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Soluble model of the phase transition of the *XY* model

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A model that resembles the time-continuum *XY* model is solved. The nature of the phase transition of the soluble model is discussed and shown to be analogous to the phase transition of the *XY* model. A second model, which describes a mixed *XY*- and Ising-spin interaction, is also discussed.

I. A SOLUBLE MODEL OF THE PHASE TRANSITION OF THE *XY* MODEL

The classical *XY* model describes the statistical mechanics of classical two-component spins $\vec{s}(\vec{x})$ on a two-dimensional lattice, $\vec{x}=(x_1, x_2)$. The spins have unit length $\vec{s}^2 = s_1^2 + s_2^2 = 1$ and the partition function is

$$Z=\sum_{\{\vec{s}(\vec{x})\}} e^{-A[\vec{s}(\vec{x})]}, \tag{1.1}$$

where the action $A[\vec{s}(\vec{x})]$ is

$$A[\vec{s}(\vec{x})] = -\sum_{\vec{x}} [\beta_2 \vec{s}(\vec{x})\cdot\vec{s}(\vec{x}+\epsilon\hat{e}_2) + \beta_1 \vec{s}(\vec{x})\cdot\vec{s}(\vec{x}+\hat{e}_1)] \tag{1.2}$$

and $\hat{e}_1, \hat{e}_2$ are unit vectors in the x_1, x_2 directions. The symmetric *XY* model has $\beta_2=\beta_1$ and $\epsilon=1$.

The symmetric *XY* model has an interesting phase structure.[1,2] For small β the spins are disordered; the spin-spin correlation function $\langle\vec{s}(\vec{x})\cdot\vec{s}(\vec{x}+\vec{r})\rangle$ tends to zero exponentially as $|\vec{r}|\to\infty$. The absence of long-range order persists to arbitrarily large β, but for β sufficiently large the spins are *critically* ordered in the sense that the correlation function decreases to zero only as a power of $|\vec{r}|$, i.e., $\langle\vec{s}(\vec{x})\cdot\vec{s}(\vec{x}+\vec{r})\rangle\sim|\vec{r}|^{-p}$ as $|\vec{r}|\to\infty$ where $p>0$. The correlation length is finite in the disordered phase and infinite in the critically ordered phase. There is a critical value β_c of β that separates the disordered phase from the critically ordered phase.

Kosterlitz and Thouless[1] explained the nature of the phase transition at β_c in the following terms. For $\beta\gg\beta_c$ the partition function sum in Eq. (1) is dominated by small spin-wave fluctuations of the spins. The spin waves create enough disorder to prevent long-range order of $\vec{s}(\vec{x})$ (Ref. 3) but not enough to produce a finite correlation length. On the other hand, for $\beta\leqslant\beta_c$ there is another important contribution to Z from special topological configurations; these are vortices in the spin field. The disorder created by the vortices produces a finite correlation length. The origin of the phase transition at β_c is that for $\beta\gtrsim\beta_c$ the vortices only contribute to Z bound in pairs and the vortex pairs do not disorder the spins as effectively as free vortices. The critical value of β is estimated to be $\beta_c\simeq 2.24/\pi$.

In recent papers the *XY* model has been used to illustrate ideas about gauge field theories.[4,5] In these papers the *XY* model is viewed as a Euclidean field theory, with the variables x_2 and x_1 identified as Euclidean time and space variables. To make the connection between the *XY* model and field theory it is useful to consider the quantity $P[\vec{\sigma}(x)]$ defined by

$$P[\vec{\sigma}(x)] = Z^{-1}\sum_{\{\vec{s}(\vec{x})\}} e^{-A[\vec{s}(\vec{x})]} \times\prod_x \delta[\vec{\sigma}(x)-\vec{s}(x,0)] \,. \tag{1.3}$$

In statistical mechanics $P[\vec{\sigma}(x)]$ is just the reduced probability distribution of the spin field along the x_1 axis. The corresponding quantity in field theory is the square of the Schrödinger wave functional of the vacuum state.[5] The functional $P[\vec{\sigma}(x)]$ is the distribution of vacuum fluctuations of the field.

