Analytic properties of the vertex function in gauge theories. II

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The analytic properties of the three-gluon vertex function for quantum chromodynamics in covariant gauges are investigated. First, a general tensor form for the vertex consistent with the Ward identity and free of kinematic singularities is constructed. The vertex is then calculated to one-loop order in the Feynman gauge. The complete expression for the off-shell one-loop vertex is expressed in terms of elementary functions plus one nonelementary function, the dilogarithm. Various kinematic limits of the vertex are considered. The most interesting results are the following. (1) Gluon mass-shell singularities occur in the transverse terms as well as the longitudinal terms. (2) The leading IR singularity is in the longitudinal part of the vertex, as is the case for QED; however, it is a pole singularity rather than the usual logarithmic singularity.

I. INTRODUCTION

The idea that the infrared (IR) singularities of quantum chromodynamics (QCD) provide the mechanism for color confinement is by now widely accepted, in spite of the somewhat limited understanding of these singularities. In this paper we investigate, via perturbation theory, the singularities of the triple-gluon vertex. In addition to providing a complete expression for the one-loop off-shell vertex, which could be of use in other calculations, we hope to shed some light on the following questions. (1) How do the singularities of this non-Abelian gauge theory differ from those of massless scalar and spinor QED discussed in the preceding paper? (2) Are the IR singularities of the vertex confined to the longitudinal vertex terms which are related to the propagator and the ghost terms by the Ward identities? (3) Is a unique form for the vertex obtained when one requires that the vertex be free of kinematic singularities?

The outline of the paper is as follows. In Sec. II we construct the most general form for the triple-gluon vertex, which is free of kinematic singularities and automatically satisfies the Ward identity. Section III contains the perturbation results which are compared with other results for special kinematic limits. Section IV deals with the small-momentum limit of the vertex. Finally, in Sec. V we summarize our results on the infrared behavior of QCD and massless QED and discuss what differences exist.

II. GENERAL FORM OF THE THREE-GLUON VERTEX

As in the preceding paper, we will construct from first principles the general form of the three-gluon vertex in covariant gauges which satisfies the Ward identity. This result in a different notation was obtained earlier by Baker and Kim.[1] Let us begin this discussion by stating that the color dependence of the vertex is given by the structure constant of the color group under consideration [SU(3) for QCD]. The momentum and color variables are as shown in Fig. 1:

$$\Gamma^{abc}_{\mu_1\mu_2\mu_3}(P_1,P_2,P_3)=f^{abc}\Gamma_{\mu_1\mu_2\mu_3}(P_1,P_2,P_3)\,. \tag{2.1}$$

Since the color dependence is factorable, we will simply discuss the Γ that appears on the right-hand side of Eq. (2.1). Our task then is constructing the most general tensor forms consistent with Bose symmetry out of three Lorentz indices and two linearly independent four-vectors. Since f^{abc} changes sign under the exchange of any two color indices, Bose symmetry requires that Γ change sign under the interchange of any two μ's and the respective P's. This can be most easily achieved by constructing tensors which are odd under the interchange of P_1, $\mu_1 \leftrightarrow P_2$, μ_2 and then making Γ invariant under cyclic permutation. Since Γ must satisfy the Ward identity, it is convenient to construct basic tensors which are transverse, i.e., orthogonal to $P_{1\mu_1}$, $P_{2\mu_2}$, and $P_{3\mu_3}$. In terms of these quantities, the transverse part of the vertex is given in terms of 4 tensors as follows:

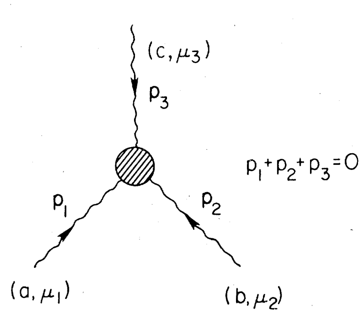

FIG. 1. The three-gluon vertex. a, b, and c are color indices. μ_1, μ_2, and μ_3 are Lorentz indices.

$$\Gamma^{(t)}_{\mu_1\mu_2\mu_3}(P_1,P_2,P_3)=F(P_1^2,P_2^2;P_3^2)(g_{\mu_1\mu_2}P_1\cdot P_2-P_{1\mu_2}P_{2\mu_1})B^3_{\mu_3}$$
$$+H[-g_{\mu_1\mu_2}B^3_{\mu_3}+\tfrac{1}{3}(P_{1\mu_3}P_{2\mu_1}P_{3\mu_2}-P_{1\mu_2}P_{2\mu_3}P_{3\mu_1})]+\text{cyclic permutations}, \quad (2.2)$$

where

$$B^3_{\mu_3}=(P_{1\mu_3}P_2\cdot P_3-P_{2\mu_3}P_1\cdot P_3)$$

and the scalar function F is symmetric in its first two arguments and H is totally symmetric in P_1^2, P_2^2, and P_3^2. The longitudinal part which contains the remaining 10 tensors can be written as

$$\Gamma^{(l)}_{\mu_1\mu_2\mu_3}(P_1,P_2,P_3)=A(P_1^2,P_2^2;P_3^2)g_{\mu_1\mu_2}(P_{1\mu_3}-P_{2\mu_3})+B(P_1^2,P_2^2;P_3^2)g_{\mu_1\mu_2}(P_{1\mu_3}+P_{2\mu_3})$$
$$+C(P_1^2,P_2^2;P_3^2)(P_{1\mu_2}P_{2\mu_1}-g_{\mu_1\mu_2}P_1\cdot P_2)(P_1-P_2)_{\mu_3}$$
$$+\frac{S}{3}(P_{1\mu_3}P_{2\mu_1}P_{3\mu_2}+P_{1\mu_2}P_{2\mu_3}P_{3\mu_1})+\text{cyclic permutations}. \quad (2.3)$$

