Qualitative analysis of homogeneous universes

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(Received 29 October 1979; revised manuscript received 23 January 1980)

We investigate the qualitative behavior of cosmological models in the case of spatially homogeneous and isotropic universes containing viscous fluids in a Stokesian nonlinear regime.

I. INTRODUCTION

Recently the method of investigating qualitatively systems of differential equations which describe certain special configurations of the gravitational field has attracted the attention of many authors.[1-8] The interest of such a method is twofold: First, it gives a very good picture of the general behavior of distinct solutions of a given set of differential equations, and second, it helps us in pointing in the direction in which the search for specific solutions should be undertaken.

It seems worthwhile to call attention to the fact that such a qualitative analysis can be effectively made only in some restricted and very special circumstances, e.g., in the case where the system of differential equations is reducible to an autonomous form of the type

$$\dot{x} = F(x, y),$$
$$\dot{y} = G(x, y),$$

where a dot represents a derivative with respect to a parameter, say the time t. The right-hand functions F and G are not explicit functions of the time coordinate but may be any linear or nonlinear function of the variables x and y. Astonishingly enough, Einstein's set of gravity equations can be reduced to such a planar autonomous system in some cases of real interest, for instance, for spatially homogeneous universes.

In the present work we will use such a method to investigate the configuration of homogeneous and isotropic universes filled with a nonlinear Stokesian fluid.

The influence of viscous phenomena in cosmology has been examined by many authors[1,2,7,9] as a model of the cosmological fluid at the drastic regions near the singularity. Hitherto such a viscous fluid has been treated only in the Cauchy linear case. One adds to the isotropic pressure p a term proportional to the expansion factor (bulk viscosity) or one introduces an anisotropic stress π_{ij} linearly related to the shear σ_{ij}. The main reason for considering, as we do in the present work, a more general nonlinear dependence of the pressure on the expansion rests on quantum effects.

Indeed, it has been suggested by many authors that the introduction of viscosity in the cosmological fluid is nothing but a phenomenological description of the effect of creation of particles by the nonstationary gravitational field of the expanding cosmos. In Ref. 10 it is shown that the quantum corrections of the macroscopic stress-energy tensor can be described by a polynomial function of the expansion factor θ.

The presence of viscosity, through such a polynomial dependence on θ, changes radically the features of the Universe. For instance, Murphy[11] has recently given a simple analytical model in which viscosity is even used to prevent a singularity region from occurring. We remark that this is in no way in contradiction with the singularity theorems once the hypotheses required by these theorems are not fulfilled by the viscous fluid.

In Sec. II we present the main equations of the gravitational field for a viscous fluid in a nonlinear Stokesian regime in an isotropic and homogeneous expanding universe. Thus the modification introduced by viscosity can appear only as a change in the isotropic pressure p to $\tilde{p} = p$ + polynomial in θ. We analyze the specific case of a quadratic regime $\tilde{p} = p + \alpha\theta + \beta\theta^2$. In Sec. II we limit α and β to be constants. We associate such a situation to the stationary case of a constant injection of new particles in the universe inducing the viscous phenomena in a steady-state regime. We then make some remarks in the general case of a more complicated polynomial dependence of pressure on θ.

In Sec. III we investigate the nonstationary regime and allow for a nonconstant quadratic coefficient β. Actually such a β can depend only on the energy, and we analyze a specific power-law dependence. We compare our results with the linear case which has been examined previously. We end with Sec. IV in which some general comments are made.

II. STEADY-STATE REGIME OF VISCOUS FLUID

We start by considering a homogeneous and isotropic cosmological model. The fundamental length, in a comoving system of coordinates in which the field velocity is

$$V^\alpha = \delta_0^\alpha ,$$

assumes the form

$$ds^2 = dt^2 - R^2(t)[dx^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta \, d\phi^2)] \tag{1}$$

in which $\sigma(\chi)$ may be χ, $\sin\chi$, or $\sinh\chi$, depending on whether the value of the three-curvature K is 0, +1, or −1, respectively.

