley, New York, 1960). However, free energies can also be minimized subject to their own constraints. The Helmholtz free energy is a minimum if the temperature is constant, for example, see Callen, p. 105.

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[17]That a modification to or addition of constraints on the Herivel approach nonrelativistically might lead to a way of taking into account viscosity was suggested by J. Serrin, in *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1959), Vol. VIII, Part 1, p. 149.

[18]C. Eckart, Phys. Rev. 58, 919 (1940); 58, 267 (1940); 58, 269 (1940). Eckart also wrote a paper on the Herivel-Lin VP in Phys. Fluids 3, 421 (1960).

[19]I. Prigogine, Physica (Utrecht) 15, 272 (1949).

[20]The argument here follows the discussion of Ref. 2, p. 148 more or less, altered to suit the relativistic formulation.

[21]Reference 6, p. 288.

[22]Reference 12, p. 165.

[23]B. H. Lavenda, *Thermodynamics of Irreversible Processes* (Wiley, New York, 1978).

[24]Reference 23, p. 98. This style of VP is quite common. See, for example, M. Kohler, Ann. Phys. (Leipzip) 124, 772 (1948), or A. H. Wilson, *Theory of Metals* (Cambridge University Press, New York, 1953), p. 300.

[25]For the quantum-mechanical version, see A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1961), Vol. I, Eq. (VII. 52).

[26]Incorporating Φ into the action is not a new idea. See, for example, M. Miche, J. de Math. 28, 151 (1949); R. Gerber, Ann. Inst. Fourier 1, 157 (1950); J. Math Pure Appl. 32, 79 (1950).

[27]By the same token, one could conclude that every term in the variational equations of (5.15) goes to zero in the same limit. Thus, one could argue that there are two branches of solutions in the limit: Eqs. (2.34) and the equation $0=0$.