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Analytic properties of the vertex function in gauge theories. I

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The analytic properties of the vertex functions for massless scalar and massless and massive spinor electrodynamics are investigated. First, a general tensor form for the vertex consistent with the Ward identities and free of kinematic singularities is constructed. All of the scalar functions that appear are then calculated to one-loop order in perturbation theory. It is found that in both massive and massless theories the infrared singularities appear in the longitudinal parts of the vertex. The massless theories have additional photon mass-shell singularities which appear in the transverse terms. Complete expressions for each vertex are obtained which contain only one nonelementary function.

I. INTRODUCTION

The idea that the infrared (IR) singularities of quantum chromodynamics (QCD) provide the mechanism for quark and color confinement is, by now, widely accepted, in spite of the somewhat limited understanding of these singularities. In comparing the IR behavior of QCD with that of QED, which is well understood, we observe that QCD has two essential complications. First, the field quanta carry a charge (color in QCD) and hence are self-coupled; and second, the quanta are coupled to massless particles (quarks and gluons). Both of these features are required in QCD although the quarks can, of course, be massive. In an attempt to make a two-step transition from massive QED to QCD, where both new features appear, we have investigated, as an intermediate theory, QED for massless particles where the photon is not self-coupled.

In this work we have studied, via perturbation theory, the analytic properties of the off-shell vertex function in gauge theories ranging from massive spinor electrodynamics to QCD in the pure gluon sector (in the Feynman gauge). The reasons for our special interest in the vertex function are the following. (1) In massless theories, functions of a single momentum such as the propagator have singularities in the dimensionless variable q^2/Λ^2 where Λ^2 is the UV cutoff. Thus, the IR singularities at the one-loop level are trivially related to the UV behavior, which is known from the renormalization requirements. In contrast, the off-shell vertex depends on three scalar variables: p^2, p'^2, and q^2. Hence, even in the one-loop result, logarithmic singularities in p^2/p'^2 or q^2/p'^2 can occur which are not coupled to the UV behavior. (2) The vertex function in a gauge theory satisfies a Ward identity, which relates the longitudinal components of the vertex to simpler functions, the inverse electron propagator in the case of QED. The fact that all of the IR singularities in the vertex function for massive QED occur in this longitudinal component allows a simple and direct investigation of the general IR properties. It has been suggested[1] that a similar procedure might be followed for QCD, provided the vertex has the proper behavior. (3) An essential step in the program described in Ref. 1 is the construction of a vertex function which automatically satisfies the Ward identity. The longitudinal part of the vertex can be represented explicitly in terms of the simpler scalar functions that appear in the Ward identity. The crucial assumption which leads to a unique form for the longitudinal vertex is that the vertex be free of kinematic singularities. While this assumption is certainly reasonable, we feel that this question requires some further study. Most kinematic singularities obviously violate the general analyticity requirements of the vertex, however, there are certain types which might naturally occur. For example, consider the two tensors $\delta_{\mu\nu}$ and $q_\mu q_\nu/q^2$. The second tensor has a kinematic singularity at $q^2=0$, but because of the gauge invariance of QED, this tensor can appear in the photon propagator. Perhaps kinematic singularities of the vertex at $q^2=0$ of the type that are obtained by multiplying an analytic vertex Γ_ν by $q_\nu q_\mu/q^2$ are also allowed because of gauge invariance.

In this paper we will study the analytic properties of the vertex functions in massless scalar electrodynamics, and massless and massive spinor electrodynamics. The more complicated case of QCD in the Feynman gauge is treated in the following paper. First, we will construct a general form for the vertex which automatically satisfies the Ward identity, is free of kinematic singularities, and in which the scalar functions that appear are free of any constraints. The one-loop perturbation-theory result is calculated in each case, verifying that the conjectured form is, in fact, produced indicating that no kinematic singularities are present. Then the nature of the singularities for various momentum limits is examined. A complete expression for each vertex is given so that the reader can examine any desired limit in addition to those which we have considered.