Raychaudhuri's equation of the evolution of the expansion factor

$$\theta = \frac{1}{\sqrt{-g}} (\sqrt{-g}\, V^\alpha)_{|\alpha}$$

is

$$\dot{\theta} + \tfrac{1}{3}\theta^2 + \tfrac{1}{2}\rho + \tfrac{3}{2}\tilde{p} = 0 . \tag{2}$$

The total pressure $\tilde{p}$ accounts for the isotropic pressure p plus viscous terms, which we will represent as a polynomial in θ:

$$\tilde{p} = p - \sum_{k=1}^{N} \alpha_k \theta^k . \tag{3}$$

In this section we will limit the α's to be constants. This should be interpreted as a steady-state regime of permanent injection of new particles in the universe, following the suggestion of some authors who try to link viscosity to the mechanism of particle creation. Indeed, it has been shown by Vereslikov *et al.*[9] that all quantities which appear in the expression of the energy-momentum tensor of the particles created by the nonstationary gravitational field have a geometrical origin, that is, can be expanded in a series on some geometrical parameter, for instance, the expansion factor.

From the conservation of energy we obtain

$$\dot{\rho} + (\rho + p)\theta = 0 . \tag{4}$$

Equations (2) and (4) together with definition (3) constitute precisely a planar autonomous system. This very simple fact seems to have been noted for the first time only recently by Belinski and Khalatnikov.[1] In order for this system to become equivalent to Einstein's equations we have to add the constraint condition

$$\rho - \frac{\theta^2}{3} - \frac{3K}{R^2} = 0 . \tag{5}$$

The main consequence of the reduction of Einstein's equations to an autonomous planar system is the possibility of submitting such a system to a qualitative investigation of the behavior of the whole set of solutions without a complete knowledge of the analytical expression of a particular solution. This introduces a great simplification and allows an investigation of properties such as the behavior of solutions near singular points and the stability, which could hardly be done by other means.

Belinski and Khalatnikov[1] examined qualitatively such a system in the case where the viscous term is a linear function of the expansion. For the quadratic dependence, which will be the case discussed here, there are modifications of the phase plane (θ, ρ) corresponding to the behavior of the universe which is not allowed to occur in the linear case.

We write equations (2) and (4) in the form

$$\dot{\theta} = P(\theta, \rho) , \tag{2'}$$

$$\dot{\rho} = L(\theta, \rho) , \tag{4'}$$

and set $\tilde{p} = p - \alpha\theta - \beta\theta^2$. We obtain

$$P(\theta, \rho) = -\tfrac{1}{2}\rho - \tfrac{3}{2}p - \tfrac{1}{3}\theta^2 + \tfrac{3}{2}\alpha\theta + \tfrac{3}{2}\beta\theta^2 , \tag{6a}$$

$$L(\theta, \rho) = -(\rho + p)\theta + \alpha\theta^2 + \beta\theta^3 . \tag{6b}$$

The singular points of the system are given by those values of θ_0 and ρ_0, in the phase plane, which annihilate simultaneously the right-hand sides of Eqs. (2′) and (4′). We see immediately that there are only two singular points: (i) the origin O (0, 0) and (ii) the point

$$B(\theta_0, \rho_0) = \left(\frac{-3\alpha}{3\beta - \gamma} , \frac{\theta_0^{\,2}}{3} \right) ,$$

in which we have assumed the equation of state $p = (\gamma - 1)\rho$ with $1 \leq \gamma \leq 2$. Then we examine the behavior of the functions $P(\theta, \rho)$ and $L(\theta, \rho)$ in the neighborhood of the singular points.

The important elements of the analysis (see the book by Andronov *et al.*[12] for a systematic treatment of the qualitative analysis) are given by the value of the determinant of the linear part of the expansion Δ and the trace σ of the matrix $\hat{\Delta}$:

$$\hat{\Delta}_0 = \begin{bmatrix} \dfrac{\partial P}{\partial \theta} & \dfrac{\partial P}{\partial \rho} \\[2ex] \dfrac{\partial L}{\partial \theta} & \dfrac{\partial L}{\partial \rho} \end{bmatrix} \quad \text{(at the singular point)} .$$

At the point B, we have

$$\Delta_B = \frac{3\alpha^2}{\gamma - 3\beta},$$

$$\sigma_B = \frac{\alpha}{3\beta - \gamma}\left(2 + \tfrac{3}{2}\gamma - \tfrac{9}{2}\beta\right).$$

Following Andronov *et al.*[12] we conclude that B is a saddle point for the system if $\gamma < 3\beta$, and it is a two-tangent node if $\gamma > 3\beta$. If $\beta = \gamma/3 - \frac{4}{9}$ then B is a one-tangent node. The stability of the solution near the node can be determined by simple inspection of the sign of the trace. For $\theta_0 > 0$, that is, $\gamma > 3\beta$, the node is stable; for $\theta_0 < 0$, the node is unstable. (We assume α and β to be positive constants.)