There is a striking analogy between the vortices in the *XY* model and the meron field configuration[6] in a non-Abelian gauge theory.[5] The contribution of vortex configurations to the vacuum functional $P[\vec{\sigma}(x)]$ produces long-range topological kinks in the spin field along the x_1 axis; a kink is a spin field $\vec{\sigma}(x)$ for which the spins rotate through 2π over some range of x. Similarly, the contribution of merons to the vacuum functional of a non-Abelian gauge theory produces long-range topological configurations that are related to magnetic monopole fields. It has been suggested that the vacuum state of quantum chromodynamics is a condensate of such monopoles and that this explains quark confinement.[7] Similarly, the disorded phase of the *XY* model resembles a condensate of kinks.

In a different context, the phase transition of the *XY* model has also been proposed as a model of the expected phase transition in Abelian lattice gauge theories between confining and nonconfining phases.[4,8]

Fradkin and Susskind provided a simplification of the model[4] by considering the x_2-continuum limit

$$\epsilon \to 0, \quad \beta_1 = \lambda\epsilon, \quad \beta_2 = 1/\epsilon . \tag{1.4}$$

In this limit the model can be reduced to a kind of quantum-mechanics problem with the Hamiltonian

$$H = \tfrac{1}{2}\sum_x \left[L^2(x) - \lambda \sum_{\delta=\pm 1} E_+(x)E_-(x+\delta)\right], \tag{1.5}$$

where x labels points of a one-dimensional lattice and the operators $L(x)$ and $E_\pm(x)$ obey the commutation relation

$$[L(x), E_\pm(x')] = \pm E_\pm(x)\delta(x, x') \tag{1.6}$$

and $E_+(x)E_-(x) = 1$. These operators can be represented as $E_\pm(x) = e^{\pm i\varphi(x)}$ and $L(x) = -i\,\delta/\delta\varphi(x)$ acting on functionals of $\varphi(x)$ that are invariant under the transformation $\varphi(x) \to \varphi(x) + 2\pi$. In terms of the original spin system, $\varphi(x)$ is the angle between $\vec{s}(x)$ and a fixed axis. The connection between the Hamiltonian H and the model in Eq. (1) is that H is the generator of translations in the x_2 direction of the x_2-continuum model. It follows that the probability distribution $P[\vec{\sigma}(x)]$ is the square of the Schrödinger wave function of the ground state of H in the representation with $\vec{\sigma}(x) = (\cos\varphi(x), \sin\varphi(x))$. The phase transition of the original model shows up in the Hamiltonian H as a special value of λ at which a qualitative change occurs in the nature of the vacuum state and spectrum of excited states. Critical properties of the original model and the Hamiltonian version are expected to be similar.[4,8]

The Hamiltonian H has not been solved.[9] A simplified Hamiltonian, which might be interesting as a model of H, can be constructed from the following considerations.

The eigenfunctions of $L(x)$ are $e^{im\varphi(x)}$ where m is an integer which can be thought of as an angular momentum quantum number. In this language $E_+(x)$ and $E_-(x)$ are raising and lowering operators that change m by one unit. Since m ranges from $-\infty$ to $+\infty$ the corresponding total angular momentum is infinite. This suggests that a simplified Hamiltonian might be introduced as a model of H by replacing $L(x)$ and $E_\pm(x)$ by angular momentum operators with finite total angular momentum. Let

$$H^{(j)} = \tfrac{1}{2}\sum_x \left([J_3(x) + \kappa]^2 - \lambda^{(j)} \sum_\delta J_+(x)J_-(x+\delta)\right), \tag{1.7}$$

where $\vec{J} = (J_1, J_2, J_3)$ is an angular momentum operator with $J^2 = j(j+1)$ and

$$J_\pm(x) = J_1(x) \pm iJ_2(x). \tag{1.8}$$

The operators $\vec{J}(x)$ and $\vec{J}(x')$ commute if $x' \neq x$. The parameter κ is 0 if j is an integer and $\frac{1}{2}$ if j is a half-integer; it is introduced so that $H^{(j)}$ has a unique ground state for $\lambda = 0$. The eigenvalues of $J_3 + \kappa$ are integers and $J_\pm$ are raising and lowering operators for eigenvectors of J_3. The commutation relation (1.6) holds with $L(x)$ identified with $J_3(x)$ and $E_\pm(x)$ with $J_\pm(x)$.