While it is clear that many of our perturbation results are contained in other papers, we found that the tensor structure dictated by the Ward identity can easily be hidden, and repeating the calculations with our goal in mind proves easier that transforming other results to the necessary form. Furthermore, the use of the Feynman parameter technique in other calculations to perform the momentum integrations produces a number of apparently unrelated integrals which cannot be expressed in terms of elementary functions. While this is not a problem when a numerical result is desired, it is a serious complication when one is investigating the analytic properties of the vertex. To avoid this problem, we employ a tensor method which allows each vertex to be expressed in terms of a single scalar integral plus elementary functions. We believe our final form to be simpler than previous results, and the reduction to a single integral could possibly be of use in higher-order calculations where the identification of the number of independent integrals becomes a major problem.

II. SCALAR ELECTRODYNAMICS

The simplest example of a gauge theory is scalar electrodynamics. While the most general form of the off-shell vertex consistent with the Ward identity is well known, we will, to illustrate the general approach, derive it from general requirements and verify that it is, in fact, consistent with the perturbation results.

The vertex must be a vector quantity and hence must have the form

$$\Gamma_\mu = A p_\mu + B p'_\mu \,, \tag{2.1}$$

where $q = p' - p$ and A and B are scalar functions of p^2, p'^2, and q^2, the "masses" associated with each particle in the vertex. The Ward identity is

$$q^\mu \Gamma_\mu = D^{-1}(p'^2) - D^{-1}(p^2) \,, \tag{2.2}$$

where D is the complete propagator of the scalar meson and hence a function of a single scalar variable. Equation (2.2) gives the following equation for A and B:

$$p\cdot q A + p'\cdot q B = D^{-1}(p'^2) - D^{-1}(p^2) \,, \tag{2.3}$$

which can be used to eliminate B,

$$\Gamma_\mu = [D^{-1}(p'^2) - D^{-1}(p^2)] \frac{p'_\mu}{p'\cdot q} + A\left(p_\mu - \frac{p\cdot q}{p'\cdot q} p'_\mu\right) . \tag{2.4}$$

We have now produced a kinematical singularity in the first term at $p'\cdot q = 0$ which must be canceled by the second term, meaning that A cannot be an arbitrary function but must satisfy the following constraint:

$$\lim_{p'\cdot q \to 0} (p\cdot q) A = D^{-1}(p'^2) - D^{-1}(p^2) .$$

As $p'\cdot q \to 0$, $p\cdot q \to (p'^2 - p^2)$, so

$$A = \frac{D^{-1}(p'^2) - D^{-1}(p^2)}{p'^2 - p^2} + (p'\cdot q) A' \tag{2.5}$$

will provide the necessary cancellation without introducing any new kinematic singularities. The new scalar function A' is now unconstrained. Substituting into Eq. (2.4) we obtain the final form

$$\Gamma_\mu = \frac{D^{-1}(p'^2) - D^{-1}(p^2)}{p'^2 - p^2} (p'_\mu + p_\mu) + A'(p'\cdot q p_\mu - p\cdot q p'_\mu) . \tag{2.6}$$

Here, the first term is the longitudinal part of the vertex and the second term is transverse as it is orthogonal to q_μ.

If we relax our requirement that the vertex be free of kinematic singularities there are many possible forms for the vertex. For example, the vertex given below satisfies the Ward identity but contains kinematic singularities in both the longitudinal and transverse parts:

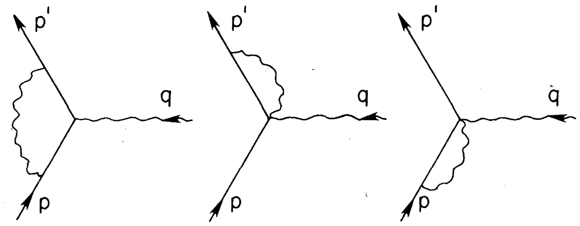
$$\Gamma'_\mu = [D^{-1}(p'^2) - D^{-1}(p^2)] \frac{q_\mu}{q^2} + \bar{A}\left(\frac{p_\mu}{p\cdot q} - \frac{p'_\mu}{p'\cdot q}\right) . \tag{2.7}$$

We now calculate these quantities in perturbation theory to see which form of the vertex is produced, how a scalar function A' with dimension p^{-2} is generated, and whether the resulting A' has any poles or zeros which might indicate some other choice for the transverse vector in Eqs. (2.6) and (2.7).

