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Renormalization of strong-coupling expansions

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We consider the renormalization of strong-coupling expansions for the Green's functions of a quantum field theory. For the $g_0\phi^4$ theory that is an expansion in powers of $g_0^{-1/2}$. In less than three space-time dimensions, we find that conventional renormalization, that is mass, coupling-constant, and wave-function renormalization, does not lead to a divergence-free renormalized strong-coupling expansion. In three or more space-time dimensions there is a renormalized strong-coupling expansion which is finite term by term, but the resultant renormalized Green's functions are trivial.

I. INTRODUCTION

Expansions in inverse powers of the coupling constant of a Lagrangian field theory have been introduced by many authors.[1-8] These expansions are at the same time expansions in integrals of products of the inverse of the Feynman propagator. Thus the integrals are much more singular (ultraviolet divergent) than the integrals of ordinary Feynman-diagram weak-coupling perturbation theory. This raises the question of regularization and renormalization. Bender, Cooper, Guralnik, and Sharp[8] suggest regularizing by putting the (Euclidean $g_0\phi^4$) field theory on a d-dimensional lattice. This defines all of the integrals but leaves the problem of getting back to the continuum limit (lattice spacing $a\to 0$).

For $d \leq 2$, considerable progress has been made by BCGS, and subsequently by Bender, Cooper, Guralnik, Moreno, Roskies, and Sharp (BCGMRS).[9] They have proceeded by finding reorganizations of the unrenormalized expansion in $1/\sqrt{g_0}$ (derived by straightforward manipulation of the functional integral) into new series which can be extrapolated to $a=0$. For example, in $d=1$, which is the quantum-mechanical anharmonic-oscillator problem, the unrenormalized strong-coupling expansion, which can be extrapolated to $a=0$, is an expansion in powers of the dimensionless parameter $m_0^2 g_0^{-2/3}$. They have also obtained rapidly converging extrapolants (for $a\to 0$) in $d=2$, in which case a mass renormalization is required.

In this paper we address the problem which is the analog of the standard renormalization problem of the weak-coupling perturbation expansion. That is, given the "canonical" strong-coupling expansion in powers of $1/\sqrt{g_0}$ and defining mass, coupling-constant, and wave-function renormalization by renormalization conditions imposed on the Green's functions, can one choose the cutoff ($1/a$) dependence of Z and the bare parameters g_0, m_0 in such a way that after elimination of $1/\sqrt{g_0}$, m_0 for $1/\sqrt{g}$, m and multiplication by appropriate Z factors, the resultant renormalized strong-coupling expansion for the renormalized Green's functions has finite limits, term by term, for $a\to 0$?

We note that the coupling constant g_0 (or g) has dimension a^{d-4}, so that the terms in the inverse-coupling-constant expansions are more singular, as $a\to 0$, as d decreases—just the opposite of the situation in the usual weak-coupling perturbation expansion. In fact, the unrenormalized canonical strong-coupling expansion is an expansion in powers of $a^{d/2-2}/\sqrt{g_0}$ so that for $d<4$ each successive term is more singular, as $a\to 0$, than the preceding term. This suggests that the strong-coupling expansion is not renormalizable, in the conventional sense described above, for $d<4$. However, examination of the renormalization conditions shows that the critical dimension is actually three. For $d<3$, the conventional renormalization procedure does not lead to a strong-coupling expansion for the renormalized Green's functions which is finite term by term in the limit $a\to 0$. This is not inconsistent with the results obtained in Refs. 8 and 9. None of the series which they extrapolate to $a=0$ are conventional renormalizations of the canonical $1/\sqrt{g_0}$ expansion. We also note that the asymptotic (in g) limit of the conventional renormalization procedure is the double limit: first $a\to 0$, then $g\to\infty$. The considerations of BCGMRS, and also of Baker and Kincaid,[10] for $d>2$, involve the opposite order of limits, first asymptotic in g_0, then $a\to 0$.

For $3\leq d\leq 4$, we find that the (almost) conventional renormalization procedure can be carried out. After the renormalization has been carried out, the renormalization conditions plus simple dimensional considerations completely determine all of the renormalized Green's functions, computed to any finite order in the renormalized strong-coupling expansion. They are essentially trivial. We will discuss this result after we have derived it.

II. THE UNRENORMALIZED STRONG-COUPLING EXPANSION

The derivation of the unrenormalized strong-coupling expansion is given in detail in BCGS, with references to earlier related work. We reproduce enough of it here to establish the notation and the series to be renormalized. We deal with the ϕ^4 theory in d Euclidean dimensions. The field ϕ is the canonical, unrenormalized, field.