One difference between the operators $H^{(j)}$ and H is that $E_+E_- = 1$ whereas $J_+J_- = j(j+1) - J_3(J_3 - 1)$. However, in the limit $j \to \infty$ with $\lambda \equiv j^2\lambda^{(j)}$ fixed, $H^{(j)}$ approaches H. Thus $H^{(j)}$ can be examined as a model of H.

The model described by $H^{(j)}$ with $j = \frac{1}{2}$ is soluble. The remainder of this section is devoted to a discussion of the solution. Some of the interesting features of the XY model are present also in the simpler model $H^{(1/2)}$. In particular, there is a transition in λ that separates a disordered phase from a critically ordered phase. In this sense $H^{(1/2)}$ may be called a model of the phase transition of the XY model.

The solution of $H^{(1/2)}$ is obtained with the help of a transformation due to Jordan and Wigner.[10] Let the operator $\xi(x)$ be defined by

$$\xi(x) = \sum_{x' < x} [J_3(x') + \tfrac{1}{2}] \tag{1.9}$$

and $d(x)$ and $d^\dagger(x)$ by

$$\begin{aligned} d(x) &= e^{i\pi\xi(x)} J_-(x), \\ d^\dagger(x) &= e^{-i\pi\xi(x)} J_+(x). \end{aligned} \tag{1.10}$$

Here $J_i = \frac{1}{2}\sigma_i$, where σ_i are the Pauli matrices. Note that $J_\pm(x)$ commutes with $\xi(x)$ since $\xi(x)$ depends only on spins at $x' < x$. It can be shown that $d^\dagger(x)$ and $d(x)$ are fermion creation and annihilation operators that obey the anticommutation relations

$$\begin{aligned} \{d(x), d(x')\} &= 0, \\ \{d(x), d^\dagger(x')\} &= \delta(x, x'). \end{aligned} \tag{1.11}$$

The occupation number operator is

$$n(x) = d^\dagger(x)d(x) = J_3(x) + \tfrac{1}{2}. \tag{1.12}$$

Written in terms of these operators the Hamiltonian $H^{(1/2)}$ is

$$H^{(1/2)} = \tfrac{1}{2}\sum_x \left[d^\dagger(x)d(x) - \lambda \sum_\delta d^\dagger(x)d(x+\delta)\right]. \tag{1.13}$$

Thus $H^{(1/2)}$ can be diagonalized by Fourier transformation of $d(x)$. Let $\hat{d}(\theta)$ be defined for $-\pi \leq \theta \leq \pi$ by

$$\hat{d}(\theta)=\sum_x e^{ix\theta}d(x)\,; \tag{1.14}$$

the inverse transformation is

$$d(x)=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\,e^{-i\theta x}\hat{d}(\theta)\,. \tag{1.15}$$

Now $\hat{d}(\theta)$ is a momentum-space fermion operator, with anticommutation relation $\{d(\theta),d^\dagger(\theta')\}= =2\pi\delta(\theta-\theta')$. The Hamiltonian $H^{(1/2)}$ is

$$H^{(1/2)}=\tfrac{1}{2}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\,\hat{d}^\dagger(\theta)\hat{d}(\theta)E(\theta)\,, \tag{1.16}$$

where

$$E(\theta)=1-2\lambda\cos\theta\,. \tag{1.17}$$

Thus $H^{(1/2)}$ describes noninteracting fermions in a band of energy levels $E(\theta)$ labeled by lattice momentum θ.