From the Feynman graphs shown in Fig. 1 for massless scalar mesons, we find exactly the form given in Eq. (2.6) with

$$D^{-1}(p^2) = -\frac{\alpha}{8\pi} p^2 \left(5 + 4 \ln \frac{\Lambda^2}{-p^2}\right) \tag{2.8}$$

and



(a) (b) (c)

FIG. 1. One-particle irreducible scalar-photon vertex at one-loop order.

$$A'=\frac{\alpha}{8\pi\Delta^2}\left[(p^2+p'^2-4p\cdot p')\left(p\cdot p' I_0+\ln\frac{q^4}{p^2p'^2}\right)+\frac{(p^2+p'^2)q^2-8p^2p'^2}{p'^2-p^2}\ln\frac{p^2}{p'^2}\right], \qquad (2.9)$$

where Λ^2 is the UV cutoff, Δ^2 is the triangle function

$$\begin{aligned}\Delta^2&\equiv(p'\cdot p)^2-p^2p'^2\\&=(p'\cdot q)^2-p'^2q^2\\&=(p\cdot q)^2-p^2q^2\\&=\tfrac{1}{4}(q^4+p^4+p'^4-2p^2p'^2-2p^2q^2-2p'^2q^2)\end{aligned} \qquad (2.10)$$

and I_0, which is a symmetric function of p^2, p'^2, and q^2 is the following integral:

$$\frac{i\pi^2}{2}I_0=\int d^4k\frac{1}{k^2(k-p)^2(k-p')^2}. \qquad (2.11)$$

Since A' is a function of only p^2, p'^2, and q^2 (no Λ^2 dependence) with dimension p^{-2} it can be written as a function of two dimensionless variables divided by momentum squared. For example, if we define $x=p^2/p'^2$ and $y=q^2/p'^2$ then

$$A'=\frac{1}{p'^2}\times(\text{function of } x \text{ and } y). \qquad (2.12)$$

Clearly, the longitudinal vertex we have obtained in the Feynman gauge is free of kinematic singularities and, as expected, does not depend on the variable q^2. By using the photon propagator in a general covariant gauge, we find the form remains the same although D^{-1} will change as it is gauge dependent.

The only possible kinematic singularity in A' would be at the zeros of Δ^2. To investigate the behavior near this point, and for other limits, it is convenient to express I_0 in terms of a special function, the dilogarithm,[2]

$$I_0=\frac{1}{\Delta}\left[f\left(\frac{p\cdot p'-\Delta}{p'^2}\right)-f\left(\frac{p\cdot p'+\Delta}{p'^2}\right)+\tfrac{1}{2}\ln\left(\frac{p\cdot p'-\Delta}{p\cdot p'+\Delta}\right)\ln\left(\frac{q^2}{p'^2}\right)\right], \qquad (2.13)$$

where the dilogarithm f is the following integral:

$$f(x)=-\int_1^x\frac{\ln t}{t-1}dt=\mathrm{Li}_2(x).$$

At this point, the symmetry of I_0 in p^2, p'^2, and q^2 is no longer obvious. The fact that the derivative of f is an elementary function makes it relatively simple to expand I_0 in a power series about $\Delta^2=0$.

The quantity within the square brackets in Eq. (2.9) is proportional to Δ^2 for small Δ and hence A' is well behaved as Δ^2 goes to zero.

The simplest limit of interest is $q^2\to0$, the mass-shell limit for the photon, for arbitrary p^2 and p'^2. In this limit $\Delta^2\to(p'^2-p^2)^2/4$ and $p\cdot p'\to\frac{1}{2}(p^2+p'^2)$. For these values the dilogarithm is well behaved and only the $\ln q^2$ term in I_0 is important. Assuming $p'^2>p^2$ we obtain

$$I_0\to\frac{2}{p'^2-p^2}\ln\frac{p^2}{p'^2}\ln\frac{q^2}{p'^2} \qquad (2.14)$$

and

$$A'\to\left(-\frac{\alpha}{4\pi}\right)\left[2\frac{(p^2+p'^2)}{(p'^2-p^2)^2}\left(2+\frac{p^2+p'^2}{p'^2-p^2}\ln\frac{p^2}{p'^2}\right)\ln\frac{q^2}{p'^2}\right]. \qquad (2.15)$$