The functional integral for the generating functional is

$$Z[J]=\mathfrak{N}\int[d\phi]\exp\left\{-\int dx[\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}m_0^2\phi^2+\tfrac{1}{4}g_0\phi^4+J\phi]\right\}. \quad (2.1)$$

The n-point Schwinger function (unrenormalized, connected, Euclidean Green's function) is

$$S_{(n)}(x_1,\ldots,x_n)=\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)}\ln Z[J]\bigg|_{J=0}. \quad (2.2)$$

The functional integral is rearranged as

$$Z[J]=\mathfrak{N}\left[\exp\left(-\tfrac{1}{2}\int\int dx\,dy\frac{\delta}{\delta J(x)}G^{-1}(x-y)\frac{\delta}{\delta J(y)}\right)\right]\times\int[d\phi]\exp\left[-\int dx(\tfrac{1}{4}g_0\phi^4+J\phi)\right], \quad (2.3)$$

where

$$G^{-1}(x-y)=(m_0^2-\partial^2)\delta(x-y). \quad (2.4)$$

The functional

$$Q[J]=\mathfrak{N}\int[d\phi]\exp\left[-\int dx(\tfrac{1}{4}g_0\phi^4+J\phi)\right] \quad (2.5)$$

may be manipulated[8] into

$$Q[J]=\mathfrak{N}\exp\left[\delta(0)\int dx\ln F(g_0^{-1/4}\delta(0)^{-3/4}J(x))\right], \quad (2.6)$$

where $\delta(0)$ is the d-dimensional δ function at zero argument. This will be interpreted, with lattice regularization, as

$$\delta(0)=a^{-d}. \quad (2.7)$$

The function F is

$$F(\xi)=\int_{-\infty}^{\infty}dt\,e^{-t^4/4-\xi t}=\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{2^n}{(2n)!}\Gamma\left(\frac{n}{2}+\frac{1}{4}\right)\xi^{2n}=\frac{1}{\sqrt{2}}\Gamma(\tfrac{1}{4})(1+R\xi^2+\cdots),$$
$$R=\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}. \quad (2.8)$$

The functional $Q[J]$ may be normalized to $Q[0]=1$ by replacing $F(\xi)$ in (2.6) by

$$\bar{F}(\xi)=F(\xi)/F(0). \quad (2.9)$$

Then

$$Z[J]=\mathfrak{N}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})^n}{n!}\left(\int\int\frac{\delta}{\delta J}G^{-1}\frac{\delta}{\delta J}\right)^n\times\sum_{m=0}^{\infty}\frac{1}{m!}\left(\delta(0)\int\ln\bar{F}\right)^m=\mathfrak{N}\left\{1+\sum_{k=0}^{\infty}\frac{1}{g_0^{k/2}}B_k[J]\right\} \quad (2.10)$$

and

$$\ln Z[J]=C_0+\sum_{k=1}^{\infty}\frac{1}{g_0^{k/2}}C_k[J]. \quad (2.11)$$

BCGS give a set of diagrammatic rules which represent the terms in these expansions. The graphs have internal lines, which are associated with an inverse propagator G^{-1} and vertices at which any even number of lines come together. It follows from (2.3) and (2.6) that, for a fixed number of external lines, each additional internal line brings along an additional power of $1/\sqrt{g_0}$. The factor for a $2p$-line vertex is

$$\lambda_{2p}=\frac{1}{g_0^{p/2}}\frac{1}{[\delta(0)]^{3p/2-1}}L_{2p}, \quad (2.12)$$

where the L_{2p} are numerical coefficients defined by the Taylor expansion of $\ln\bar{F}(x)$,

$$\ln\bar{F}(\xi)=\sum_{k=1}^{\infty}\frac{L_{2k}}{(2k)!}\xi^{2k}. \quad (2.13)$$

There is an integral over the coordinates of all internal vertices, and each graph has a symmetry number. At this point the character of every term in the expansions is clear. There is a $(1/\sqrt{g_0})^k$ and an integral over products of inverse propagators. In momentum space, the inverse propagator is

$$\tilde{G}^{-1}(p)=p^2+m_0^2. \quad (2.14)$$

So, in momentum space, each term in the expansion is an integral over internal momenta q, of polynomials in the external p's and internal q's, with powers of m_0^2 and a^2 making up the required dimensions.

We now write out specifically some terms in the expansion of S_2 and S_4, which are the starting points for the renormalization procedure. The graphs for S_2, through order $1/g_0^2$ (three internal lines), are given by BCGS. We reproduce the first three orders in Fig. 1. The $1/\sqrt{g_0}$ expansion for S_2, including the $1/g_0^2$ terms (three lines) not shown in Fig. 1, is