The characteristic roots, which are the eigenvalues of matrix $\hat{\Delta}_B$, take the values $\lambda_1 = \frac{1}{2}\theta_0(3\beta - \gamma)$ and $\lambda_2 = -\frac{2}{3}\theta_0$. The investigation of the behavior of the solution for $t \to \pm\infty$ can be easily made in both cases (see Figs. 1–4).

The examination of the integral curves near the origin O is somewhat more complicated due to the fact that O is a nonelementary singular point (that is, the corresponding determinant Δ_0 vanishes identically at the origin). We will not give here all the long and tedious calculations that constitute the analysis of the system at this point. Instead, we will present only the final results (see Figs. 1–4).

Although it is not our purpose here to extend the analysis to higher than the quadratic dependence of the pressure on the expansion factor, let us make some comments for the case of a higher power. We set

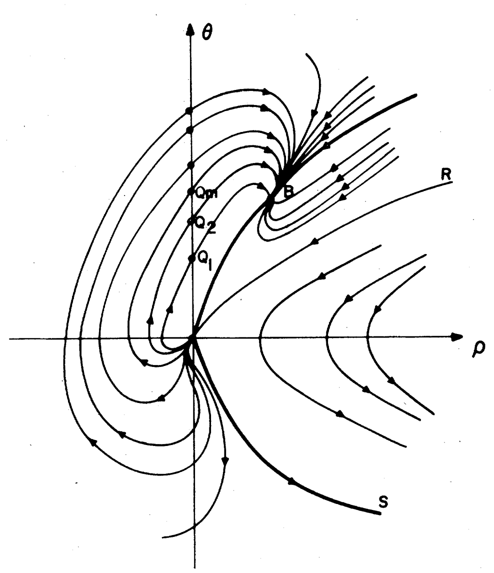

FIG. 1. Case in which α and β are constants. Point B is a two-tangent node. The curve is drawn for $\gamma/3 - \frac{4}{9} < \beta < \gamma/3$.

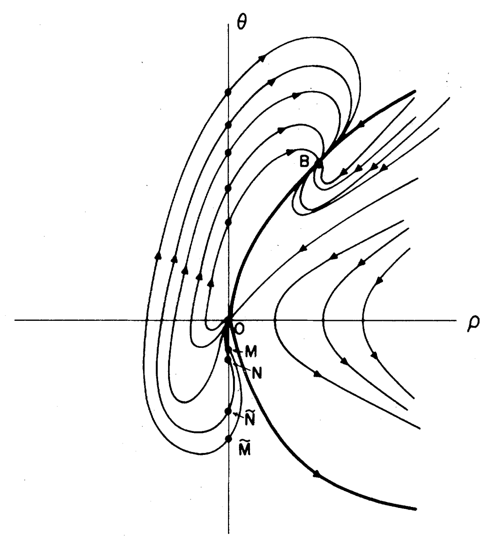

FIG. 2. α and β are constants. Point B is a one-tangent node. $\beta = \gamma/3 - \frac{4}{9}$.

$$\tilde{p} = p - \psi\theta^n$$

for $n > 0$ and $\psi \neq 0$. The singular point of the system, besides the origin, is given by the simultaneous solution of the equations

$$\rho_0 = \frac{\psi}{\gamma}\theta_0{}^n,$$

$$\theta_0{}^{n-2} = \frac{\gamma}{3\psi}.$$

The determinant of the linearized matrix $\hat{\Delta}$

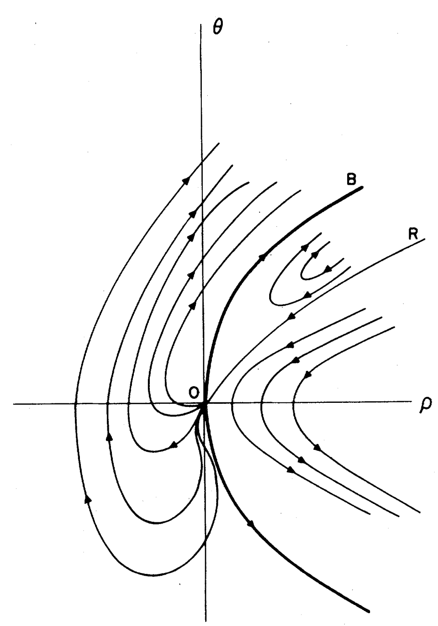

FIG. 3. α and β are constants. For the case in which $\beta = \gamma/3$ there is only one singular point at the origin **O**.