Thus, we see that the $q^2\to0$ limit for $p^2\neq p'^2$ A' is logarithmically infinite. The $q^2\to0$ limit of the transverse vector which is the coefficient of A' in the vertex becomes

$$\frac{p'^2-p^2}{2}q_\mu$$

and

$$\Gamma^t_\mu\to\left(-\frac{\alpha}{4\pi}\right)q_\mu\frac{(p^2+p'^2)}{(p'^2-p^2)}\left(2+\frac{p^2+p'^2}{p'^2-p^2}\ln\frac{p^2}{p'^2}\right)\ln q^2. \qquad (2.16)$$

If we now take the limit $p^2\to p'^2$, the coefficient of $\ln q^2$ is finite, though not zero, and A' diverges for $q^2\to0$ as before. However, the transverse vertex now vanishes as $p^2\to p'^2$ because of the p'^2-p^2 in the vector, leaving only the longitudinal vertex, which is finite. If the order of the limits is reversed, the same result is obtained. This combination of limits is what one conventionally calls the infrared limit, i.e., $q_\mu\to0$ which requires both $q^2\to0$ and $p^2\to p'^2$, and, of course, is the $q^2\to0$ limit in Euclidean space. The actual divergences associated with the infrared limit occur when one takes the scalar mesons to the mass-shell limit (in this case $p'^2\to0$), and these singularities occur in the longitudinal terms. This behavior is essentially the same as that of a massive theory.

Before proceeding to the spinor case, we will summarize our scalar results. In the mass-shell limit for the photon the vertex is singular for the scalar mesons being unequally off shell ($p'^2 \neq p^2$) and this singularity occurs in the transverse vertex. It is in this photon mass-shell limit that massless QED has unusual behavior not present in massive theories. This could be an indication in perturbation theory of some basic disease of massless scalar electrodynamics.

The IR limit behaves like that of massive electrodynamics with divergences occurring only in the longitudinal vertex as the mesons approach the mass shell. Finally, we note that in Euclidean space the only singularities that occur are the usual IR singularities in the longitudinal vertex.

III. SPINOR ELECTRODYNAMICS

The additional degrees of freedom introduced by having two spin-$\frac{1}{2}$ particles coupled to the photon considerably complicate the tensor decomposition of the vertex. In this case there are three four-vectors: γ_μ, p_μ, and p'_μ, and four types of scalars proportional to 1: $\not{p}$, $\not{p}'$, and $p^\mu p'^\nu \sigma_{\mu\nu}$. The resulting 12 spin amplitudes have been enumerated elsewhere[3]; however, since our goal is producing a vertex which is free of kinematic singularities and which automatically satisfies the Ward identity, we will introduce eight tensors that give no contribution to the Ward identity [the generalization of the second term in Eq. (2.6)]. The remaining four tensors will be completely determined by the Ward identity.

The Ward identity for spinor electrodynamics is

$$q_\mu \Gamma^\mu = S_F^{-1}(p'^2) - S_F^{-1}(p^2),$$
$$= F(p'^2)\not{p}' - F(p^2)\not{p} + G(p'^2) - G(p^2), \quad (3.1)$$

where F and G are the scalar functions that determine the electron propagator. Note that one of the twelve amplitudes will be identically zero due to the fact that the scalar $p^\mu p'^\nu \sigma_{\mu\nu}$ does not occur on the right side of Eq. (3.1). The portion of the vertex which "solves" Eq. (3.1) and is free of kinematic singularities is

$$\Gamma_0^\mu = \frac{(\not{p}+\not{p}')}{2}(p'+p)^\mu \frac{F(p'^2)-F(p^2)}{p'^2-p^2} + \frac{F(p'^2)+F(p^2)}{2}\gamma^\mu + \frac{G(p'^2)-G(p^2)}{p'^2-p^2}(p+p')^\mu . \quad (3.2)$$

Note that the straightforward generalization of the scalar result

$$\Gamma_0^\mu = \frac{S_F^{-1}(p'^2) - S_F^{-1}(p^2)}{p'^2-p^2}(p+p')^\mu \quad (3.3)$$

has kinematic singularities and is therefore unacceptable.