Here the scalar functions A and C are symmetric in their first two arguments, while B is anitsymmetric, and S is antisymmetric under exchange of any pair of arguments. The complete vertex has now been expressed in terms of six independent scalar functions, four of which appear in the longitudinal vertex and hence can be determined via the Ward identity.

The Ward identity for the triple-gluon vertex in the covariant gauge is

$$P_2^{\mu_2}\Gamma_{\mu_1\mu_2\mu_3}(P_1,P_2,P_3)=-J(P_3^2)G(P_2^2)(g^{\mu_2}_{\mu_3}P_3^2-P_3^{\mu_2}P_{3\mu_3})\Gamma_{\mu_2\mu_1}(P_3,P_2;P_1)$$
$$+J(P_1^2)G(P_2^2)(g^{\mu_2}_{\mu_1}P_1^2-P_{1\mu_1}P_1^{\mu_2})\Gamma_{\mu_2\mu_3}(P_1,P_2;P_3), \quad (2.4)$$

where J and G are scalar functions appearing in the gluon and ghost propagators and the Γ's with two indices are related to the ghost-ghost-gluon vertices as shown in Fig. 2. The gluon propagator is the following:

$$-iD_{\mu\nu}(P)=-\frac{i}{P^2}\left[\left(g_{\mu\nu}-\frac{P_\mu P_\nu}{P^2}\right)\frac{1}{J(P^2)}+\alpha_0\frac{P_\mu P_\nu}{P^2}\right], \quad (2.5)$$

where α_0 is the gauge parameter. The ghost propagator is simply

$$\Delta(P)=i\,\frac{G(P^2)}{P^2}\,. \quad (2.6)$$

Before actually "solving" the Ward identity, let us examine the number of linear equations which this relation represents. The single Eq. (2.4) is a tensor equation with 2 Lorentz indices and two linearly independent four-vectors and hence is 5 linear equations for the coefficients of the independent tensor components. These equations plus cyclic permutations yield 15 linear equations. The left-hand side contains 10 scalar functions, the 3 independent orderings of the arguments for A, B, and C plus S. However, note that if the tensor indices on the right-hand side of Eq. (2.4) are contracted with $P_{1\mu_1}$ and $P_{3\mu_3}$, the result is zero, yielding one (actually 3 of the 15 total as this equation is invariant under cyclic permutation) equation relating the 10 scalars appearing in the longitudinal three-gluon vertex. To proceed further we must decompose the ghost-ghost-gluon vertex into its basic tensor forms as follows:

$$\Gamma_{\mu_2\mu_1}(P_3,P_2;P_1)=g_{\mu_1\mu_2}a(P_1,P_2,P_3)-P_{2\mu_1}P_{1\mu_2}b(P_1,P_2,P_3)$$
$$+P_{1\mu_1}P_{3\mu_2}c(P_1,P_2,P_3)+P_{1\mu_2}P_{3\mu_1}d(P_1,P_2,P_3)+P_{3\mu_1}P_{3\mu_2}e(P_1,P_2,P_3)\,. \quad (2.7)$$

With these definitions only the functions a, b, and d will contribute to the Ward identity.

Of the 12 remaining equations, 9 determine the 9 independent scalars of the three-gluon vertex and 3 are equations of constraint on the scalars in the ghost-ghost-gluon vertex. Before presenting the resulting expression it will be convenient to streamline our notation somewhat as follows:

$$a_{ijk}=G(P_j^2)J(P_k^2)a(P_i^2,P_j^2,P_k^2)\,,$$
$$b_{ijk}=G(P_j^2)J(P_k^2)b(P_i^2,P_j^2,P_k^2)\,, \quad (2.8)$$
$$d_{ijk}=G(P_j^2)J(P_k^2)d(P_i^2,P_j^2,P_k^2)\,.$$

In terms of these functions we obtain the following expression for the scalars in the three-gluon vertex:

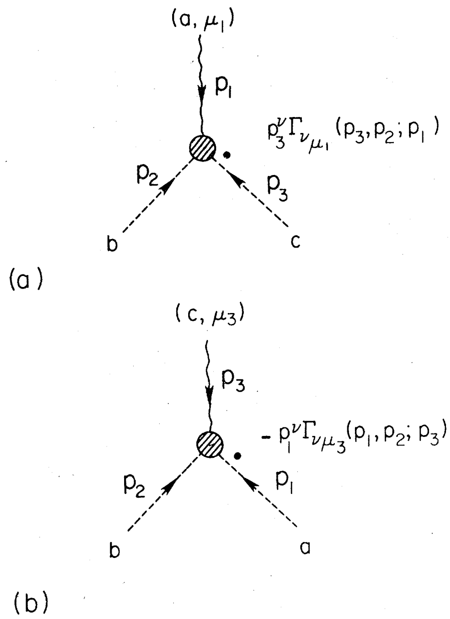

FIG. 2. The ghost-ghost-gluon vertex. The color dependence is given by f^{abc}.

$$A(P_1^2, P_2^2; P_3^2) = \tfrac{1}{4}[2(a_{312} + a_{321}) + P_3^2(b_{123} + b_{213}) + 2P_1 \cdot P_3 d_{321} + 2P_2 \cdot P_3 d_{312} + (P_1^2 - P_2^2)(b_{231} + b_{312} - b_{132} - b_{321})], \tag{2.9a}$$

$$B(P_1^2, P_2^2; P_3^2) = \tfrac{1}{4}[2(a_{321} - a_{312}) + 2P_1 \cdot P_3 d_{321} - 2P_2 \cdot P_3 d_{312} + P_3^2(b_{321} - b_{312} + b_{132} - b_{231}) - (P_1^2 - P_2^2)(b_{123} + b_{213})], \tag{2.9b}$$

$$C(P_1^2, P_2^2; P_3^2) = \frac{1}{P_1^2 - P_2^2}(a_{231} - a_{132} + P_2 \cdot P_3 d_{132} - P_1 \cdot P_3 d_{231}), \tag{2.9c}$$

and

$$S(P_1^2, P_2^2, P_3^2) = -\tfrac{1}{2}(b_{123} + b_{231} + b_{312} - b_{132} - b_{213} + b_{321}). \tag{2.9d}$$

The equations of constraint on the ghost-ghost-gluon vertex are the following equation and cyclic permutations of this equation:

$$a_{123} - a_{213} - P_1 \cdot P_2(b_{123} - b_{213}) + P_1 \cdot P_3 d_{123} - P_2 \cdot P_3 d_{213} = 0. \tag{2.10}$$

After obtaining the one-loop expression for the three-gluon vertex, we calculate the one-loop ghost-ghost-gluon vertex and the necessary self-energy functions to allow us explicitly to verify the relations given in Eqs. (2.9) and (2.10). This will provide a check on the longitudinal part of the three-gluon vertex.

III. PERTURBATION RESULTS

In these calculations, we will consider only the pure gluon theory, leaving out the contribution from quark loops. A further simplification is obtained by using the Feynman gauge. The Feynman diagrams that contribute to one-loop order are shown in Fig. 3. Our procedure, described in detail in the Appendix, is to reduce all tensor integrals to elementary functions and a single scalar integral

$$I_0 = \frac{2}{\pi^2 i} \int d^4k \frac{1}{k^2(k - P_1)^2(k + P_3)^2}. \tag{3.1}$$

This integral can be expressed in terms of the dilogarithm as follows:

$$I_0 = \frac{1}{\Delta}\left[\mathrm{Li}_2\left(-\frac{P_1 \cdot P_3 + \Delta}{P_3^2}\right) - \mathrm{Li}_2\left(-\frac{P_1 \cdot P_3 - \Delta}{P_3^2}\right) + \frac{1}{2}\ln\left(\frac{P_1 \cdot P_3 + \Delta}{P_1 \cdot P_3 - \Delta}\right)\ln\frac{P_2^2}{P_3^2}\right], \tag{3.2}$$

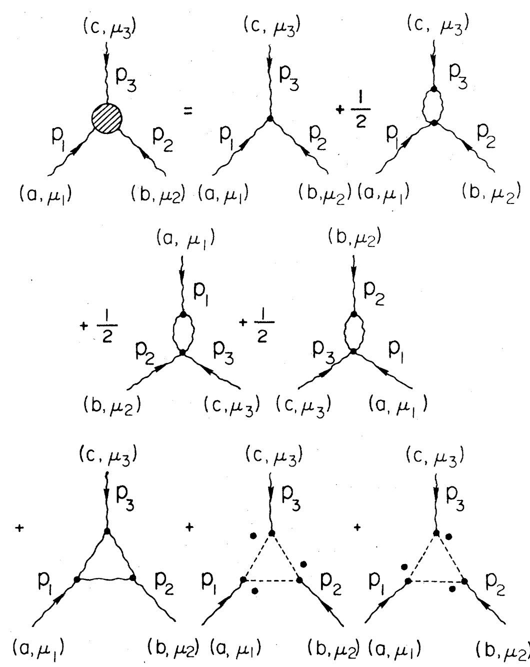

FIG. 3. The Feynman diagrams contributing to one-loop order of the three-gluon vertex.

where Δ is the triangle function

$$\Delta^2 = (P_1 \cdot P_2)^2 - P_1^2 P_2^2 = (P_1 \cdot P_3)^2 - P_1^2 P_3^2 = (P_2 \cdot P_3)^2 - P_2^2 P_3^2$$

and

$$\mathrm{Li}_2(x) = -\int_1^x \frac{\ln t}{t-1}\, dt$$

is the dilogarithm.

