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Functional integration through inverse scattering variables. II

H. J. de Vega
Laboratoire de Physique Théorique et Hautes Energies, Université Pierre et Marie Curie, Tour 16-1er étage, 4, place Jussieu, 75230 Paris Cedex 05, France

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We continue to develop the method of functional integration over spectral variables introduced in a previous paper. The usual functional integration variables are taken as potentials of an auxiliary linear problem whose spectral data become new integration variables. We deal in this paper with the quantum pendulum and the one- and two-dimensional anharmonic oscillators. We find the functional integration measure and the integration bounds for the spectral variables associated with these quantum systems. This integration measure is valid semiclassically. We compute with it the functional integral for the systems mentioned before. We get in this way integral representations for the ground-state energies. These integral representations turn out to be exact in the semiclassical limit in all cases. They possess the correct large-order behavior of the perturbative expansions (both in Borel-summable and in the non-Borel-summable cases). They also exhibit correctly the tunnel-effect features.

I. INTRODUCTION

In a previous paper[1] we developed inverse scattering techniques to compute functional integrals of the form

$$Z(\hbar) = \int \mathfrak{D}v \exp\left(-\frac{1}{\hbar} S[v(\cdot)]\right). \quad (1.1)$$

Basically, we considered an auxiliary linear problem where the original integration variable $v(x)$ is considered as a potential. We take the spectral data of this linear problem as new integration variables. This change of variables can be recast as a canonical transformation in several cases.[2-5] It is profitable only if it completely separates the action S. This means that one must look for an auxiliary linear system whose spectral variables (SV) completely separate the action. One still needs the integration measure in the SV and the integration bounds. In I, we explicitly found the integration measure (in the semiclassical regime) and the integration bounds for the SV of the Schrödinger equation. (That is the linear problem associated with the Korteweg–De Vries equation.)

In the present paper we find the integration measure and the integration bounds in terms of spectral variables for three systems: the pendulum and the one- and two-dimensional anharmonic oscillators. Using these inverse scattering variables we compute the functional integral for these three quantum systems.

As is well known the generating function for the pendulum reads like Eq. (1.1) with

$$S[v] = \int_{-\infty}^{+\infty} dx\left[\frac{1}{2}\left(\frac{dv}{dx}\right)^2 + 1 - \cos v(x)\right]. \quad (1.2)$$

Here x stands for the imaginary time. We write this action S in terms of the SV of the Dirac-type linear problem associated with the sine-Gordon equation. S completely separates in such variables. The measure and integration bounds we find for these spectral variables can be used for the two-dimensional sine-Gordon field theory. Here we focus our attention on the quantum pendulum and we compute its ground-state energy (lower edge of the first allowed energy band). Our final result reads

$$E_G(g) = E_I(g) + E_D(g) + E_C(g). \quad (1.3)$$

Here

$$E_I(g) = -\frac{4}{g}\int_{-\infty}^{+\infty} dx \cosh x \exp\left(-\frac{8}{g}\cosh x\right) F_1(x,g) \quad (1.4)$$

and

$$E_D(g) = -\frac{2^4}{\pi g^2}\int_{-\infty}^{+\infty} dy \cosh y \times\int_0^{\pi/2} d\theta \sin\theta(1+\sin\theta) \times \exp\left(-\frac{16}{g}\cosh y \sin\theta\right) F_2(y,\theta,g), \quad (1.5)$$

where

$$F_1(0,g) = 1 + O(g), \quad F_2(\theta,0,g) = 1 + O(g). \quad (1.6)$$

The first term $E_I(g)$ correctly gives the nonanalytic part of $E_G(g)$ as a function of g for small g [see Eqs. (2.23)–(2.25)]. That is the tunnel-effect contribution. In our approach $E_I(g)$ comes out from the canonical spectral variables (CSV) associated with soliton configurations. The second term $E_D(g)$ accounts from the CSV associated with doublet configurations. $E_D(g)$ is analytic at

$g=0$, and the large orders of its expansion in powers of g coincide with the known behavior for $E_G(g)$.[6] We recall that this series is not Borel summable.