The remaining eight tensor forms must satisfy

$$q_\mu T_i^\mu = 0, \quad i = 1, 2, \ldots, 8 .$$

A set of independent T's which have this property is

$$\begin{aligned}
T_1^\mu &= Q^\mu = p^\mu(p'\cdot q) - p'^\mu(p\cdot q),\\
T_2^\mu &= Q^\mu(\not{p}+\not{p}'),\\
T_3^\mu &= q^2\gamma^\mu - q^\mu \not{q},\\
T_4^\mu &= Q^\mu p^\lambda p'^\nu \sigma_{\lambda\nu},\\
T_5^\mu &= \sigma^{\mu\lambda} q_\lambda, \qquad\qquad (3.4)\\
T_6^\mu &= \gamma^\mu(p'^2-p^2) - (p+p')^\mu \not{q},\\
T_7^\mu &= \frac{p'^2-p^2}{2}[\gamma^\mu(\not{p}+\not{p}') - p^\mu - p'^\mu]\\
&\quad + (p+p')^\mu p^\nu p'^\lambda \sigma_{\nu\lambda},\\
T_8^\mu &= -\gamma^\mu p^\nu p'^\lambda \sigma_{\nu\lambda} + p^\mu \not{p}' - p'^\mu \not{p} .
\end{aligned}$$

The complete vertex can then be written

$$\Gamma^\mu = \Gamma_0^\mu + \sum_{i=1}^{8} A_i T_i^\mu . \quad (3.5)$$

Other than simplicity, the only criteria we have for choosing this set of T's rather than some linear combinations of these is the perturbation result. It was found that, if instead of T_3 given above, we used $Q^\mu \not{q}$, which is a linear combination of T_2, T_3, and T_6, a kinematical singularity appeared in A_6, while for the set above, all of the A's are separately analytic. Since A_4, A_5, and A_7 are zero to second order in perturbation theory, we cannot be sure that higher-order calculations might not require particular linear combinations of T_4, T_5, and T_7 rather than the forms given above. The results of the second-order calculations are

$$\begin{aligned}
F &= \frac{\alpha}{4\pi}\left[\ln\frac{\Lambda^2}{m^2} + \frac{3}{2} + \frac{m^2}{p^2} + \left(\frac{m^4}{p^4}-1\right)\ln\frac{m^2-p^2}{m^2}\right],\\
G &= -\frac{\alpha m}{\pi}\left[\left(\frac{m^2}{p^2}-1\right)\ln\frac{m^2-p^2}{m^2} + 1 + \ln\frac{\Lambda^2}{m^2}\right],
\end{aligned} \quad (3.6)$$

$$A_4 = A_5 = A_7 = 0, \quad (3.7)$$

$$A_1 = -\frac{\alpha m}{\pi\Delta^2}\left[\frac{m^2+p\cdot p'}{2}J_0 + 2S - \frac{(p\cdot p'+p'^2)L' - (p^2+p\cdot p')L}{p'^2-p^2}\right], \quad (3.8a)$$

$$A_8=\frac{\alpha}{4\pi\Delta^2}\left[\frac{(m^2+p\cdot p')}{2}q^2J_0+2q^2S-(p^2L+p'^2L')+p\cdot p'(L+L')\right], \tag{3.8b}$$

$$A_2=-\frac{3}{4}\frac{(p\cdot p'+m^2)}{\Delta^2}A_8+\frac{\alpha}{4\pi\Delta^2}\left[\frac{(q^2-4m^2)}{8}J_0-\frac{1}{2}-\frac{p\cdot p'm^2}{2p^2p'^2}+\frac{1}{4}(L+L')\right.$$
$$\left.+\frac{(p^2+p\cdot p')(1+m^2/p^2)L-(p'^2+p\cdot p')(1+m^2/p'^2)L'}{2(p'^2-p^2)}\right], \tag{3.8c}$$

$$A_6=\frac{p'^2-p^2}{2}A_2, \tag{3.8d}$$

$$A_3=-\frac{3(m^2+p\cdot p')}{32m\Delta^2}(p^2-p'^2)^2A_1$$
$$-\frac{\alpha}{8\pi\Delta^2}\left\{\left[-\Delta^2-\frac{(p^2+p'^2-2m^2)^2}{8}\right]J_0-2(m^2+p\cdot p')S+\frac{p'\cdot p}{2}\left[\left(\frac{1-m^2}{p^2}\right)L+\left(1-\frac{m^2}{p'^2}\right)L'\right]\right.$$
$$\left.+\frac{p^2-p'^2}{4}(L-L')+(m^2+p\cdot p')+\frac{(m^2p\cdot p'+p^2p'^2)}{2p^2p'^2}(p'^2+p^2)\right\}, \tag{3.8e}$$