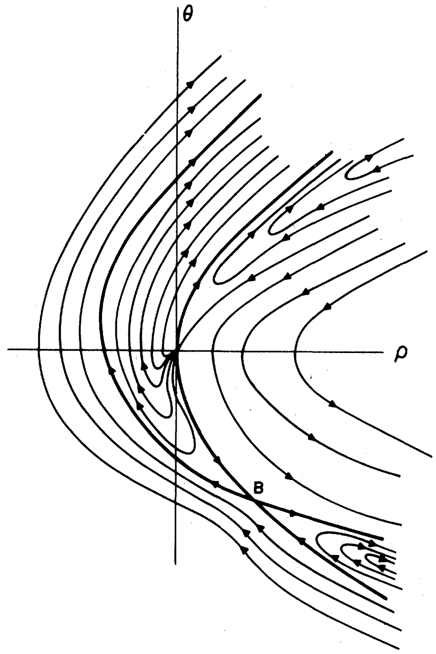

FIG. 4. α and β are constants. Point B is a saddle point. This case occurs for $\beta > \gamma/3$.

near the point $B(\theta_0, \rho_0)$ reduces to

$$\Delta_M = \gamma\theta_0^2\tfrac{1}{3}(2-n)\,.$$

We conclude that for any $n > 2$ point B is a saddle point for the autonomous system. For $n = 2$, there is no singularity other than the origin, unless the coefficients of viscosity ψ and γ are related by the expression $\gamma = 3\psi$. Finally, for $n = 1$, B is a node. For the analysis of the origin all the features are very similar to the quadratic case.

Let us make some comments on Figs. 1–4. We start by noting that we have drawn the integral curves in the whole domain of ρ, even for $\rho < 0$, although a universe filled with negative total energy is devoid of physical meaning, at least classically. We remark that the parabola BOS in Fig. 1 (and the corresponding parabolas in the subsequent figures) divides the positive sector ($\rho > 0$) into three regions, each of which is characterized by the corresponding value of the constant K of Eq. (5): Within this parabola we have $K = 1$ which corresponds to the closed models; outside it we have the open models for $K = -1$.

The character of the singular point B (node or saddle) depends on the sign of θ. Such a situation does not occur in the linear case, since in that case the singularity can be located only in the first quadrant ($\theta > 0$). Further, in the steady-state linear case, the singular point B can be only a node. This makes a great difference between the linear case and the quadratic one.

Let us comment on Fig. 1. For the constant $K = -1$ we can distinguish two general behavior patterns.

(i) The universe starts at $t = -\infty$ with an infinite radius and negative θ. The universe contracts from this dilute phase with zero matter energy. Then as the universe contracts a negative energy starts to appear. Its absolute value increases until the contraction attains a minimum. Then the contraction begins decelerating and after a while it changes its sign and θ becomes positive. The energy remains negative. Now the universe expands and after a certain (finite) time enters a region of positive energy which increases with the expansion. Finally, after a maximum of θ is attained a deceleration of θ occurs (although the sign of θ does not change) until the solution enters the singularity B, in which once again the radius of the universe becomes infinity.

For an observer who sees only the classical positive region, the universe starts with positive expansion $q_1, q_2, \ldots,$ or q_n and zero energy. Then it follows the path from q_1 to B in Fig. 1.

In all these models, which represent only part of the integral curves limited by the requirement of positivity of the energy, the universe starts abruptly with an arbitrary expansion θ_i and zero energy and ends at B.

(ii) The universe starts in the same condition as in case (i). However, after a certain (finite) time the negative energy decreases until it attains once again the value zero, after which it becomes positive. The energy increases and the contraction of the universe accelerates. After a certain finite time the energy attains a maximum and begins to decrease until it vanishes. The curve enters a region of negative energy, after which the behavior of such a universe follows the same lines as in the previous case (i). The universe has a classical meaning ($\rho > 0$) in the region $M\tilde{M}$, $N\tilde{N}$, and so on. In a typical behavior: it starts at $(\theta, \rho) = (M, 0)$ and ends after a finite lapse of time, at $(\tilde{M}, 0)$. During this period of time it experiences no singularity at all.