The calculation of the triple-gluon vertex was very tedious, in spite of extensive use of REDUCE for the algebraic manipulations. In this calculation dimensional regularization has been employed. Dimensional regularization proves superior to the ultraviolet-cutoff method in that gauge invariance is automatically preserved. Our procedure was the following. All algebraic manipulations and integrals were evaluated in n dimensions using the convention

$$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^nk}{(2\pi)^4}\,.$$

Different conventions for going to n dimensions such as $(2\pi)^4 \to (2\pi)^n$ can produce differing results for constant terms, but only in the function A which contains all of the UV-divergent terms. These constants are usually removed in the process of defining the renormalized quantities. This is not always the case when the minimal subtraction prescription is employed. We will note any remaining ambiguities as they appear.

The one-loop expressions for the scalar amplitudes of the three-gluon vertex are as follows:

$$A(P_1^2,P_2^2;P_3^2) = 1 - \frac{g_0^2 C_A}{64\pi^2}\left\{\frac{8}{3}\left(\frac{-2}{\epsilon} + \ln\frac{\mu^2}{-P_3^2} + C + \frac{1}{6}\right) - \frac{10}{3}\ln\frac{P_1^2P_2^2}{P_3^4} - 2P_1\cdot P_2 I_0 + \frac{P_3^2 + 3P_1\cdot P_2}{2\Delta^2}\left[(P_2^2 - P_1^2)\ln\frac{P_2^2}{P_1^2} + P_3^2\left(\ln\frac{P_1^2P_2^2}{P_3^4} + P_1\cdot P_2 I_0\right)\right]\right\}, \tag{3.3a}$$

$$B(P_1^2,P_2^2;P_3^2) = -\frac{g_0^2 C_A}{128\pi^2\Delta^2}\left\{\left[\frac{26\Delta^2}{3} + (P_1^2 - P_2^2)^2\right]\ln\frac{P_2^2}{P_1^2} + P_3^2(P_2^2 - P_1^2)\left(\ln\frac{P_1^2P_2^2}{P_3^4} + P_1\cdot P_2 I_0\right) + \Delta^2(P_1^2 - P_2^2) I_0\right\}, \tag{3.3b}$$

$$C(P_1^2,P_2^2;P_3^2) = -\frac{g_0^2 C_A}{64\pi^2\Delta^2}\left\{\left[\frac{26\Delta^2}{3(P_1^2 - P_2^2)} + \tfrac{3}{2}(P_2^2 - P_1^2)\right]\ln\frac{P_2^2}{P_1^2} + \frac{3P_3^2}{2}\left(\ln\frac{P_1^2P_2^2}{P_3^4} + P_1\cdot P_2 I_0\right)\right\}, \tag{3.3c}$$

$$F(P_1^2,P_2^2;P_3^2) = -\frac{g_0^2 C_A}{64\pi^2\Delta^2}\left\{(6\Delta^2 - 5P_3^2 P_1\cdot P_2)\frac{P_3^2}{3\Delta^2} I_0 + \frac{16}{3} + \frac{16P_1^2P_2^2}{\Delta^2} - \frac{20P_3^2 P_1\cdot P_2}{3\Delta^2} + \left[-3 + P_3^2\frac{(2P_3^2 + 9P_1^2 + 9P_2^2)}{3\Delta^2} + \frac{5P_1^2P_2^2P_3^4}{\Delta^4}\right]\left(\ln\frac{P_1^2P_2^2}{P_3^4} + P_1\cdot P_2 I_0\right) + \left[13(P_3^2 - 2P_1^2 - 2P_2^2) + \frac{(P_2^2 - P_1^2)^2}{\Delta^2}\left(7P_3^2 + 4P_1^2 + 4P_2^2 + \frac{15P_1^2P_2^2P_3^2}{\Delta^2}\right)\right]\frac{\ln(P_2^2/P_1^2)}{3(P_2^2 - P_1^2)}\right\}, \tag{3.3d}$$

$$H = -\frac{g_0^2 C_A}{64\pi^2}\left\{\left(\frac{5P_1^4P_2^4P_3^4}{3\Delta^6} + \frac{7}{2}\frac{P_1^4P_2^2P_3^2}{\Delta^4} + \frac{P_1^4}{\Delta^2} - \frac{4}{3}\right)I_0 + \frac{1}{9\Delta^2}\ln\frac{P_2^2P_3^2}{P_1^4}\left[\frac{30P_1^4P_2^2P_3^2 P_2\cdot P_3}{\Delta^4} + P_1^2 P_2\cdot P_3\left(\frac{P_1^2 + 21P_2^2 + 21P_3^2}{\Delta^2}\right) + 2(5P_1^2 - 3P_2^2 - 3P_3^2)\right] + \frac{20}{9}\left[\frac{(P_1\cdot P_2)(P_2\cdot P_3)(P_3\cdot P_1)}{\Delta^4}\right] + \text{cyclic permutations}\right\}, \tag{3.3e}$$

$$S = 0, \tag{3.3f}$$

where $\epsilon = n - 4$, μ is an arbitrary mass scale introduced so that the renormalized coupling constant remains dimensionless for all n, $C = -\gamma - \ln\pi$ (γ = Euler's constant), and C_A is the Casimir eigenvalue for the adjoint representation of the color group [$C_A = N$ for SU(N)]. The constant C is regularization dependent and will be different for other possible continuations from 4 to n dimensions.