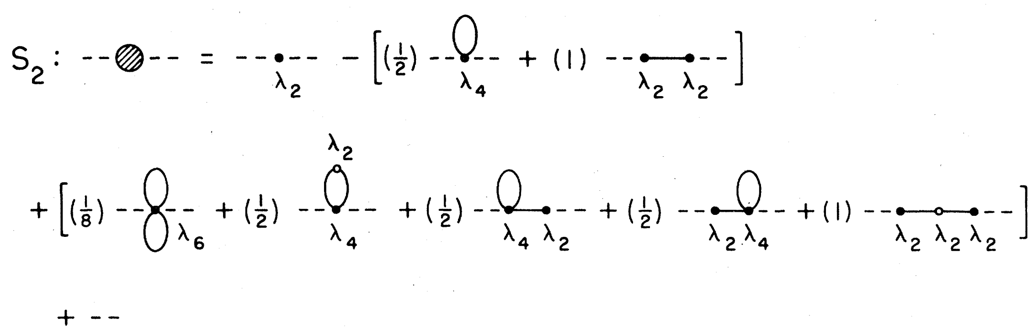

FIG. 1. Diagrams associated with the $1/\sqrt{g_0}$ expansion of the two-point function. Solid lines go with a factor $G^{-1}(z_i, z_j)$. The dashed lines go with a factor $\delta(x_i - z_k)$. The internal coordinates (z's) are integrated over. The λ_{2p} are the vertex factors, and the numbers in parentheses are the combinatorial factors for the diagrams.

$$S_2 = \lambda_2 1 - \left[\frac{\lambda_4}{2} A1 + \lambda_2^2 G^{-1}\right] + \left[\frac{\lambda_6}{8} A^2 1 + \lambda_4\lambda_2(\tfrac{1}{2}B1 + AG^{-1}) + \lambda_2^3 H\right]$$
$$- \left[\frac{\lambda_8}{48} A^3 1 + \frac{\lambda_6\lambda_2}{4}(AB1 + A^2G^{-1}) + \lambda_4^2(\tfrac{1}{6}L + \tfrac{1}{4}AB1 + \tfrac{1}{4}A^2G^{-1}) + \lambda_4\lambda_2^2(BG^{-1} + \tfrac{1}{2}C1 + \tfrac{3}{2}AH) + \lambda_2^4 I\right] + \cdots . \quad (2.15)$$

In (2.15), A, B, and C are constants (independent of $x-y$, or p^2) which do depend on the lattice constant a, and are divergent in the limit $a \to 0$. They are evaluated on the lattice by BCGS,

$$A = \frac{2d + m_0^2 a^2}{a^{2+d}} ,$$
$$B = \frac{(2d + m_0^2 a^2)^2 + 2d}{a^{4+d}} , \quad (2.16)$$
$$C = \frac{(2d + m_0^2 a^2)^3 + 6d(2d + m_0^2 a^2)}{a^{6+d}} .$$

1 is $\delta(x-y)$ in coordinate space and 1 in momentum space. G^{-1} is given by (2.4) and (2.14), and transcribed into the lattice as[8]

$$G^{-1}(x-y) \to \frac{1}{a^6}\left[-\sum_{\nu=1}^{d}(\delta_{+,\nu} + \delta_{-,\nu}) + (2d + m_0^2 a^2)\delta_0\right] . \quad (2.17)$$

Notice that the lattice Fourier transform of (2.17), in the limit[11] $a \to 0$, just gives (2.14). H and I are convolutions of G^{-1} in coordinate space, hence products in momentum space:

$$\tilde{H}(p) = (p^2 + m_0^2)^2 , \quad \tilde{I}(p) = (p^2 + m_0^2)^3 , \quad (2.18)$$

and

$$L(x,y) = [G^{-1}(x-y)]^3 . \quad (2.19)$$

After transcription onto the lattice, as (2.17), BCGS evaluate this as

$$L(x,y) = \frac{1}{a^{4+2d}} G^{-1}(x-y) + \frac{(2d + m_0^2 a^2)^3 - (2d + m_0^2 a^2)}{a^{6+2d}} \delta(x-y) . \quad (2.20)$$

The vertex factors are

$$\lambda_{2p} = \frac{a^{(3p/2-1)d}}{(\sqrt{g_0})^p} L_{2p} .$$

In momentum space, the higher-order terms in (2.14) include terms which are iterations of lower-order terms. These are most conveniently sorted out by writing

$$S_2 = 1/S_2^{-1} \quad (2.21)$$

and inverting the series in (2.15) to obtain

$$\tilde{S}_2^{-1} = \frac{1}{\lambda_2} + \left(\frac{\lambda_4}{2\lambda_2^2} A + \tilde{G}^{-1}\right) + \left[\left(-\frac{\lambda_6}{8\lambda_2^2} + \frac{\lambda_4^2}{4\lambda_2^3}\right) A^2 + \frac{\lambda_4}{2\lambda_2} B\right]$$
$$+ \left[\frac{\lambda_4^2}{6\lambda_2^2}\tilde{L} + \left(\frac{\lambda_8}{48\lambda_2^2} - \frac{\lambda_6\lambda_4}{8\lambda_2^3} + \frac{\lambda_4^3}{8\lambda_2^4}\right) A^3 + \left(\frac{\lambda_6}{4\lambda_2} - \frac{\lambda_4^2}{4\lambda_2^2}\right) AB + \frac{\lambda_4}{2} C\right] + \cdots . \quad (2.22)$$