The vacuum state, i.e., the ground state of $H^{(1/2)}$, is the state in which all of the fermion levels with negative energy are occupied. For $\lambda<\frac{1}{2}$ there are no negative-energy levels so the ground state is the fermion vacuum state. For $\lambda>\frac{1}{2}$ all levels with $|\theta|<\theta_0$ have negative energy where

$$\cos\theta_0=1/2\lambda\,. \tag{1.18}$$

Thus there are two phases separated by a phase transition at the critical value $\lambda_c=\frac{1}{2}$. In the low-λ phase ($\lambda\leq\frac{1}{2}$) there is a gap between the energies of the vacuum state and the first excited state; in the high-λ phase ($\lambda\geq\frac{1}{2}$) there is no energy gap.

The low-λ phase is analogous to the disordered phase of the *XY* model in Eq. (1). For $\lambda\leq\frac{1}{2}$ the vacuum state $|\Omega\rangle$ is an eigenstate of $J_3(x)$ for all x:

$$J_3(x)|\Omega\rangle=-\tfrac{1}{2}|\Omega\rangle\,. \tag{1.19}$$

Therefore, the (1,2) components of the spins $\vec{J}(x)$ are *maximally disordered*: Each spin is equally likely to be in the eigenstate of $\hat{n}\cdot\vec{J}$ with eigenvalue $+\frac{1}{2}$ as in that with eigenvalue $-\frac{1}{2}$ where $\hat{n}$ is any unit vector in the (1,2) plane. This statement can be expressed most exactly in terms of a probability distribution $P[\sigma(x)]$. Consider the set of states in which each spin is in an eigenstate of either J_1 or J_2; denote these states by $|\sigma_{1,2}(x)\rangle$ where $\sigma_1(x)$ or $\sigma_2(x)$ is the eigenvalue of $J_1(x)$ or $J_2(x)$. Then the vacuum probability distribution $P[\sigma(x)]$ is

$$P[\sigma(x)]=|\langle\sigma(x)|\Omega\rangle|^2\,. \tag{1.20}$$

The state $|\Omega\rangle$ is maximally disordered because $P[\sigma(x)]$ is independent of $\sigma(x)$. The operator J_3 plays the role of a disorder parameter in this discussion. In an eigenstate of J_3, the components J_1 and J_2 are disordered because of the uncertainty principle that follows from the noncommutativity of J_3 and $J_{1,2}$.

The nature of the vacuum state in the high-λ phase can first be discussed in terms of the expectation value of $J_3(x)$. It can be shown that

$$\begin{aligned}\langle J_3(x)\rangle&=-\tfrac{1}{2}\quad\text{for }\lambda\leq\tfrac{1}{2}\,,\\ \langle J_3(x)\rangle&=-\tfrac{1}{2}+\frac{\theta_0}{\pi}\quad\text{for }\lambda\geq\tfrac{1}{2}\,.\end{aligned} \tag{1.21}$$

In the high-λ phase the vacuum is not an eigenstate of $J_3(x)$ so some kind of partial ordering of the (1,2) spin components is possible. However, the fact that $\langle J_3\rangle$ is nonzero shows that the spins are not simply aligned in some direction in the (J_1,J_2) plane. The expectation value $\langle J_3(x)\rangle$ does tend to zero in the limit $\lambda\to\infty$. More can be learned by considering the correlation function

$$\begin{aligned}\Gamma_3(n)&\equiv\langle J_3(x)J_3(x+n)\rangle-\langle J_3(x)\rangle^2\\ &=-\frac{1}{(n\pi)^2}\sin^2 n\theta_0\,.\end{aligned} \tag{1.22}$$

Even for the limit $\lambda\to\infty$ there remains a correlation between the values of J_3 at different sites, so there is no long-range alignment of spins in the (J_1,J_2) plane.