In Sec. III we consider the one- and two-dimensional anharmonic oscillators. Besides its own interest, it could be noted that the two-dimensional anharmonic oscillator is equivalent to a hydrogen atom in a uniform electric field (Stark effect).[7]

We compute $Z(g)$ for the anharmonic oscillator by using the CSV of the Dirac-type linear problem associated with the nonlinear Schrödinger equation. We do that with the aid of the integration measure and integration bounds appropriate to these CSV. Finally, we get the ground-state energy. For the one-dimensional anharmonic oscillator, we find

$$E_G^{(1)}(-h)=-\frac{4i\sqrt{2}\,\mu^3}{\pi}\int_0^\infty dp\,e^{-\beta(p,-p)}D^{(1)}(p)+F^{(1)}(h) \tag{1.7}$$

and for the two-dimensional one

$$E_G^{(2)}(-h)=-\frac{16\sqrt{2}\,\mu^5}{\pi}\int_{\mathrm{Im}\,p\geq 0} d^2p\,e^{-\beta(p,p^*)}D^{(2)}(p,p^*)+F^{(2)}(h). \tag{1.8}$$

Here $-h\equiv g<0$ stands for the anharmonic coupling constant and

$$\beta(p,p^*)=-i\mu^2[p-p^*+\tfrac{4}{3}h^2\mu^4(p^3-p^{*3})]. \tag{1.9}$$

In conclusion, we have obtained integral representations for the energy and/or for its imaginary part (for negative coupling) via functional integration over CSV. The integral representations are exact for small coupling. In this regime, we obtain the correct nonanalytic exponent and the correct pre-exponential factor with the correct power of the coupling constant.

This suggests that we are in presence of the leading term of a new kind of perturbative expansion. High-order terms can in principle be computed by working on a system made discrete by inverse spectral techniques.

II. THE QUANTUM PENDULUM (MATHIEU'S EQUATION)

We consider in this section the quantum pendulum. As is well known its Lagrangian can be written as

$$\mathcal{L}(x)=\frac{1}{g}\left[\frac{1}{2}\left(\frac{du}{dx}\right)^2+1-\cos u(x)\right], \tag{2.1}$$

where x is the imaginary time. The generating function reads, as usual,

$$Z(g)=\int \mathfrak{D}u\,\exp\left(-\int \mathcal{L}\,dx\right). \tag{2.2}$$

We can add a supplementary integration on a variable $\pi(x)$ in order to build a canonical structure:

$$Z(g)=Z_0^{-1}\int \mathfrak{D}u\,\mathfrak{D}\pi\,\exp(-S_{\rm eff}[\pi,u]), \tag{2.3}$$

$$S_{\rm eff}[\pi,u]=\frac{1}{2g}\int dx\,\pi(x)^2+\int dx\,\mathcal{L}(x). \tag{2.4}$$

Here Z_0 is such that $Z(0)=e^{-(2L)/2}$. The Poisson brackets are defined by

$$\{\alpha,\beta\}=\int dx\left[\frac{\delta\alpha}{\delta u(x)}\frac{\delta\beta}{\delta\pi(x)}-\frac{\delta\beta}{\delta u(x)}\frac{\delta\alpha}{\delta\pi(x)}\right]. \tag{2.5}$$

We compute now the functional integral (2.3) for $Z(g)$ by using the CSV of the sine-Gordon model.[3] We recognize that $S_{\rm eff}$ has the same form as the Hamiltonian of the sine-Gordon field if we identify our imaginary time x with the spatial coordinate in the sine-Gordon model. This is not surprising because the quantum pendulum is the sine-Gordon theory at fixed time.