The *structure* of (2.22) is

$$\tilde{S}_2^{-1} \sim \frac{1}{a^2}\left[\frac{\sqrt{g_0}}{a^{d/2-2}} + (p^2a^2, 1, m_0^2, a^2) + \frac{a^{d/2-2}}{\sqrt{g_0}}(1, m_0^2a^2; m_0^4a^4) + \left(\frac{a^{d/2-2}}{\sqrt{g_0}}\right)^2 \times (p^2a^2, 1, m_0^2a^2, m_0^4a^4, m_0^6a^6) + \cdots\right] . \quad (2.22')$$

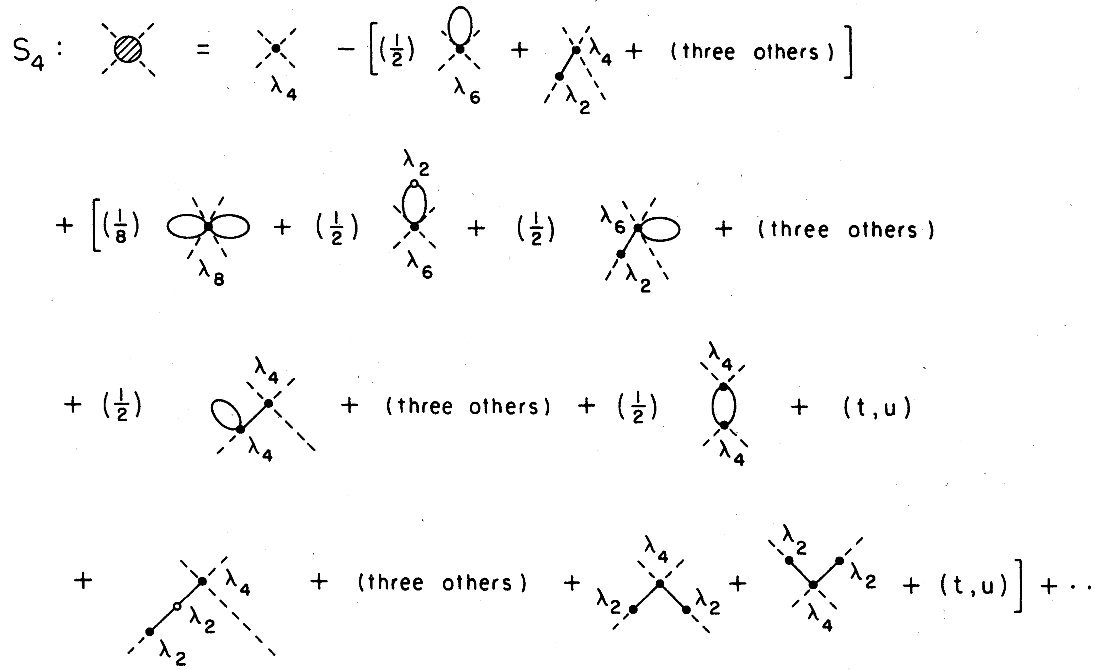

FIG. 2. Diagrams associated with the $1/\sqrt{g_0}$ expansion of the four-point function.

Terms including $(p^2a^2)^2$ first occur in one-particle-irreducible graphs with six internal lines.[12]

The 0-, 1-, 2-line graphs for S_4 are given in Fig. 2. After writing out the expression corresponding to each of the graphs in Fig. 2 and multiplying by the 0-, 1-, 2-line terms from (2.22) for each external leg, we arrive at the expansion for the amputated four-point function

$$-\Gamma_4=\prod_{i=1}^{4}S_2^{-1}(p_i)S_4 \tag{2.23}$$

$$=\left(\frac{\lambda_4}{\lambda_2^{\,4}}\right)\left\{1+\left[\left(-\frac{1}{2}\frac{\lambda_6}{\lambda_4}+2\frac{\lambda_4}{\lambda_2}\right)A\right]\right.$$
$$+\left[\left(\frac{1}{8}\frac{\lambda_8}{\lambda_4}-\frac{3}{2}\frac{\lambda_6}{\lambda_2}+\frac{5}{2}\frac{\lambda_4^{\,2}}{\lambda_2^{\,2}}\right)A^2+\left(\frac{1}{2}\frac{\lambda_6\lambda_2}{\lambda_4}-2\lambda_4\right)B\right.$$
$$\left.\left.+\frac{\lambda_4}{2}\left(K(s)+K(t)+K(u)\right)\right]+\cdots\right\}. \tag{2.24}$$