It is also instructive to consider the action of a rotation operator on the vacuum state of each phase. Let $U(\alpha,x)$ be

$$U(\alpha,x)=e^{i\alpha J_3(x)} \tag{1.23}$$

which rotates the spin at x by the angle α in the (J_1,J_2) plane. The square of the expectation value of $U(\alpha,x)$ is

$$\begin{aligned}&|\langle U(\alpha,x)\rangle|^2=1\quad\text{for }\lambda\leq\tfrac{1}{2}\,,\\ &|\langle U(\alpha,x)\rangle|^2=1-2\,\frac{\theta_0}{\pi}\left(1-\frac{\theta_0}{\pi}\right)(1-\cos\alpha)\quad\text{for }\lambda\geq\tfrac{1}{2}\,.\end{aligned} \tag{1.24}$$

The first equation means that the low-λ vacuum is invariant under $U(\alpha,x)$; the spins are totally disordered so rotating the spin at x does not change the state. The second equation measures the degree of order in the high-λ vacuum. The overlap of the vacuum state $|\Omega\rangle$ with the state $U(\alpha,x)|\Omega\rangle$ decreases as the coupling constant λ increases but is nonzero unless α and λ are equal to π and ∞. This demonstrates that there is some kind of partial order of the (1,2) components of the spins.

The Hamiltonian $H^{(1/2)}$ is invariant under the global rotation of all spins by any angle α in the (J_1,J_2) plane. This transformation is produced by the operator $U(\alpha)\equiv\prod_x U(\alpha,x)$. The transformations $U(\alpha)$ form a continuous group with $0\leq\alpha\leq 2\pi$. It is the invariance of the theory under this group that is distinctive of the *XY* model. It can be argued that if $|\Omega\rangle$ is *not* invariant under the action of $U(\alpha)$ then there must be massless excitations. The former possibility is realized in the low-λ phase, and the latter is realized in the high-λ phase since there is no energy gap in that phase. The massless excitations are analogs of the spin waves of the critically ordered phase of the classical XY model.

The operator that creates topological kinks of the spins is a product over spin sites of $U(\alpha(x), x)$. The low-λ vacuum is invariant under this operator. Thus the low-λ phase might be described as a kink condensate.

So far the discussion has dealt with the disorder parameter $J_3(x)$. The natural way to discuss the spin order is directly in terms of correlation functions of the operator $J_\pm(x)$. First note that the expectation value $\langle J_\pm(x)\rangle$ vanishes for either phase; there is no long-range order. Now define the correlation function $\Gamma(n)$ by

$$\Gamma(n) = \langle J_+(x)J_-(x+n)\rangle . \tag{1.25}$$

In the low-λ phase $\Gamma(n)$ is equal to zero for all n since $J_-(x)$, like $d(x)$, annihilates the vacuum. This lack of correlation indicates again the complete disorder of the vacuum state for $\lambda \leq \frac{1}{2}$.

In the high-λ phase the value of $\Gamma(n)$ for nearest-neighbor sites is

$$\Gamma(1) = \frac{1}{\pi}\sin\theta_0 = \frac{1}{\pi}\left(1-\frac{1}{4\lambda^2}\right)^{1/2} . \tag{1.26}$$

The correlation is nonzero but small; for comparison the expectation value of $J_+(x)J_-(x+1)$ is equal to 1 in a state with all spins aligned in the (J_1, J_2) plane [i.e., a state in which each spin is in an eigenstate of $\hat{n}\cdot\vec{J}(x)$, $\hat{n}$ in the (1,2) plane].

The formula for $\Gamma(n)$ with arbitrary n and λ is not known to me. It can be shown that in the limit $\lambda \to \frac{1}{2}+$, $\Gamma(n)$ is asymptotically

$$\Gamma(n) \sim \frac{2}{\pi}(\lambda-\tfrac{1}{2})^{1/2} \quad \text{as } \lambda \to \tfrac{1}{2} \tag{1.27}$$

for all n. In the limit $\lambda\to\infty$ the first few values of $\Gamma(n)$ are

$$\Gamma(n;\lambda=\infty) = (2/\pi)^n c_n , \tag{1.28}$$

where

$$c_1=\tfrac{1}{2},\quad c_2=\tfrac{1}{2},\quad c_3=\tfrac{2}{3},$$
$$c_4=\tfrac{8}{9},\quad c_5=\tfrac{512}{405},\quad c_6=\tfrac{32768}{18225}. \tag{1.29}$$

The function $\Gamma(n)$ approaches zero as $n\to\infty$ since $\langle J_\pm(x)\rangle = 0$. The interesting question is how fast it approaches zero. Equations (1.27)–(1.29) suggest that $\Gamma(n)$ does not decrease particularly rapidly in the high-λ phase. This would be analogous to the behavior of the correlation function $\langle\vec{s}(\vec{x})\cdot\vec{s}(\vec{x}+\vec{r})\rangle$ in the critically ordered phase of the XY model.