Let us turn now to the case in which the constant $K = +1$, that is, the region inside the parabola BOS in Fig. 1. The separatrix OR divides into two regions. The region ROB contains solutions in which the universe starts with an infinite expansion and infinite density. Then the expansion θ decreases, the energy decreases until a minimum, after which it increases again and finally ends at the singularity B. The region ROS contains solutions with the same behavior as the closed Friedmann model.

Finally, for $K = 0$ we can have three solutions corresponding to the regions OB, BL, and OS, the interpretations of which are evident.

Figure 2 does not present any new features. Figure 3 has a similar behavior at negative values of θ, but a different feature for $\theta > 0$. This is due to the absence of the singular point B. Thus, all curves which end at B in graphs 1 and 2 now go to infinity. There is a region (BOR) with a saddle-point behavior (actually, the origin is a saddle node). This region represents universes which start with $(\theta,\rho)=(\infty,\infty)$, the expansion decreases together with the density, attains a minimum, and starts increasing again without limit.

Finally, Fig. 4 contains a combination of these previous models. Point B, which appears in the contracting region ($\theta<0$) is a saddle point. Two particular classes of models, which seem worthy of mention, are described below.

(a) A model which starts with an infinite density and infinite contraction ($\rho=-\theta=\infty$). As time goes on the density diminishes, attains a minimum, and then starts again increasing without limit. During this entire period the universe keeps contracting.

(b) A model which starts with $\rho=+\infty$ and $\theta=-\infty$. The density diminishes (and also the contraction diminishes) until it arrives at the value $\rho=0$. If we continue to follow this curve, we enter the region of negative energy (with negative θ). Beyond this region the universe emerges, for positive values of θ, with positive energy. Classically these two regions are disconnected, the model ends at $\rho=0$ for negative θ, and it starts with $\rho=0$ and positive θ. However, if we allow for negative values of the energy we see that there is a continuation from the model which goes from ($\theta=-\infty$, $\rho=\infty$), passes a negative-energy region, and ends at ($\theta=+\infty$, $\rho=+\infty$).

III. QUADRATIC REGIME OF VISCOUS FLUID

Nonstationary case

Let us discuss a more realistic model of the viscous fluid by allowing the coefficients α and β to become functions of the total energy ρ. In order to examine the effects of the quadratic dependence without contamination of the linear factor, we set $\alpha=0$. Assuming a power-law dependence $\beta=M\rho^{\mu}$ (M and μ are constants) as in Ref. 1, we write

$$P(\theta,\rho)=\rho-\tfrac{1}{3}\theta^2+\tfrac{3}{2}M\rho^{\mu}\theta^2-\tfrac{3}{2}\gamma\rho\,,$$

$$L(\theta,\rho)=M\theta^3\rho^{\mu}-\gamma\rho\theta\,.$$

The singular points now (besides the origin) are doubled, appearing symmetrically with respect to an inversion of θ. We will call these symmetric singular points $B_{(+)}$ (for $\theta>0$) and $B_{(-)}$ (for $\theta<0$). They are given by the conditions

$$\rho_0=\tfrac{1}{3}\theta_0^{\,2}\,,$$

$$3M\rho_0^{\,\mu}=\gamma\,.$$

Developing $P(\theta,\rho)$ and $L(\theta,\rho)$ in the neighborhood of these points we obtain

$$\begin{pmatrix}P(\theta,\rho)\\ L(\theta,\rho)\end{pmatrix}\simeq\begin{pmatrix}-\tfrac{3}{2}\theta_0+\gamma\theta_0 & 1+\tfrac{3}{2}\gamma(\mu-1)\\ \tfrac{2}{3}\gamma\theta_0^{\,2} & \gamma\theta_0(\mu-1)\end{pmatrix}\begin{pmatrix}\theta\\ \rho\end{pmatrix}$$

$$+\text{higher powers of }\theta,\rho\,.$$

Thus, the determinant Δ of the linear part is given by

$$\Delta=-\tfrac{2}{3}\gamma\mu\theta_0^{\,2}$$

and its trace σ by

$$\sigma=\theta_0(\gamma\mu-\tfrac{2}{3})\,.$$

Thus, we obtain the following results: If $\mu>0$ then points $B_{(\pm)}$ are saddle points; if $\mu<0$ and $4\Delta-\sigma^2<0$, points $B_{(\pm)}$ are two-tangent nodes; and if $\mu=-2/3\gamma$, points $B_{(\pm)}$ are one-tangent nodes. Furthermore, if $\theta_0>0$ the node is stable and if $\theta_0<0$ the node is unstable.