The corresponding one-loop expressions for J and G are

$$J(P^2) = 1 - \frac{\mu^\epsilon g_0^2 C_A}{16\pi^2}\left\{\frac{5}{3}\left[-\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{-P^2}\right) + C\right] + \frac{1}{9}\right\}$$

and

$$G(P^2)=1+\frac{\mu^\epsilon g_0{}^2C_A}{32\pi^2}\left[-\frac{2}{\epsilon}+\ln\left(\frac{\mu^2}{-P^2}\right)+C\right].$$

To obtain the ghost-ghost-gluon vertex to one-loop order, we must evaluate the Feynman diagrams shown in Fig. 4. The resulting values of the scalar amplitudes are

$$a(P_1,P_2,P_3)=\mu^\epsilon\left\{1-\frac{g_0{}^2C_A}{64\pi^2}\left[-2\left(-\frac{2}{\epsilon}+\ln\frac{\mu^2}{-P_2{}^2}+C\right)+\ln\frac{P_1{}^2}{P_3{}^2}+\frac{2P_3{}^2-P_2{}^2}{2}I_0\right]\right\},$$

$$b(P_1,P_2,P_3)=-\frac{\mu^\epsilon g_0{}^2C_A}{64\pi^2\Delta^2}\left[P_3{}^2P_1\cdot P_2I_0+(P_1{}^2-P_2{}^2)\ln\frac{P_1{}^2}{P_2{}^2}+P_3{}^2\ln\frac{P_1{}^2P_2{}^2}{P_3{}^4}\right],$$

$$c(P_1,P_2,P_3)=-\frac{g_0C_A\mu^\epsilon}{128\pi^2\Delta^2}\left[(P_2{}^2-P_3{}^2)P_2\cdot P_3I_0+(3P_3{}^2-P_1{}^2+P_2{}^2)\ln\frac{P_2{}^2}{P_3{}^2}+2(P_2{}^2-P_3{}^2)\ln\frac{P_2{}^2}{P_1{}^2}\right],$$

$$d(P_1,P_2,P_3)=-b(P_1,P_2,P_3)+\frac{g_0C_A\mu^\epsilon}{64\pi^2\Delta^2}\left[-\frac{P_1\cdot P_3(2P_2{}^2+P_1{}^2-P_3{}^2)}{2}I_0+\frac{(4P_1{}^2-P_2{}^2)}{2}\ln\frac{P_3{}^2}{P_1{}^2}\right.$$
$$\left.+\left(\frac{P_3{}^2-P_1{}^2-2P_2{}^2}{2}\right)\ln\frac{P_1{}^2P_3{}^2}{P_2{}^4}\right],$$

$$e(P_1,P_2,P_3)=-\frac{g_0{}^2C_A\mu^\epsilon}{64\pi^2\Delta^2}\left[-(P_1{}^2P_2\cdot P_3)I_0+P_1{}^2\ln\frac{P_1{}^2}{P_3{}^2}+\left(\frac{P_2{}^2-P_3{}^2-P_1{}^2}{2}\right)\ln\frac{P_3{}^2}{P_2{}^2}\right].$$

The fact that these quantities, which are calculated independently, are consistent with the Ward identity Eqs. (2.8) and (2.9) provides a check on our expressions for the longitudinal part of the triple-gluon vertex.

At this time the most general published result on the one-loop three-gluon vertex is that of Celmaster and Gonsalves.[2] They obtained the equal leg limit ($P_1{}^2=P_2{}^2=P_3{}^2$) of the triple-gluon vertex for a general covariant gauge. When compared with the results of Celmaster and Gonsalves specialized to the Feynman gauge, we find our results are in complete agreement provided we change the value of C, as they used $(2\pi)^n$ in their dimensional regularization.

The other check on our results is the comparison with the work of Baker and Kim.[1] They used the Ward identity to obtain an expression for the longitudinal vertex from the one-loop calculations of the ghost-ghost-gluon vertex, and the gluon and ghost self-energies. The last term in their expressions Eq. (4.22) is in apparent disagreement with our results. However, this term actually vanishes when the cyclic permutations are included, due to the symmetry of their function I_A. When corrections are made for their use of a UV cutoff rather than dimensional regularization we find that we are in complete agreement.

The renormalization of the various quantities which we have calculated can now be easily performed. The results for various renormalization prescriptions and the resulting relationship between coupling constants and renormalization constants will be the subject of a forthcoming paper.