A function that is simpler to compute than $\Gamma(n)$ is $\tilde{\Gamma}(n)$ defined by

$$\tilde{\Gamma}(n) = \langle d^\dagger(x)d(x+n)\rangle \tag{1.30}$$

whose value is

$$\tilde{\Gamma}(n) = 0 \quad \text{for } \lambda \leq \tfrac{1}{2},$$
$$\tilde{\Gamma}(n) = \frac{1}{n\pi}\sin n\theta_0 \quad \text{for } \lambda \geq \tfrac{1}{2}. \tag{1.31}$$

The slow decrease of $\tilde{\Gamma}(n)$ as $n\to\infty$ shows that there are long-range correlations between the excitations created by d and $d^\dagger$.

Up to now only spatial correlations of the spins have been described. Temporal correlations can also be considered, for example, the function

$$\tilde{\Gamma}(T) = \langle e^{HT}d(x)e^{-HT}d^\dagger(x)\rangle ; \tag{1.32}$$

the generator of translation in (Euclidean) time is $H^{(1/2)}$. The function $\tilde{\Gamma}(T)$ can be shown to be

$$\tilde{\Gamma}(T) = \int_{\theta_0}^{\pi}\frac{d\theta}{\pi}\, e^{-TE(\theta)} , \tag{1.33}$$

and in the limit $T\to\infty$ it approaches

$$\tilde{\Gamma}(T) \sim e^{-T(1-2\lambda)}(4\pi\lambda T)^{-1/2} \quad \text{for } \lambda \leq \tfrac{1}{2}, \tag{1.34a}$$

$$\tilde{\Gamma}(T) \sim [\pi(4\lambda^2-1)^{1/2}T]^{-1} \quad \text{for } \lambda \geq \tfrac{1}{2}. \tag{1.34b}$$

Thus $\tilde{\Gamma}(T)$ is exponentially decreasing as $T\to\infty$ if there is an energy gap between the ground state and first excited state, as in the disordered phase; and decreases as T^{-1} in the high-λ phase where there is no energy gap.

Power-law decrease of a correlation function is associated with critical phenomena. For this reason, points for which the Hamiltonian has massless excitations are identified with critical points of the system. In the two-dimensional Ising model there is only one isolated critical point, as will be discussed in Sec. II. But in the XY model $H^{(1/2)}$, all $\lambda \geq \frac{1}{2}$ correspond to critical points. Thus, the point $\lambda = \frac{1}{2}$ separates a disordered phase from a critically ordered phase, like the transition of the XY model. This is the basis for the statement that $H^{(1/2)}$ is a model of the phase transition of the XY model (1.1).

The phase transition at $\lambda = \frac{1}{2}$ can be studied further by analyzing the response of the spin system to external fields as a function of λ. For instance, the magnetic susceptibility to a constant magnetic field in the (J_1, J_2) plane diverges at $\lambda = \frac{1}{2}$.

It should also be remarked that the model described by $H^{(1/2)}$ can be viewed in a different way as a one-dimensional array of spins that interact with an external magnetic field in the J_3 direction. If the parameter λ is factored out of $H^{(1/2)}$ the magnitude of the external field is identified as $1/\lambda$. Such one-dimensional models are interesting in their own right and have been studied for arbitrary values of j.[11]

The x_2-continuum version of the two-dimensional Ising model presents an interesting contrast to the model considered here. The Ising Hamiltonian is a one-dimensional array of Ising variables in a transverse magnetic field[4] given by

$$H=\tfrac{1}{2}\sum_x \sigma_3(x)-\kappa\sum_x \sigma_1(x)\sigma_1(x+1)\,, \tag{1.35}$$

where again σ_3, σ_1 are Pauli matrices. This model is also soluble.[12] There is a phase transition at $\kappa=\frac{1}{2}$ between an ordered phase and a disordered phase. The energy gap vanishes only at the transition point $\kappa=\frac{1}{2}$, which is therefore identified as a critical point. This transition is a model of the phase transition of the two-dimensional Ising model.