Let us make some comments on Figs. 5–8. For $K=-1$, there are models which start at $B^{(-)}$ and end at $\tilde{Q}_i$. They represent universes which start with infinite radius and finite energy. These models contract until the matter energy becomes zero at $\tilde{Q}_i$.

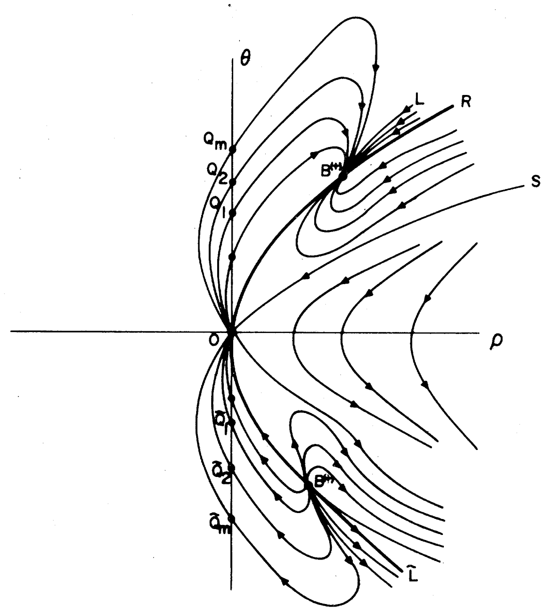

FIG. 5. $\tilde{p}=p-M\rho^{\mu}\theta^2$; M and μ are constants. The figure is drawn for the case in which $-2/3\gamma<\mu<0$. $B_{(\pm)}$ are two-tangent nodes.

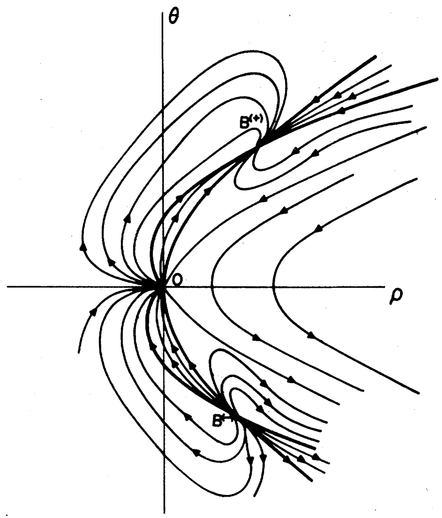

FIG. 6. $\tilde{p} = p - M\rho^{\mu}\theta^2$; M and μ are constants. The figure shows the case in which $\mu < -2/3\gamma$. $B_{(\pm)}$ are two-tangent nodes.

If we follow these integral curves into the negative-energy region $\rho < 0$, then we see that the model runs into the singular point at the origin with zero expansion and zero energy. A symmetric situation occurs for curves going into $B_{(+)}$ with $K = -1$.

In Fig. 8 the elliptic sector characterizes the integral curves of our system for $\rho < 0$. These curves represent unphysical configurations of universes which start at $t = -\infty$ with zero expansion and zero matter energy. The universe has at its beginning an infinite radius and enters a decelerating era until it attains the epoch of

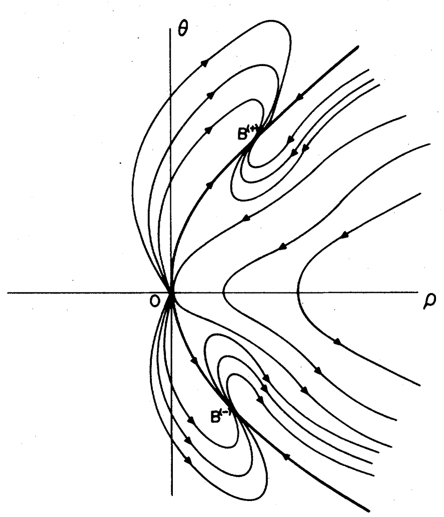

FIG. 7. $\tilde{p} = p - M\rho^{\mu}\theta^2$; M and μ are constants. The figure shows the case $\mu = -2/3\gamma$. $B_{(\pm)}$ are one-tangent nodes.