IV. THE SMALL-MOMENTUM LIMIT OF THE THREE-GLUON VERTEX

The explicit expression for the one-loop vertex can now be used to investigate what singularities occur for various kinematical limits. Proceeding as in the previous paper, we will first consider the limit where one gluon goes to the mass shell, for example $P_2{}^2\to 0$. In this limit, both the transverse vertex and the longitudinal vertex are logarithmically divergent. In this limit the divergent parts of the functions F and H are

$$F(P_1{}^2,P_2{}^2;P_3{}^2)\sim\frac{4\ln P_2{}^2}{P_1{}^2-P_3{}^2}\left\{\left[3-\frac{8P_1{}^2P_3{}^2}{(P_1{}^2-P_3{}^2)^2}\right]\frac{\ln(P_1{}^2/P_3{}^2)}{(P_1{}^2-P_3{}^2)^2}+\frac{4}{(P_1{}^2-P_2{}^2)}-\frac{13}{3}\frac{1}{P_3{}^2}+\frac{8P_3{}^2}{(P_1{}^2-P_3{}^2)^2}\right\},\tag{4.1a}$$

$$F(P_3{}^2,P_1{}^2;P_2{}^2)\sim\frac{12\ln P_2{}^2}{(P_1{}^2-P_3{}^2)^2}\left(2-\frac{P_1{}^2+P_3{}^2}{P_1{}^2-P_3{}^2}\ln\frac{P_1{}^2}{P_3{}^2}\right),\tag{4.1b}$$

and

$$H\sim\frac{8\ln P_2{}^2}{(P_1{}^2-P_3{}^2)^2}\left[P_1{}^2+P_3{}^2-\frac{2P_1{}^2P_3{}^2}{P_1{}^2-P_3{}^2}\ln\frac{P_1{}^2}{P_3{}^2}\right].\tag{4.1c}$$

In spite of the $P_1^2 - P_3^2$ denominators in the above expression, the limit $P_1^2 \to P_3^2$ (P_2^2 fixed) is finite and the vertex remains logarithmically singular as $P_2^2 \to 0$.

Ward identity in QED does not allow any photon mass-shell singularities in the longitudinal vertex. The behavior of the QCD longitudinal three-gluon vertex is different, both because of the symmetry requirement and because the Ward identity is less of a constraint. As a result, the longitudinal QCD vertex has gluon mass-shell singularities much like the meson mass-shell singularities in the massless scalar QED vertex.

Both parts of the vertex remain logarithmically divergent even in the equal off-shell limit. Finally, if we consider $P_{2\mu} \to 0$, we find that the vertex is finite in this limit. The remaining singularities of the vertex are all in the longitudinal vertex and take the following form:

$$\Gamma_{\mu_1\mu_2\mu_3} \to \mu^{\epsilon}\Big\{(g_{\mu_3\mu_2}P_{1\mu_1} + g_{\mu_1\mu_2}P_{1\mu_3} - 2g_{\mu_1\mu_3}P_{1\mu_2})\left[1 - \frac{g_0^2 C_A}{24\pi^2}\left(-\frac{2}{\epsilon} + \ln\frac{\mu^2}{-P_1^2} + \frac{1}{6}\right)\right]$$
$$- \frac{g_0^2 C_A}{16\pi^2}\,\frac{1}{6}\left(2g_{\mu_1\mu_3}P_{1\mu_2} + 3P_{1\mu_1}g_{\mu_2\mu_3} + 3P_{1\mu_3}g_{\mu_1\mu_2} - \frac{8P_{1\mu_1}P_{1\mu_2}P_{1\mu_3}}{P_1^2}\right)\Big\}. \qquad (4.2)$$

The logarithmic singularity in the first term in Eq. (4.3) is the usual IR singularity that appears in QED when one takes the photon momentum to zero and then the other legs to their respective mass-shell value. The surprising behavior of this limit is the pole singularity present in the last term. This type of singular term is common in vector particle propagators and is usually associated with the gauge freedom for these quantities. Such terms do not appear in vertex functions for QED, and hence are something peculiar to QCD. We note that in Euclidean space, where many calculations are actually performed, this particular tensor form is not singular.

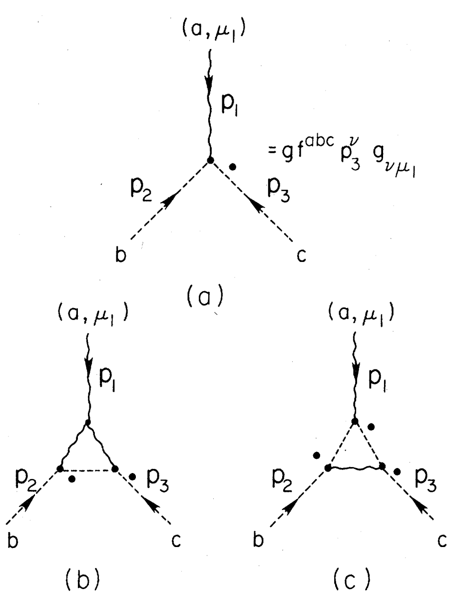

FIG. 4. The Feynman diagrams contributing to one-loop order of the ghost-ghost-gluon vertex.