The CSV of the sine-Gordon field follows from the spectral data of the following linear problem[3]:

$$\frac{i\partial\phi}{\partial x}=\left[-\frac{i}{4}\sigma_3\left(\pi+\frac{du}{dx}\right)+\frac{1}{16\lambda}(\sigma_2\cos u+i\sigma_1\sin u)-\lambda\sigma_2\right]\phi(x). \tag{2.6}$$

Here σ_1, σ_2, and σ_3 are the Pauli matrices and $\phi(x)$ is a two-component spinor. Equation (2.6) has Jost-type matrix solutions $\psi_\pm$ with asymptotic behavior

$$\psi_\pm(x,\lambda)\underset{x\to\pm\infty}{\sim}\begin{pmatrix}e^{i\sigma x} & e^{-i\sigma x}\\ ie^{i\sigma x} & -ie^{-i\sigma x}\end{pmatrix},$$

where $\sigma\equiv\lambda-1/(16\lambda)$. The solutions $\psi_+(x)$ and $\psi_-(x)$ are linearly connected by the transition matrix $T(\lambda)$:

$$\psi_+(x,\lambda)=\psi_-(x,\lambda)T(\lambda). \tag{2.7}$$

Here

$$T(\lambda)=\begin{pmatrix}a(\lambda) & -b(\lambda)^*\\ b(\lambda) & a(\lambda)^*\end{pmatrix} \tag{2.8}$$

with $|a(\lambda)|^2+|b(\lambda)|^2=1$. It is easy to show that $a(\lambda)$ is analytic in $\mathrm{Im}\lambda>0$ and that its zeros in that region are the eigenvalues of the linear system (2.6). These eigenvalues are purely imaginary or lie in pairs symmetrically to the imaginary axis.

Finally the CSV for the sine-Gordon field can be written as

$$P(\lambda)=-\frac{8}{\pi g\lambda}\ln|a(\lambda)|,\quad Q(\lambda)=-\arg b(\lambda),\qquad 0\leq\lambda<+\infty \tag{2.9}$$

$$p_l=\frac{1}{g}\ln(4\kappa_l),\quad q_l=8\ln|c_l|,$$
$$\xi_k=\frac{4}{g}\ln|4\lambda_k|,\quad \eta_k=4\ln|d_k|, \tag{2.10}$$

$$\theta_k=\frac{4}{g}\arg\lambda_k,\quad \varphi_k=-4\arg d_k. \tag{2.11}$$

Here the $i\kappa_l (l=1,\ldots,n_1)$ are purely imaginary zeros of $a(\lambda)$ in $\mathrm{Im}\lambda>0$. The $\lambda_k=\pm S_k+it_k$ with $S_k>0<t_k$ $(k=1,\ldots,n_2)$ are pairs of complex zeros of $a(\lambda)$. The configurations associated with them are solitons and doublets (breathers), respectively. Finally

$$c_l\equiv -ib(i\kappa_l),\quad d_k\equiv -ib(\lambda_k). \tag{2.12}$$

That is, c_l and d_k are normalization coefficients for the eigenfunctions of the linear problem (2.6). The asymptotic behavior of the first column of $\psi_+(x)$ at an eigenvalue reads

$$\psi_+(x)_1\underset{x\to+\infty}{\sim}\begin{pmatrix}1\\ i\end{pmatrix}\exp\left[i\left(\lambda-\frac{1}{16\lambda}\right)x\right],$$
$$\psi_+(x)_1\underset{x\to-\infty}{\sim}\begin{pmatrix}1\\ -i\end{pmatrix}b(\lambda)\exp\left[-i\left(\lambda-\frac{1}{16\lambda}\right)\right]x, \tag{2.13}$$

where $\lambda=\lambda_k$ or $\lambda=i\kappa_l$.

The CVS (2.9)–(2.11) are useful to compute the functional integral for $Z(g)$ [Eq. (2.3)] because S_{eff} separates completely when expressed in terms of them:

$$S_{\mathrm{eff}}[\pi,u]=\tfrac{1}{2}\int_0^\infty d\lambda\,P(\lambda)\left(4\lambda+\frac{1}{4\lambda}\right)+\frac{8}{g}\sum_{l=1}^{n_2}\cosh(gp_l)+\frac{16}{g}\sum_{k=1}^{n_2}\sin(\theta_k)\cosh(\tfrac{1}{4}g\xi_k). \tag{2.14}$$