A and B are the constants given in (2.16). $K(p^2)$ is the Fourier transform of $(G^{-1}(x-y))^2$, evaluated on the lattice by BCGS as

$$K(p^2)=-\frac{1}{a^{2+d}}(P^2+m_0^{\,2})$$
$$+\frac{(2d+m_0^{\,2}a^2)+(2d+m_0^{\,2}a^2)^2}{a^{4+d}}. \tag{2.25}$$

The structure of (2.24) is

$$-\Gamma_4\sim g_0\left[1+\frac{a^{d/2-2}}{\sqrt{g_0}}(1,m_0^{\,2}a^2)\right.$$
$$\left.+\left(\frac{a^{d/2-2}}{\sqrt{g_0}}\right)^2(sa^2,ta^2,ua^2,1,m_0^{\,2}a^2,m_0^{\,4}a^4)+\cdots\right]. \tag{2.24'}$$

III. RENORMALIZATION

The renormalization of the two-point function is

$$\mathcal{S}_2=\frac{1}{Z}S_2 \tag{3.1}$$

with physical renormalization conditions

$$\mathcal{S}_2^{-1}(p^2=-m^2)=0,\quad \frac{d}{dp^2}\mathcal{S}_2^{-1}(p^2)\Big|_{p^2=-m^2}=1. \tag{3.2}$$

In (3.2) $p^2=-m^2$ because p_μ is Euclidean four-momentum. It is convenient to use, in place of (3.2), off-shell renormalization conditions

$$\mathcal{S}_2^{-1}(p^2=0)=M^2,\quad \frac{d}{dp^2}\mathcal{S}_2^{-1}(p^2)\Big|_{p^2=0}=1. \tag{3.2'}$$

Equations (3.1) and (3.2′) define the mass and wave-function renormalization. The coupling-constant renormalization is defined by a renormalization condition imposed on the amputated four-point function. Define

$$-\Gamma_4(p_1,p_2,p_3,p_4)=\prod_{i=1}^{4}(\mathcal{S}_2(p_i))^{-1}\hat{\mathcal{S}}_4(p_1,p_2,p_3,p_4), \tag{3.3}$$

where $\hat{S}_4$ is the Fourier transform of S_4 with the momentum-conservation δ function factored out. The renormalization of the amputated four-point function is

$$\tilde{\Gamma}_4 = Z^2 \Gamma_4 \tag{3.4}$$

and the renormalization condition is

$$(\tilde{\Gamma}_4)_{sp} = g\ . \tag{3.5}$$

It is convenient to choose the off-shell symmetry point $p_1 = p_2 = p_3 = p_4 = 0$.

Substituting (2.22) into (3.1) and (3.2′), we see that the wave-function renormalization has the form

$$\frac{1}{Z} = \frac{d}{dp^2} S_2^{-1}(p^2)\bigg|_{p^2=0} = 1 + \left(\frac{a^{d/2-2}}{\sqrt{g_0}}\right)^2 \frac{L_4^2}{6L_2^2} + \cdots\ . \tag{3.6}$$

So to this order (through three lines),

$$Z = 1 + \sum_k \left(\frac{a^{d/2-2}}{\sqrt{g_0}}\right)^k Z_k\ , \tag{3.7}$$

$$Z_1 = 0,\quad Z_2 = -\frac{L_4^2}{6L_2^2}\ . \tag{3.8}$$

When one goes to higher orders in (2.22′), one will encounter terms

$$\left(\frac{a^{d/2-2}}{\sqrt{g_0}}\right)^{L-1} (p^2a^2)^R (m_0^2a^2)^N\ , \tag{3.9}$$

where L is the number of internal lines. With L inverse propagators and R powers of p^2, one would expect values of N from zero to $L-R$. Then

$$Z = 1 + \sum_{k=0} Z_k^{(0)} x^k + \eta \sum_{k=1} Z_k^{(1)} x^k + \cdots + \eta^N \sum_{k=N} Z_k^{(N)} x^k + \cdots\ , \tag{3.7′}$$

where

$$x = \frac{a^{d/2-2}}{\sqrt{g_0}}\ ,\quad \eta = m_0^2 a^2\ . \tag{3.10}$$

We note that all of the one-particle-irreducible S_2 diagrams that we have calculated, using the lattice techniques of BCGS, have the property that the term with $R=1$ and the maximum value of $N = L-1$ is absent. [For example, see Eqs. (2.19) and (2.20).] If this is true in general, then in the term in (3.7′) with η^N, the sum on k starts with $k > N$. Then $Z = 1 + O(x)$. If it is not true in general, then $Z = \text{constant} + O(x)$.