In Sec. II a mixed Ising- and XY-spin model will be described to study the difference between the two transitions.

II. A MIXED XY AND ISING MODEL

The transition point of the XY model separates a disordered phase from a critically ordered phase. This aspect of the transition can be illustrated further with the Hamiltonian model of Sec. I by considering a mixed XY and Ising model. Let the Hamiltonian be

$$\begin{aligned} H=&\tfrac{1}{2}\sum_x \sigma_3(x)\\ &-\frac{\lambda}{4}\sum_x[\sigma_+(x)\sigma_-(x+1)+\sigma_-(x)\sigma_+(x+1)]\\ &-\kappa\sum_x \sigma_1(x)\sigma_1(x+1)\,, \end{aligned}\tag{2.1}$$

where σ_i are Pauli matrices and $\sigma_\pm=\sigma_1\pm i\sigma_2$. The parameters λ and κ will be taken to be positive. If $\kappa=0$ the Hamiltonian reduces to $H^{(1/2)}$ of Sec. I; if $\lambda=0$ it becomes the t-continuum Ising model equation (1.35).

The critically ordered phase of the XY model gives rise to a *line* of critical points in the mixed theory.

The Hamiltonian H is again solved by the Jordan-Wigner transformation (1.10). In terms of the momentum-space fermion operators $\hat{d}(\theta)$ the Hamiltonian is

$$\begin{aligned} H=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}\{&[1-2(\lambda+\kappa)\cos\theta]\hat{d}^\dagger(\theta)\hat{d}(\theta)\\ &-i\kappa\sin\theta[\hat{d}(\theta)\hat{d}(-\theta)+\hat{d}^\dagger(\theta)\hat{d}^\dagger(-\theta)]\}\,. \end{aligned}\tag{2.2}$$

This is brought to diagonal form by introducing new fermion operators $b(\theta)$ and $b^\dagger(\theta)$ defined by

$$\begin{aligned} b(\theta)&=\cos f(\theta)\hat{d}(\theta)-i\sin f(\theta)\hat{d}^\dagger(-\theta)\,,\\ b^\dagger(\theta)&=\cos f(\theta)\hat{d}^\dagger(\theta)+i\sin f(\theta)\hat{d}(-\theta)\,, \end{aligned}\tag{2.3}$$

where $f(\theta)$ is given by

$$\begin{aligned} \sin 2f(\theta)&=\frac{2\kappa\sin\theta}{\{[1-2(\lambda+\kappa)\cos\theta]^2+(2\kappa\sin\theta)^2\}^{1/2}}\,,\\ \cos 2f(\theta)&=\frac{1-2(\lambda+\kappa)\cos\theta}{\{[1-2(\lambda+\kappa)\cos\theta]^2+(2\kappa\sin\theta)^2\}^{1/2}}\,. \end{aligned}\tag{2.4}$$

If $\lambda=0$ this is just the transformation that diagonalizes the Ising Hamiltonian (1.35).[4,12]

It can be shown that H is

$$H=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}F(\theta)b^\dagger(\theta)b(\theta)+E_0\,, \tag{2.5}$$

where

$$F(\theta)=\{[1-2(\lambda+\kappa)\cos\theta]^2+(2\kappa\sin\theta)^2\}^{1/2}\,. \tag{2.6}$$

The c-number constant term E_0, which depends on λ and κ but not on the operators $b(\theta)$ and $b^\dagger(\theta)$, is a vacuum energy and can be dropped.