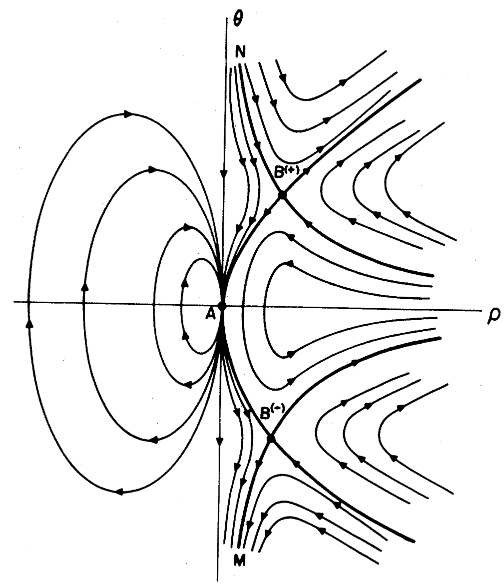

FIG. 8. Case $\tilde{p} = p - M\rho^{\mu}\theta^2$; M and μ are constants, with $\mu > 1$. Points $B_{(\pm)}$ are saddle points.

maximum contraction, after which the expansion becomes positive. By the middle of its life, it enters a region of expansion and keeps expanding (with increasing θ) until it arrives at a maximum value θ_{max}. After that, its expansion starts decelerating until it comes back to the original state ($\theta = \rho = 0$).

Let us now turn to the physical region ($\rho > 0$). The behavior of the integral curves for expanding universes in the quadratic viscous regime has almost the same features as in the linear case. The singular point $B_{(+)}$ is a saddle point which distinguishes four regions of distinct behavior:

region I, from $(\theta, \rho) = (+\infty, 0)$ to $(0, 0)$;

region II, from $(+\infty, 0)$ to $(+\infty, +\infty)$;

region III, from $(0, +\infty)$ to $(+\infty, +\infty)$;

region IV, from $(0, +\infty)$ to values of negative θ.

All these regions are equally presented in the linear case and have been discussed previously by Belinski and Khalatnikov.[1]

Let us turn to the case of negative θ. Here the situation changes drastically. The existence of a new singular point $B_{(-)}$ which turns out to be a saddle point (Fig. 8) introduces an infinite barrier represented by the separatrix $AB_{(-)}M$. Thus, contrary to the linear case in which any curve which passes through points near the origin of the ρ axis goes to $(-\infty, +\infty)$, in the quadratic case, due to the existence of the boundary $AB_{(-)}M$, these curves can only end with an infinite contraction and vanishing total energy. This represents universes which start with zero expansion, zero energy, and infinite radius ($R\to\infty$). After that, the energy increases, attains a maximum (near the saddle point $B_{(-)}$), and diminishes indefinitely. The curves from region IV, crossing the ρ axis, go just near $B_{(-)}$ and then are repelled by the saddle point. Such models represent a cosmos that starts from a highly condensed phase ($R\to 0$) with an infinite energy. Then as the universe expands (slowly) the energy decreases, until a minimum value $\rho_{\min}$ (different for each model). Beyond that point the sign of the function θ changes: The universe enters a contracting era and keeps contracting forever, increasing indefinitely the value of the matter energy.

V. CONCLUSION

The purpose of the present paper is to make use of the method of qualitative analysis of planar autonomous systems of differential equations, in order to investigate some spatially homogeneous cosmological models. We have discussed in Secs. II and III the case of the homogeneous and isotropic cosmos filled with a Stokesian fluid in a quadratic regime.

We have shown how the quadratic term can cause deviations from the usual models, in some cases very drastically. We have presented these new features in a self-explanatory series of graphs. There are some unusual behaviors which seem to demand an interpretation in the light of quantum theory. The investigation of these models should be a very interesting matter of research.

ACKNOWLEDGMENT

The work of R. A. A. was supported by a grant from Conselho National de Pesiquisa, Brazil.

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