The integration bounds for the CSV corresponding to

$$-\infty<u(x)<+\infty$$

for all finite x and

$$u(\pm\infty)=0\pmod{2\pi}$$

follows from Eqs. (2.9)–(2.11):

$$0\leq P(\lambda)<\infty,\quad |Q(\lambda)|<2\left(\lambda-\frac{1}{16\lambda}\right)L, \tag{2.15}$$

$$-\infty<p_l<+\infty,\quad -\infty<\xi_k<+\infty, \tag{2.16}$$

$$0\leq\theta_k<2\pi/g. \tag{2.17}$$

The variables q_l and η_k can take in principle any real value from $-\infty$ to $+\infty$. If we make a translation $x\to x+X$ the only CSV that change are $Q(\lambda)$, q_l, φ_k, and η_k:

$$q_l\to q_l+8X\cosh(gp_l),$$
$$\eta_k-i\varphi_k\to\eta_k-i\varphi_k-4iXe^{i\theta_k}\cosh(\tfrac{1}{4}g\xi_k). \tag{2.18}$$

Then, if we now consider the system in a large box of length $2L$ in imaginary time, we find

$$-8L\cosh(gp_l)<q_l<8L\cosh(gp_l),$$
$$\int d\eta_k\,d\varphi_k=-8L\cosh(\tfrac{1}{4}g\xi_k)f_2(\theta_k,\xi_k,g). \tag{2.19}$$

The integration measure in CSV can be written for the present case as

$$d\mu=\frac{1}{n_1!\,n_2!}\prod_{\alpha=1}^{2M-n_1-2n_2}\frac{\pi}{L}\,dP(\lambda_\alpha)dQ(\lambda_\alpha)\times\prod_{l=1}^{n_1}dp_l\,dq_l\prod_{k=1}^{n_2}d\xi_k\,d\xi_k\,d\theta_k\,d\varphi_k, \tag{2.20}$$

where we have made imaginary time discrete as a lattice of $2M$ points with spacing $\Delta=L/M$. We integrate over $2M-n_1-2n_2$ points in order to preserve the total number $2M\gg1$ of independent variables. The factorials $n_1!\,n_2!$ avoid double counting of configurations. As it was discussed in I the measure (2.20) should be multiplied by the Jacobian of the transformation from (π,u) to CSV. This Jacobian tends to one for $\Delta\to0$, $L\to\infty$.

The computation of $Z(g)$ is now straightforward using Eqs. (2.3), (2.14)–(2.20):

$$Z(g)=\lim_{M\to\infty}\sum_{n_1,n_2=0}^{M}\frac{1}{n_1!n_2!}\prod_{\alpha=1}^{2M-n_1-2n_2}\frac{\lambda_\alpha-1/16\lambda_\alpha}{\lambda_\alpha+1/16\lambda_\alpha}\left[\int_{-\infty}^{+\infty}dp\exp\left(-\frac{8}{g}\cosh(gp)\right)16L\cosh(gp)\right]^{n_1}$$
$$\times\left\{8L\int_{-\infty}^{+\infty}d\xi\int_0^{2\pi/g}d\theta f_2(g,\theta,\xi)\cosh\left(\frac{g}{4}\xi\right)\times\exp\left[-\frac{16}{g}\sin\left(\theta\right)\cosh\left(\frac{g}{4}\xi\right)\right]\right\}^{n_2} . \tag{2.21}$$

Finally, the ground-state energy reads

$$E_G(g)=-\lim_{L\to\infty}\frac{1}{2L}\ln Z(g)=-\frac{4}{g}\int_{-\infty}^{+\infty}dx\cosh x\exp\left(-\frac{8}{g}\cosh x\right)f_1(x,g)$$
$$-\frac{16}{g^2}\int_{-\infty}^{+\infty}dy\cosh y\int_0^{\pi/2}d\theta\sin\theta\exp\left(-\frac{16}{g}\sin\theta\cosh y\right)f_1(y,\theta,g)+c(g), \tag{2.22}$$

where $x\equiv gp$, $y\equiv g\xi/4$, and we have inserted the functions f_1 and f_2 instead of one in order to take into account quantum effects associated with the Jacobian. The last term in Eq. (2.22) stands for contributions from the integration over the continuum variables P_α, Q_α. $c(0)$ is such that $E_G(0)=\frac{1}{2}$.