So again H reduces to noninteracting fermions in a band of energy levels $F(\theta)$ with $-\pi\le\theta\le\pi$. Here $F(\theta)$ is positive for all λ,κ. In contrast the energy $E(\theta)$ in Eq. (1.17) is negative for $|\theta|\le\theta_0$. The reason for this difference is that $b^\dagger(\theta)$, which is defined in such a way that it creates positive-energy states, creates *hole* states in the filled negative-energy sea; the ground state $|\Omega\rangle$ obeys $b(\theta)|\Omega\rangle=0$.

Critical points of the model, points where correlation functions do not decrease exponentially with separation, are identified as points at which there is a zero-energy excitation, as discussed

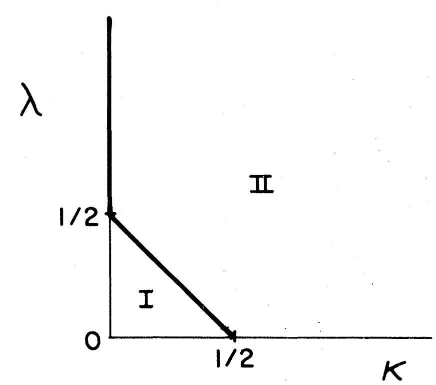

FIG. 1. Phase diagram of the mixed XY- and Ising-spin model. The λ axis is the pure XY model and the κ axis is the pure Ising model. Regions I and II are the disordered and ordered phases, respectively. The dark line is the line of critical points on which the energy gap is zero.

following Eq. (1.34). In this model there are two critical line segments in the (λ, κ) plane:

$$\lambda \leqslant \tfrac{1}{2} \text{ and } \kappa = \tfrac{1}{2} - \lambda \ , \tag{2.7a}$$

$$\lambda \geqslant \tfrac{1}{2} \text{ and } \kappa = 0 \ . \tag{2.7b}$$

The lattice momentum of the zero-energy excitation is at $\theta = 0$ for the line $\lambda + \kappa = \frac{1}{2}$, and is at $\cos\theta = 1/2\lambda$ for the line $\lambda \geqslant \frac{1}{2}$, $\kappa = 0$; for these values of λ, κ, and θ, the energy $F(\theta)$ vanishes.

Figure 1 is a (λ, κ) phase diagram of the system described by H. The critical lines are indicated. Regions I and II are disordered and ordered phases, respectively.

The t-continuum Ising model corresponds to the κ axis. There is a transition point at $\kappa = \frac{1}{2}$ separating a disordered phase $(\kappa < \frac{1}{2})$ from an ordered phase $(\kappa > \frac{1}{2})$. The energy gap vanishes only at the critical point $\kappa = \frac{1}{2}$.

The *XY* model of Sec. I corresponds to the λ axis. The transition at $\lambda = \frac{1}{2}$ is of a different nature than the Ising transition in that it separates a disordered phase $(\lambda < \frac{1}{2})$ from a phase with critical order $(\lambda > \frac{1}{2})$.

The qualitative distinction between these models is the existence of a continuous symmetry in the *XY* model, namely, rotation of all spins by any angle α, that is not present if $\kappa \neq 0$. The origin of the line of critical points along the λ axis with $\lambda \geqslant \frac{1}{2}$ is the restoration of this symmetry as $\kappa \to 0$.

This symmetry under a group of continuous transformations is the distinctive feature of the *XY* model. It is present in both the t-continuum *XY* Hamiltonian in Eq. (1.5) and the simplified spin-$\frac{1}{2}$ model $H^{(1/2)}$ in Eq. (1.7). This accounts for the similarity of the phase transitions in these two models.

The model considered here is soluble in the sense that the Hamiltonian can be diagonalized. However, certain important quantities are not easily computed. In particular, the order parameter $\langle \sigma_1(x) \rangle$ cannot be computed because the operator $\sigma_1(x)$ is a complicated nonlocal function of the fermion operators $b(\theta)$. It would be interesting to study this model in terms of the original spin variables $\sigma_1(x)$ using approximate methods that have been applied previously to similar t-continuum models, for example, strong-coupling expansions[13] or renormalization-group techniques.[14] The aim would be to reproduce the known critical lines in Fig. 1.

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