V. CONCLUSIONS

In comparing the vertex singularities of QCD to those of massless QED, let us first consider the $q^2 \to 0$ limit, i.e., the gluon or photon mass-shell limit. In this limit, $\ln q^2$ singularities are present in the transverse part of the vertex. While there are some differences in detail, the behavior of all these gauge theories is quite similar, with singularities that do not survive in the $q_\mu \to 0$ limit or in Euclidean space.

In the conventional IR limit the QCD vertex has a pole singularity in the longitudinal part and would appear to be a much stronger singularity than the usual logarithmic behavior of QED. The significance of these singularities, which are present in Minkowski space but in fact disappear in Euclidean space, is not clear. We have not calculated a physical process to see how unitarity is preserved.

Finally, our results provide some support for the assumption that the $q_\mu \to 0$ singularities of the three-gluon vertex are all in the longitudinal terms. The fact that the transverse parts are singular at $q^2 = 0$ casts some doubt on the assumption that the longitudinal part is completely dominant. In any case, this assumption is used in the study of the IR behavior of the running coupling constant in axial-gauge QCD and because of the differences in the Ward identity and the absence of the ghost field in the axial gauge it is certainly not obvious that our covariant gauge results can be generalized to this case. A clearer answer to this question requires an analysis of the axial-gauge vertex. While we have made some progress on the one-loop triple-gluon vertex in the axial gauge, no results other than the UV-divergent terms have been obtained.

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APPENDIX

The calculation of the one-loop triple-gluon vertex requires the integration over the loop momentum. The resulting integrals can be classified by whether they have two momentum denominators or three.

Those with two are all UV divergent and can be evaluated in terms of elementary functions with the aid of dimensional regularization. The necessary integrals of this type are

$$\int d^4k \frac{1}{(k-P)^2(k-P')^2} = \pi^2 i \mu^\epsilon \left(-\frac{2}{\epsilon} + \ln\frac{\mu^2}{-q^2} + C\right), \tag{A1}$$

$$\int d^4k \frac{k_\nu}{(k-P)^2(k-P')^2} = \frac{\pi^2 i}{2}(P+P')_\nu \mu^\epsilon \times\left(-\frac{2}{\epsilon} + \ln\frac{\mu^2}{-q^2} + C\right), \tag{A2}$$

$$\int \frac{d^4k\, k_\lambda k_\nu}{(k-P)^2(k-P')^2} = \frac{\pi^2 i}{6}\mu^\epsilon\left[\left(-\frac{q^2}{2}g_{\nu\lambda} 2P_\nu P_\lambda + 2P'_\nu P'_\lambda + P_\nu P'_\lambda + P'_\nu P_\lambda\right) \times\left(-\frac{2}{\epsilon} + \ln\frac{\mu^2}{-q^2} + C\right) - \tfrac{1}{3}(q^2 g_{\lambda\nu} - q_\lambda q_\nu)\right], \tag{A3}$$

where $q = P' - P$ and $C = -\gamma - \ln\pi$.

The integrals with three denominators are I_0, defined in Eq. (3.1) and those with one, two, and three momentum factors in the numerator. The necessary integrals are given below:

$$I_{1\mu} = \int d^4k \frac{k_\mu}{k^2(k-P)^2(k-P')^2}, \tag{A4}$$

$$I_{2\mu\nu} = \int d^4k \frac{k_\mu k_\nu}{k^2(k-P)^2(k-P')^2}, \tag{A5}$$

$$I_{3\mu\nu\lambda} = \int d^4k \frac{k_\mu k_\nu k_\lambda}{k^2(k-P)^2(k-P')^2}. \tag{A6}$$

By using a tensor decomposition of these integrals we can show that all can be expressed in terms of the integrals given in (A1), (A2), and (A3) and the basic integral I_0.

First, we will consider $I_{1\mu}$, which must be a symmetric function of P and P' hence

$$I_{1\mu} = I_1(P,P')P_\mu + I_1(P',P)P'_\mu. \tag{A7}$$

If we now form $P^\mu I_{1\mu}$ and $P'^\mu I_{1\mu}$ we obtain two equations for $I_1(P,P')$ and $I_1(P',P)$. The fact that

$$P\cdot k = \tfrac{1}{2}[P^2 + k^2 - (P-k)]^2 \tag{A8}$$

and the corresponding expression for $P'\cdot k$ allows $P\cdot I_1$ and $P'\cdot I_1$ to be expressed in terms of the integral in (A1) and I_0. Hence, I_1 is now expressible in terms of I_0:

$$I_1(P,P') = \frac{1}{\Delta^2}\left[P'^2\ln\left(\frac{q^2}{P'^2}\right) - P\cdot P'\ln\frac{q^2}{P^2} + \frac{P'^2 P\cdot q}{2}I_0\right]. \tag{A9}$$