The first term in Eq. (2.22) is the contribution from the (p_I, g_I). That means tunneling between two consecutive minima of the potential $1-\cos u$. It gives for small g

$$E_I(g)\underset{g\to 0^+}{\sim}-\frac{4f_1(0,0)}{\pi^{1/2}\sqrt{g}}e^{-8/g}[1+O(g)], \tag{2.23}$$

where the subscript I stands for instanton. On the other hand the usual semiclassical methods[8] give for this tunneling contribution

$$E_I(g)\underset{g\to 0^+}{\sim}-4(g\pi)^{-1/2}e^{-8/g}[1+O(g)]. \tag{2.24}$$

Hence, we should set

$$f_1(0,0)=1. \tag{2.25}$$

It must be noted that the integration over CSV gives not only the correct exponent $8g^{-1}$ in $E_1(g)$ but also the correct power $g^{-1/2}$ with correct sign. This power of g is a quantum feature of $E_1(g)$ related to the zero modes of the instanton.[8]

Let us analyze now the second term $E_D(g)$ in Eq. (2.22). This term is analytic in g at $g=0$ and admits an expansion in powers of g. The behavior of its coefficients is determined, for large orders, by $f_2(\theta,0,0)$. If we insert a constant instead of $f_2(\theta,0,0)$ we get from Eq. (2.22) for large K

$$\operatorname{coeff}(g^{2K})=-\text{ const }\frac{(2K)!}{16^{2K}}\left[1+O\left(\frac{1}{2K}\right)\right]. \tag{2.26}$$

This should be compared with the known large orders of the expansion[6]

$$E_G(g)=\sum_{K=0}^{\infty}A_K g^K, \tag{2.27}$$

$$A_K\underset{K\to\infty}{\sim}-\frac{2}{\pi}\frac{K!}{16^K}\left[1+O\left(\frac{1}{K}\right)\right]. \tag{2.28}$$

Equations (2.26)–(2.28) clearly show that the large orders in g of $E_G(g)$ are obtained from the term $E_D(g)$ and that we can set

$$f_2(\theta,0,0)=(1+\sin\theta)/\pi. \tag{2.29}$$

The term in $\sin\theta$ takes into account the odd power of g. As before in $E_I(g)$, inverse scattering integration provides besides the classical factor 16^{-K} a quantum effect, i.e., the correct leading power of K in Eq. (2.26).

To conclude, we can write the ground-state energy of the pendulum, that is, the lower edge of the first allowed band as in Eqs. (1.4) and (1.5) where

$$F_1(x,g)=f_1(x,g),$$

$$F_2(\theta,y,g)=\frac{\pi f_2(\theta,y,g)}{1+\sin\theta}.$$

In the present case we cannot determine the large-g behavior of $E_G(g)$ as we did in I for the N-dimensional anharmonic oscillator, because we do not know at present F_1 and F_2 for large x and g. However, it is interesting to note from Eqs. (1.3)–(1.5) that $E_D(g)$ behaves like g^{-1} for $g\to\infty$ as $E_G(g)$ really does.[9]

The expression we got by integration over CSV in the present case corresponds to the physical region $g>0$ because the original functional integral (2.3) converges in this domain for real $u(x)$.

III. ONE- AND TWO-DIMENSIONAL ANHARMONIC OSCILLATORS

We consider in this section the one- and two-dimensional isotropic anharmonic oscillators. In paper I, we transformed the quantum N-dimensional anharmonic oscillator to CSV via the "α representation." Here we shall go to CSV without appealing to the α representation.

The CSV associated with the nonlinear Schrödinger equation will be useful as new integration variables in the functional integral for these two anharmonic oscillators.