Again substituting (2.22′) into (3.1) and (3.2′), the mass renormalization has the form

$$\begin{aligned} m &= M^2 a^2 \\ &= a^2 Z S_2^{-1}(p^2=0) \\ &= \left(1 + \sum_{k=0} Z_k^{(0)} x^k + \eta \sum_{k=1} Z_k^{(1)} x^k + \cdots\right) \\ &\quad \times \left(\sum_{k=-1} \alpha_k^{(0)} x^k + \eta \sum_{k=0} \alpha_k^{(1)} x^k + \eta^2 \sum_{k=1} \alpha_k^{(2)} x^k + \cdots\right)\ , \end{aligned} \tag{3.11}$$

where the $Z_k^{(n)}$ and $\alpha_k^{(n)}$ are finite numerical coefficients (functions of the L_{2p}). Substituting (2.24′) into (3.4) and (3.5), the coupling-constant renormalization has the form

$$\begin{aligned} \frac{1}{\xi^2} &= g a^{4-d} \\ &= a^{4-d} Z^2 (\Gamma_4)_{sp} \\ &= \frac{1}{x^2}\left(1 + \sum_{k=0} Z_k^{(0)} x^k + \eta \sum_{k=1} Z_k^{(1)} x^k + \cdots\right) \\ &\quad \times \left(\sum_{k=0} \beta_k^{(0)} x^k + \eta \sum_{k=1} \beta_k^{(1)} x^k + \cdots\right)\ . \end{aligned} \tag{3.12}$$

Now if we can determine the dependence on a of g_0, m_0 (x,η) so that (3.11) and (3.12) are satisfied as $a \to 0$, for fixed g, M, then we expect that after eliminating g_0, m_0 from (2.22) and (2.24) in favor of g, M and multiplication by the appropriate Z's, the resulting renormalized S_2^{-1}, $\tilde{\Gamma}_4$ will have finite limits for $a \to 0$. In the limit $a \to 0$, $m = M^2 a^2$ goes to zero, and for $d < 4$, so does $1/\xi^2 = g a^{d-4}$. (We will discuss the case $d = 4$ separately.) After truncation at any finite order (maximum power of the expansion parameter x), we can think of "solving" (3.11) and (3.12) in three steps. First solve (3.11) for $\eta(x,m)$. Substitute the resulting $\eta(x,m)$ into (3.12) and solve for $x(\xi,m)$. Substitute $x(\xi,m)$ back into $\eta(x,m)$ to obtain $\eta(\xi,m)$. Already at the first step we learn something significant about the strong-coupling expansion. Since $\alpha_{-1}^{(0)} = 1/L_2 \neq 0$, Eq. (3.11) will have a solution in the (bare) strong-coupling limit $x \to 0$ only if

$$\eta \sim 1/x\quad (\text{for } x \to 0)\ . \tag{3.13}$$

This marks a breakdown of the (bare) strong-coupling expansion. For $\eta \sim 1/x$, there are an infinite number of terms in (3.11), and in (3.12), which are of order x^{-1}, an infinite number of order x^0, etc. To truncate Eqs. (3.11) and (3.12) at some finite number of terms, we need a new definition of order, which we take to be the number of internal lines of the associated graph. This is the same as the formal expansion in powers of $1/\sqrt{g_0}$. For example, Eq. (2.15) includes all terms with from zero to three internal lines, and Eq. (2.24) includes all terms with from zero to two internal lines.

We can now discuss the behavior of x, η in the limit of $a \to 0$, as determined by (3.11) and (3.12) truncated at any finite order, as defined above.

For $a \to 0$, we have $m \to 0$, and (3.11) becomes an equation to determine $\eta(x) = \eta(x, m=0)$. When this result is substituted into (3.12), with $1/\xi^2 = 0$, the resulting equation is an equation to determine x, in the limit $a=0$. We can make this more explicit as follows: In any finite order, (3.11) is an algebraic equation for η, depending on x, but not on m, in the limit $a=0$. Thus it will have some number of solutions, depending on x, $\eta_r(x)$, each of which behaves as $1/x$ for $x \to 0$. Then we may expand $x\eta(x)$ in a power series in x, and truncate it at the appropriate finite order. Now multiply Eq. (3.12) by x^2, set $1/\xi^2 = 0$ (the limit $a=0$) and substitute the polynomial from $x\eta_r(x)$ into the truncated (3.12). Multiply out the polynomials and again truncate at the indicated order. The result is another algebraic equation, for x, $0 = P_{N,r}(x)$, with solutions $x_{r,s}$, which are independent of m, ξ. Thus the various "solutions" x, η of the renormalization conditions (3.11) and (3.12) all have the property that they have finite limits, independent of ξ, m, for $a \to 0$. For $a \neq 0$, x, η do depend on ξ, m in just the way required to satisfy the renormalization conditions (3.11) and (3.12). The multiplicity of solutions of the renormalization conditions [arising from the breakdown of the strong-coupling expansion implied by (3.13)] is not a problem for the renormalization program. When we construct the renormalized Green's functions we require only the knowledge that x, η have finite limits for $a \to 0$, and that they satisfy the renormalization conditions (3.11) and (3.12).