where the functions S, L, L', and J_0 are the following:

$$S=\left(1-\frac{4m^2}{q^2}\right)^{1/2}\sinh^{-1}\left(\frac{-q^2}{4m^2}\right)^{1/2}$$
$$=\frac{1}{2}\left(1-\frac{4m^2}{q^2}\right)^{1/2}\ln\frac{1+(1-4m^2/q^2)^{1/2}}{1-(1-4m^2/q^2)^{1/2}}, \tag{3.9a}$$

$$L=\left(1-\frac{m^2}{p^2}\right)\ln\left(\frac{m^2-p^2}{m^2}\right), \tag{3.9b}$$

$$L'=\left(1-\frac{m^2}{p'^2}\right)\ln\left(\frac{m^2-p'^2}{m^2}\right), \tag{3.9c}$$

and

$$J_0=\frac{2}{\pi^2{}_i}\int d^4k\frac{1}{k^2[(k-p)^2-m^2][(k-p')^2-m^2]}. \tag{3.9d}$$

While the formulas for the A's appear to be quite complicated, they are much simpler than any previous results in that they are all expressed in terms of elementary functions and a single scalar integral. We emphasize again that the fact that only one integral appears is well disguised if Feynman parameters are introduced to evaluate the various vector and tensor integrals. However, if these same integrals are decomposed into scalars, the existence of a single integral J_0 becomes obvious (see the Appendix).

The absence of kinematic singularities at $\Delta^2=0$ can again be shown, as all integrals can be evaluated analytically in this limit.

The massless electron limit can now be obtained by letting $m\to 0$ in Eqs. (3.6)–(3.9). The resulting scalar amplitudes in this case are

$$F=\frac{\alpha}{4\pi}\left[\ln\left(\frac{\Lambda^2}{-p^2}\right)+\frac{3}{2}\right], \tag{3.10a}$$

$$G=0,\quad A_1=A_4=A_5=A_7=0, \tag{3.10b}$$

$$A_8=\frac{\alpha}{8\pi\Delta^2}\{q^2[p\cdot p'I_0+\ln(q^4/p^2p'^2)]+(p'^2-p^2)\ln(p^2/p'^2)\}, \tag{3.11a}$$

$$A_2=-\frac{3}{4}\frac{p\cdot p'}{\Delta^2}A_8+\frac{\alpha}{8\pi\Delta^2}\left(\frac{q^2}{4}I_0+\frac{p'^2+p^2+2p\cdot p'}{p'^2-p^2}\ln\frac{p^2}{p'^2}-1\right), \tag{3.11b}$$

$$A_3=\frac{\alpha}{8\pi\Delta^2}\left\{\left[\frac{3}{8}\frac{(p\cdot p')^2(p'^2-p^2)^2}{\Delta^2}-\Delta^2-\frac{1}{8}(p^2-p'^2)^2\right]I_0\right.$$
$$+\frac{1}{4}(p^2-p'^2)\left[1-\frac{3}{2}\frac{(p\cdot p')(p^2+p'^2+2p\cdot p')}{\Delta^2}\right]\ln\left(\frac{p^2}{p'^2}\right)$$
$$\left.+\frac{(p\cdot p')}{2}\left[\frac{3}{4}\frac{(p'^2-p^2)^2}{\Delta^2}-1\right]\ln\left(\frac{q^4}{p^2p'^2}\right)+\frac{p^2+p'^2+2p\cdot p'}{2}\right\}, \tag{3.11c}$$

$$A_6=\frac{p'^2-p^2}{2}A_2. \tag{3.11d}$$

As for the scalar case, the A's are divergent in the photon mass-shell limit $q^2 \to 0$. Employing Eq. (2.13) we find that

$$A_3 = \frac{\alpha}{2\pi} \frac{1}{(p'^2 - p^2)^2} \left[p^2 + p'^2 - \frac{2p^2 p'^2}{p^2 - p'^2} \ln\left(\frac{p^2}{p'^2}\right)\right] \ln q^2 \tag{3.12}$$

is the only divergent term. In this limit, T_3 becomes

$$T_3 = -q^\mu \not{q} \,. \tag{3.13}$$

Finally, taking the $p^2 = p'^2$ limit, we find that

$$A_3 = \frac{\alpha}{6\pi} \frac{\ln(q^2)}{p^2} \,. \tag{3.14}$$

Here, in contrast to the scalar case, the tensor form does not vanish, and the transverse vertex has a logarithmic divergence at $q^2 = 0$.