The integral $I_{2\mu\nu}$ is symmetric in μ and ν as well as under $P \leftrightarrow P'$ and hence has the following tensor decomposition:

$$I_{2\mu\nu} = g_{\mu\nu}I_A + \left(P_\mu P_\nu - \frac{g_{\mu\nu}}{n}P^2\right)I_B(P,P') + \left(P_\mu P'_\nu + P'_\mu P_\nu - \frac{2g_{\mu\nu}}{n}P\cdot P'\right)I_C + \left(P'_\mu P'_\nu - \frac{g_{\mu\nu}}{n}P'^2\right)I_B(P',P), \tag{A10}$$

where I_A and I_C are symmetric under $P \leftrightarrow P'$ and the n is the trace of $g_{\mu\nu}$ in n dimensions. The trace of $I_{2\mu\nu}$, or I_A, which contains all the UV divergence is now given by the integral in Eq. (A1). The remaining scalars can be obtained by noting that $P^\mu I_{2\mu\nu}$ and $P'^\mu I_{2\mu\nu}$ can be expressed in terms of $I_{1\nu}$ and the integrals in (A1) and (A2). The scalars in $I_{2\mu\nu}$ are as follows:

$$I_A = \frac{\mu^\epsilon \pi^2 i}{4}\left(-\frac{2}{\epsilon} + \ln\frac{\mu^2}{-q^2} + C + \tfrac{1}{2}\right), \tag{A11a}$$

$$I_B(P,P') = \frac{\pi^2 i}{4}\frac{1}{\Delta^2}\left[\left(P^2 P\cdot P' - \frac{3P^2P'^2}{2}\right)I_1(P,P') + P'^2 - \frac{P'^4}{2}I_1(P',P) + P\cdot P'\ln\left(\frac{P^2}{q^2}\right)\right], \tag{A11b}$$

$$I_C = \frac{\pi^2 i}{4\Delta^2}\left\{\left[\tfrac{3}{2}P^2(P\cdot P') - P^2P'^2\right]I_1(P,P') + P'^2\frac{(P\cdot P')}{2}I_1(P',P) - P\cdot P' - P^2\ln\frac{P^2}{q^2}\right\}. \tag{A11c}$$

Finally, the integral $I_{3\mu\nu\lambda}$, which is completely symmetric in $\mu\nu\lambda$, has the following tensor decomposition:

$$I_{3\mu\nu\lambda}=[T(P)_{\mu\nu\lambda}+T(P')_{\mu\nu\lambda}]I_D+[T(P)_{\mu\nu\lambda}-T(P')_{\mu\nu\lambda}]I_E+\left[P_\mu P_\nu P_\lambda-\frac{P^2}{6}T(P)_{\mu\nu\lambda}\right]I_F(P,P')$$
$$+\left[P_\mu P_\nu P'_\lambda+P_\lambda P_\mu P'_\nu+P_\nu P_\lambda P'_\mu-\frac{P\cdot P'}{3}T(P)_{\mu\nu\lambda}-\frac{P^2}{6}T(P')_{\mu\nu\lambda}\right]I_G(P,P')$$
$$+\left[P_\mu P'_\nu P'_\lambda+P_\lambda P'_\mu P'_\nu+P_\nu P'_\lambda P'_\mu-\frac{P\cdot P'}{3}T(P')_{\mu\nu\lambda}-\frac{P'^2}{6}T(P)_{\mu\nu\lambda}\right]I_G(P',P)$$
$$+\left[P'_\mu P'_\nu P'_\lambda-\frac{P'^2}{6}T(P)_{\mu\nu\lambda}\right]I_F(P',P), \tag{A12}$$

where

$$T(P)_{\mu\nu\lambda}=g_{\mu\nu}P_\lambda+g_{\lambda\mu}P_\nu+g_{\nu\lambda}P_\mu\,.$$

Here the scalar function I_D is symmetric under exchange of P and P' while I_E is odd. If we form $g^{\mu\nu}I_{3\mu\nu\lambda}$ we obtain the integral given in Eq. (A2), thus determining that

$$I_E=0 \tag{A13a}$$

and

$$I_D=\frac{\pi^2 i}{12}\mu^\epsilon\left(-\frac{2}{\epsilon}+\ln\frac{\mu^2}{-q^2}+C+\tfrac{1}{3}\right). \tag{A13b}$$

The remaining scalars can now be obtained by forming $P^\mu I_{3\mu\nu\lambda}$ and comparing with (A3), (A10), and (A11). This yields

$$I_F(P,P')=\frac{i\pi^2}{6}\left[\left(2-\frac{P^2P'^2}{2\Delta^2}\right)I_B(P,P')+\frac{(P\cdot P')^4}{2P^2\Delta^2}I_B(P',P)-\left(\frac{1}{P^2}+\frac{2P'\cdot q}{\Delta^2}\right)P'^2I_C-\frac{P'^2P\cdot q}{2P^2\Delta^2}\right] \tag{A13c}$$

and

$$I_G(P,P')=\frac{i\pi^2}{6\Delta^2}\left[P^2\frac{(P'\cdot P)}{2}I_B(P,P')-\frac{P'^4}{2}I_B(P',P)+2P'^2P\cdot qI_C+\tfrac{1}{2}(P'\cdot q)\right]. \tag{A13d}$$

All the necessary integrals have now been expressed in terms of elementary functions and I_0 which contains dilogarithms.

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