In four dimensions, the only difference is that $1/\xi^2 = g a^{4-d} = g$ is independent of a; hence the (finite) limits of x and η as $a \to 0$ do depend on ξ. The asymptotic limit ($g_0 \to \infty$, $x \to 0$) of (3.12) gives, in this case,

$$\frac{1}{g} \sim (\text{constant}) \frac{1}{g_0} \quad (\text{independent of } a \text{ for } d=4). \qquad 1/g_0 \to 0 \tag{3.14}$$

The constant in (3.14) can be evaluated to any order in internal lines, but this has no numerical significance because of the circumstance of η being of order $1/x = \sqrt{g_0}$. It is striking that the coupling-constant renormalization is finite in $d=4$ in the strong-coupling expansion. This happens because the strong-coupling expansion involves only integrals over polynomials,[11] hence it never produces any logarithms. For the same reason, the dimensionless wave-function renormalization (3.7′) is finite in the strong-coupling expansion.

Now that we have determined the behavior as $a \to 0$ of the solutions (x, η) of the renormalization conditions (3.11) and (3.12), let us use these results to determine the renormalized Green's functions to any finite order (in the number of internal lines). Start with $\mathcal{S}_2$. The generalization of (2.22′) to any finite order is

$$S_2^{-1}(p^2) = \sum_{j=0}^{j_{\max}} A_j(x,\eta)(p^2)^j (a^2)^{j-1}. \tag{3.15}$$

The $A_j(x,\eta)$ are multinomials in x, η. Since x, η have finite limits as $a \to 0$, so do the $A_j(x,\eta)$. The renormalization conditions (3.2′) determine

$$A_1 = 1/Z, \tag{3.16a}$$

$$\frac{A_0}{A_1}\frac{1}{a^2} = M^2. \tag{3.16b}$$

[The η which solve (3.11) determine $A_0(x,\eta) \sim m \sim a^2$ for $a \to 0$.] Thus

$$\mathcal{S}_2^{-1}(p^2) = Z S_2^{-1}(p^2) = \lim_{a\to 0} \sum_j \frac{A_j}{A_1}(p^2)^j (a^2)^{j-1} = M^2 + p^2. \tag{3.17}$$

For $\tilde{\Gamma}_4$, the generalization of (2.24′) to any finite order is

$$\Gamma_4(s_{kl}) = a^{d-4}\frac{1}{x^2}\sum A_{j_1\cdots j_\sigma}(x,\eta) \times (s_{k_1 l_1})^{j_1}\cdots(s_{k_\sigma l_\sigma})^{j_\sigma}(a^2)^j, \tag{3.18}$$

where

$$s_{kl} = p_k \cdot p_l, \quad j_1 + \cdots + j_\sigma = j, \tag{3.19}$$

and the $A_{j_1\cdots j_\sigma}(x,\eta)$ are monomials in x, η, hence they have finite limits for $a \to 0$. The renormalization conditions (3.4) and (3.5) determine

$$Z^2 a^{d-4}\frac{1}{x^2}A_{0\cdots 0}(x,\eta) = g. \tag{3.20}$$

[The x, η which solve (3.11) and (3.12) determine $A_{0\cdots 0}(x,\eta) \sim 1/\xi^2 \sim a^{4-d}$ for $a \to 0$.] Thus

$$\tilde{\Gamma}_4(s_{kl}) = Z^2\Gamma_4(s_{kl}) = \begin{cases} g & \text{for } d>2, \\ g + \left[\frac{1}{x^2}\sum_{k,l} A_{1,kl}(x,\eta)\right]_{a=0} s_{kl} & \text{for } d=2, \\ \infty & \text{for } d<2, \end{cases} \tag{3.21}$$

i.e., the renormalization of the strong-coupling expansion fails for $d<2$. Recall the earlier remark that the ultraviolet singularities of the strong-coupling expansion became more severe as the dimension decreases. The coupling-constant renormalization has eliminated the leading divergence (as $a \to 0$) from $\tilde{\Gamma}_4$, but for $d<2$ there are nonleading divergences, the $a^{d-4}\, s_{kl} a^2$ terms, which are not removed by the coupling-constant renormalization. If we go on to consider the six-point function, for which there is no renormalization condition in the ϕ^4 theory, we find that the renormalization fails for any $d<3$. From the first few terms in the (bare) strong-coupling expansion and simple dimensional considerations we determine

$$\Gamma_{2p} \sim a^{p(d-2)-d} \frac{1}{x^p} \times \sum A^{(p)}_{j_1 \cdots j_\sigma}(x,\eta)(s_{k_1 l_1} a^2)^{j_1} \cdots (s_{k_\sigma l_\sigma} a^2)^{j_\sigma} \quad (3.22)$$

and

$$\tilde{\Gamma}_{2p} = Z^p \Gamma_{2p} . \quad (3.23)$$

Since Z is a finite function of x, η, it is sufficient to consider (3.22). We already know that renormalization fails for $d<2$. For $d>2$, the most singular term, as $a \to 0$, is the smallest p for which there is no renormalization condition, i.e., Γ_6. We see that

$$\tilde{\Gamma}_6 \underset{a\to 0}{\sim} a^{2d-6} = \begin{cases} 0 & \text{for } d>3 , \\ \text{constant} & \text{for } d=3 , \\ \infty & \text{for } d<3 . \end{cases} \quad (3.24)$$