The generating function for the two-dimensional anharmonic oscillator reads

$$Z^{(2)}(g)=\int \mathfrak{D}\psi\mathfrak{D}\psi^* e^{-S[\psi,\psi^*]}, \tag{3.1}$$

where

$$\psi(x)=\psi_1(x)+i\psi_2(x)$$

and

$$S[\psi,\psi^*]=\tfrac{1}{2}\int dx\left(\left|\frac{\partial\psi}{\partial x}\right|^2+\mu^2|\psi|^2+g\mu^3|\psi|^4\right). \tag{3.2}$$

Similar expressions hold for the one-dimensional anharmonic oscillator provided one imposes $\psi(x)=\psi(x)^*$, i.e., the constraint $\Pi_x\delta(\psi(x)-\psi(x)^*)$. We will focus our attention on the region $g<0$ and call $h\equiv -g>0$. The linear problem relevant in this case is[4]

$$\left[i\sigma_3\frac{\partial}{\partial x}+(h\mu^3)^{1/2}(\sigma_1\psi_1+\sigma_2\psi_2)\right]\Phi(x)=\lambda\Phi(x). \tag{3.3}$$

Matrix Jost-type solutions $\Phi_\pm(x)$ defined by

$$\Phi_\pm(x)\underset{x\to\pm\infty}{\sim} e^{-i\lambda\sigma_3 x} \tag{3.4}$$

are connected by the transition matrix $T(\lambda)$ through the relation

$$\Phi_-(x)=\Phi_+(x)T(\lambda). \tag{3.5}$$

Here

$$T(\lambda)=\begin{pmatrix} a(\lambda) & -b^*(\lambda)\\ b(\lambda) & a^*(\lambda)\end{pmatrix} \tag{3.6}$$

and

$$|a(\lambda)|^2+|b(\lambda)|^2=1.$$

The eigenvalues of the linear system (3.3) are given by the zeros of $a(\lambda)$ in $\mathrm{Im}\lambda>0$. The angle-action variables for the nonlinear Schrödinger equation follow from the spectral data of system (3.3).[4] We shall use these canoncial variables in the functional integral (3.1). They can be written as

$$P(\lambda)=\frac{1}{h\mu^3}\ln|a(\lambda)|^{-2},\quad Q(\lambda)=\frac{1}{\pi}\arg b(\lambda) \tag{3.7}$$

$$p_n=\frac{1}{h\mu^3}\lambda_n,\quad q_n=\ln c_n^{-2},\quad n=1,\ldots,N_B. \tag{3.8}$$

Here $c_n\equiv b(\lambda_n)$ and N_B stands for the number of eigenvalues. The transformation from

$$\{\psi(x),\psi(x)^*,x\in\mathbf{R}\}$$

to

$$\{P(\lambda),Q(\lambda),\lambda\in\mathbf{R};p_n,q_n,1\le n\le N_B\}$$

is canonical. Poisson brackets are defined through

$$\{\alpha,\beta\}=i\int_{-\infty}^{+\infty}dx\left[\frac{\delta\alpha}{\delta\psi(x)}\frac{\delta\beta}{\delta\psi(x)^*}-\frac{\delta\beta}{\delta\psi(x)}\frac{\delta\alpha}{\delta\psi^*(x)}\right]. \tag{3.9}$$

The CSV given by Eqs. (3.7) and (3.8) are appropriate for the two-dimensional anharmonic oscillator because they separate completely its Euclidean action (3.2).

By using the two first trace identities given by Ref. 4 it follows that

$$S[\psi,\psi^*]=\frac{\mu^2}{2\pi}\int_{-\infty}^{+\infty}d\lambda\,P(\lambda)\left[1+4\frac{\lambda^2}{\mu^2}\right] -i\mu^2\sum_{n=1}^{N_B}\left[p_n-p_n^*+\tfrac{4}{3}h^2\mu^4(p_n^{\,2}-p_n^{*3})\right]. \tag{3.10}$$

To deal with the one-component anharmonic oscillator we simply impose $\psi=\psi^*$. This implies that the λ_n and hence the p_n are purely imaginary.