If we now consider the IR limit, the next step is taking $q_\mu \to 0$. In this limit T_3 vanishes and the vertex has a finite limit. Finally, taking the mass-shell limit $p^2 \to 0$, we find that the transverse part is finite and that the longitudinal vertex is logarithmically divergent, just as in the case of massive electrodynamics. It is clear that this behavior is also obtained in Euclidean space where the $q^2 = 0$ implies $q_\mu \to 0$ and $p^2 \to p'^2$.

IV. CONCLUSIONS

It is clear that no kinematic singularities are present in our one-loop perturbation calculations. We would expect higher-order calculations of the longitudinal vertex terms to satisfy the Ward identity by the same mechanism as the first-order terms, and hence to produce the same result as would be obtained by using a higher-order propagator in our general expressions Eqs. (2.6) and (3.2).

The situation with regard to the small-q singularities is more complicated, with the transverse vertex terms having a logarithmic divergence at $q^2 = 0$. These singularities disappear for scalar QED when the meson legs are equally off shell $p^2 = p'^2$ and for massless spinor QED when $q_\mu \to 0$. The conventional IR singularities of massive QED are obtained by letting $q_\mu \to 0$ and then taking the remaining momenta to their mass-shell limit. In this limit, all of the singularities are directly related to those in the massive particle propagator and no singularities appear in the photon propagator, which produces the following results: (1) the power of the logarithmic singularities of the propagator builds up with each increasing order of α in just such a way that the leading logarithms can be exponentiated, (2) there are no nonleading logarithms and (3) the logarithms in the vertex also exponentiate, as they are the same as those in the propagator. When we investigate massless QED in this limit the behavior is the same as in the massive theory, and all of the magic described above should work except for problems with the photon propagator. In massless QED the one-loop photon self-energy has a $\ln q^2$ term; however, because the transverse vertex also contains $\ln q^2$ terms, it seems likely that nonleading logarithms will also appear. Because of these complications, the photon wave-function renormalization cannot be performed at $q^2 = 0$, but must be done at some finite mass μ^2 and an $\alpha(\mu^2)$ must be introduced. While we have no idea what the resulting meson or photon self-energy might be, it is sufficiently complicated that it might move the $q^2 = 0$ poles in these propagators to some finite value. If this speculation turns out to be the case, then the singularities we observe might be the first indication in perturbation theory that these theories have spontaneous symmetry breaking as proposed by Coleman and Weinberg.[4] We are at present studying the buildup of these singularities and hope to have something more definite to say in the near future.

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APPENDIX

The integrals which appear in the lowest-order meson-photon vertex are

$$I^{(0)} = \int d^4k \frac{1}{k^2(k-p)^2(k-p')^2} \,, \tag{A1}$$

$$I_\mu^{(1)} = \int d^4k \frac{k_\mu}{k^2(k-p)^2(k-p')^2} \,, \tag{A2}$$

and

$$\int d^4k \frac{1}{(k-p)^2(k-p')^2} = i\pi^2 \left[\ln\left(\frac{\Lambda^2}{-q^2}\right) + 1\right] . \tag{A3}$$

Clearly, the vector integral $I_\mu^{(1)}$ can only have components in the p_μ and p'_μ directions,

$$I_\mu^{(1)} = \frac{i\pi^2}{2} (p_\mu I_A + p'_\mu I_B) \,. \tag{A4}$$

By forming the scalar products $I^{(1)} \cdot p$ and $I^{(1)} \cdot p'$ and solving for I_A and I_B we obtain

$$I_A = -\frac{1}{\Delta^2}\left[p'^2 \ln\left(\frac{p'^2}{q^2}\right) + p\cdot p' \ln\left(\frac{q^2}{p^2}\right) - \frac{p'^2}{2}(p\cdot q)I_0\right], \tag{A5}$$

$$I_B = I_A(p \leftrightarrow p'), \tag{A6}$$

where $I_0 = (2/\pi^2 i) I^{(0)}$. Note that this reduction to the integral I_0 and elementary functions is possible because

$$p\cdot k = -\tfrac{1}{2}[(k-p)^2 - k^2 - p^2]$$

and

$$p'\cdot k = -\tfrac{1}{2}[(k-p')^2 - k^2 - p'^2]. \tag{A7}$$

Thus, only integrals of the form of (A3) and I_0 will occur. Clearly, the reduction of second- and third-rank tensor integrals can be carried out in exactly the same manner, and only elementary functions and I_0 will be needed to express these integrals in terms of the basic tensors formed from p_μ, p'_μ, and $g_{\mu\nu}$.