Finally, in four dimensions

$$\tilde{\Gamma}_{2p}(s_{kl}) \underset{a\to 0}{\to} 0 \quad \text{for all } 2p>4 \quad (d=4). \quad (3.25)$$

Equations (3.17), (3.21), and (3.25) are the bases of the assertion that, to any finite order (in internal lines), the renormalized strong-coupling expansion gives essentially trivial Green's functions in four dimensions (in fact, for any $d \geq 3$).

IV. DISCUSSION

We have studied the technical question: Can one carry out the renormalization of the canonical unrenormalized strong-coupling expansion for the Green's functions of the ϕ^4 theory and obtain a renormalized strong-coupling expansion which is finite order by order as the cutoff is removed (lattice spacing $a \to 0$)? The problem is complicated, compared to the analogous problem of renormalization of the weak-coupling expansion, by the circumstances that the two-point function in lowest order in the strong-coupling expansion is not the free two-point function and, in less than four dimensions, the degree of ultraviolet singularity of the terms of the canonical strong-coupling expansion increases as the order increases. We have found that the standard renormalization conditions (for mass, wave-function, and coupling-constant renormalization) do not lead to a finite renormalized strong-coupling expansion in less than three (Euclidean) dimensions. In three or more dimensions the renormalization procedure does lead to an expansion which is finite order by order, where the order is defined by the number of internal lines in the associated diagrams. However, the resultant renormalized Green's functions, to any finite order, are trivial.

We believe that these results have a certain intrinsic interest, but we conclude that the conventionally renormalized strong-coupling expansion does not tell us much (or anything?) about the ϕ^4 theory. In low numbers of dimensions ($d \leq 2$), the failure of renormalizability of the strong-coupling expansion is completely misleading, since the ϕ^4 theory is known to exist. In one Euclidean dimension, the unrenormalized theory is finite and solvable, and it is known that the true strong-coupling expansion is an expression in powers of the dimensionless parameter $m_0^2 g_0^{-2/3}$. Thus, in this case, the question of the renormalizability of the canonical strong-coupling expansion (which is in powers of $g_0^{-1/2}$) is irrelevant.[13] In higher dimensions ($d \geq 3$), our results superficially resemble the results of BCGRS[9] and of Baker and Kincaid[10] (see also the earlier work of Wilson[14]) who claim numerical evidence for the triviality of ϕ^4 theory in four dimensions. However, the two statements are really very different. BCGRS and Baker and Kincaid do not carry out coupling-constant renormalization. They attempt to compute g, defined as $(\tilde{\Gamma}_4)_{\text{sp}}$, in an asymptotic limit, first $g_0 \to \infty$, then $a \to 0$, and find some numerical evidence that the renormalized coupling constant g goes to zero in this limit. In our study of the renormalization of the canonical strong-coupling expansion, the renormalized coupling constant g is an input fixed parameter, and the corresponding asymptotic limit is first $a \to 0$, then $g \to \infty$. From the standpoint of the renormalized theory, the analogous calculation would be to compute

$$\lim_{a\to 0} g_0(g, M; a)_{g, M \text{ fixed}}$$

and to see if the limit g_0 had some unacceptable value, i.e., negative or complex. In fact, the renormalized strong-coupling expansion cannot be used for this because every term in the expansion with order defined by the number of internal lines is of the same order in $1/\sqrt{g}$, so there is no numerical significance to any finite order. Finally, we remark that the triviality of the renormalized strong-coupling Green's functions (in $d \geq 3$) is a direct result of an essentially unphysical feature of the strong-coupling expansion—to any order of approximation, in the unrenormalized theory with finite cutoff, or in the renormalized expansion with $1/a \to \infty$, none of the amputated Green's functions have any imaginary part. In the limit $a \to 0$ they are approximated by polynomials in the momenta. (For finite a they are entire functions of the momenta.[11])

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[12]For finite a, Euclidean rotationally noninvariant terms involving $p_1^4 + p_2^4 + \cdots + p_d^4$ also occur. For dimensional considerations these are not different than the rotationally invariant $(p^2)^2$, and in the limit $a \to 0$ the Euclidean rotational invariance must be restored. See the discussion in Ref. 8. Henceforth, we shall ignore this complication.
[13]The successful numerical results of BCGS for $d=1$ are obtained by Pade-type extrapolation of the singular terms of the unrenormalized $m_0^2 g_0^{-2/3}$ expansion.
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