In the one-loop electron vertex the integrals that occur are

$$J^{(0)} = \int d^4k \frac{1}{k^2[(k-p)^2 - m^2][(k-p')^2 - m^2]}, \tag{A8}$$

$$J_\mu^{(1)} = \int d^4k \frac{k_\mu}{k^2[(k-p)^2 - m^2][(k-p')^2 - m^2]}, \tag{A9}$$

$$J_{\mu\nu}^{(2)} = \int d^4k \frac{k_\mu k_\nu}{k^2[(k-p)^2 - m^2][(k-p')^2 - m^2]}, \tag{A10}$$

$$\int d^4k \frac{1}{[(k-p)^2 - m^2][(k-p')^2 - m^2]} = \pi^2 i\left[\ln\left(\frac{\Lambda^2}{m^2}\right) + 1 - 2\left(1 - \frac{4m^2}{q^2}\right)^{1/2} \sinh^{-1}\left(\frac{-q^2}{4m^2}\right)^{1/2}\right]$$
$$= \pi^2 i\left[\ln\left(\frac{\Lambda^2}{m^2}\right) + 1 - 2S\right], \tag{A11}$$

and

$$\int d^4k \frac{1}{k^2[(k-p)^2 - m^2]} = \pi^2 i\left[\ln\left(\frac{\Lambda^2}{m^2}\right) + 1 + \left(\frac{m^2}{p^2} - 1\right)\ln\frac{m^2 - p^2}{m^2}\right]$$
$$= \pi^2 i[\ln(\Lambda^2/m^2) + 1 - L]. \tag{A12}$$

Again we decompose $J_\mu^{(1)}$ and $J_{\mu\nu}^{(2)}$ into all possible components:

$$J_\mu^{(1)} = \frac{\pi^2 i}{2}(p_\mu J_A + p'_\mu J_B), \tag{A13}$$

$$J_{\mu\nu}^{(2)} = \frac{\pi^2 i}{2}[g_{\mu\nu}A + (p_\mu p_\nu - g_{\mu\nu}p^2/4)J_c + (p'_\mu p_\nu + p_\mu p'_\nu - \tfrac{1}{2}p\cdot p' g_{\mu\nu})J_D + (p'_\mu p'_\nu - \tfrac{1}{4}p'^2 g_{\mu\nu})J_E]. \tag{A14}$$

The scalar coefficients can be obtained by taking the trace of $J_{\mu\nu}^{(2)}$, and by saturating the indices with p and p' as before. The resulting expressions are

$$A = \tfrac{1}{2}[\ln(\Lambda^2/m^2) + 1 - 2S], \tag{A15}$$

$$J_A = \frac{1}{\Delta^2}\{\tfrac{1}{2}[p'^2(p\cdot q) + m^2(p'\cdot q)]J_0 + 2p'\cdot qS - p'^2 L' + p\cdot p' L\}, \tag{A16}$$

$$J_B = J_A(p \leftrightarrow p'), \tag{A17}$$

$$J_C = \frac{1}{2\Delta^2}\left\{\frac{(p\cdot p')}{2}\left[(p'^2 - m^2)J_A - 2S - \left(\frac{m^2}{p^2} - 1\right)L + \frac{m^2}{p^2}\right] - \frac{p'^2}{2}[3(p^2 - m^2)J_A + (p'^2 - m^2)J_B - 2]\right\}, \tag{A18}$$

$$J_D = \frac{1}{2\Delta^2}\left\{\frac{(p\cdot p')}{2}[3(p^2 - m^2)J_A + (p'^2 - m^2)J_B - 2] - p^2\left[(p'^2 - m^2)J_A - 2S - \left(\frac{m^2}{p^2} - 1\right)L + \frac{m^2}{p^2}\right]\right\}, \tag{A19}$$

and

$$J_E = J_c(p \leftrightarrow p'). \tag{A20}$$

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