Let us now write the integration measure for the CSV variables (3.7) and (3.8) in semiclassical approximation. Following the same procedure as in I and Sec. II, we get

$$d\mu=\frac{1}{N_B!}\left(\frac{L}{\pi}\right)^{2N_B}\prod_{\alpha=1}^{M-2N_B}dQ(\lambda_\alpha)dP(\lambda_\alpha)\prod_{j=1}^{N_B}d^2q_j\,d^2p_j. \tag{3.11}$$

Here

$$d^2q_j=d(\mathrm{Re}q_j)d(\mathrm{Im}q_j),$$
$$d^2p_j=d(\mathrm{Re}p_j)d(\mathrm{Im}p_j).$$

The integration bounds read in the present case

$$\begin{aligned}&0\le \mathrm{Im}p_j<+\infty,\quad j=1,\ldots,N_B\\ &-\infty<\mathrm{Re}p_j<+\infty,\\ &0\le P(\lambda)<+\infty,\quad 0\le Q(\lambda)<2,\quad \lambda\in\mathbf{R}.\end{aligned} \tag{3.12}$$

The variable q_j needs as usual a bit of care. Under a translation $x\to x+X$ on $\psi(x)$, it is the only CSV that changes. It transforms in the following way:

$$q_j \to q_j - 4ip_j X . \tag{3.13}$$

If we put the system in a large box of length $2L$ the domain of q_j in the complex plane will have an area proportional to L for fixed p_j. We can then set

$$\int d^2 q_j = 2L\mu\, \mathbf{C}^{(2)}(p_j, p_j^*) , \tag{3.14}$$

where μ stands for dimensional reasons and $\mathbf{C}^{(2)}$ is proportional to $|p|$ for large p.

We can easily obtain the integration bounds if $\psi = \psi^*$ (one-dimensional anharmonic oscillator).

In this case, the p_j become purely imaginary and the q_j real:

$$\begin{aligned} &0 \leq -ip_j \leq +\infty ,\\ &-\mu L\, \mathbf{C}^{(1)}(p_j) \leq q_j \leq \mu L\, \mathbf{C}^{(1)}(p_j) . \end{aligned} \tag{3.15}$$

One should also set $\Pi_{\alpha=1}^{M-N_B}$ instead of $\Pi_{\alpha=1}^{M-2N_B}$ in the measure (3.11). The bounds on $P(\lambda)$ and $Q(\lambda)$ are the same as before.

We turn now to the computation of the functional integral for $Z^{(2)}(g)$. It results as follows:

$$Z^{(2)}(g) = Z_0^{-1} \lim_{M\to\infty} \sum_{N_B=0}^{M} \frac{1}{N_B!}\left(\frac{L}{\pi}\right)^{2N_B} \left(\int dQ\right)^{M-2N_B}$$
$$\times \prod_{\alpha=1}^{M-2N_B} \int_0^\infty dP(\lambda_\alpha) \exp\left[-\frac{P(\lambda\alpha)}{L}\left(1+\frac{4\lambda_\alpha^2}{\mu^2}\right)\right] \left(\int_{\mathrm{Im}\,p\geq 0} d^2 p\, e^{-\beta(p,p^*)} \int d^2 q\right)^{N_B} . \tag{3.16}$$

And then

$$E_G^{(1)}(-h) = -\frac{\mu^5}{(2\pi)^2} \times \int_{\mathrm{Im}\,p\geq 0} d^2 p\, \mathbf{C}^{(2)}(p,p^*)\, e^{-\beta(p,p^*)} + D(h) , \tag{3.17}$$

where $D(h)$ stands for possible contributions coming from the continuous spectrum of CSV. $D(0)$ is such that $E^{(2)}G(0) = 1$ and $\beta(p, p^*)$ is given by Eq. (1.9).

For $h = -g \to 0^+$, $E_G(-h)$ given by Eq. (2.17) is dominated by the saddle points of the integral. That is, the points where

$$\left.\frac{\partial\beta}{\partial p}\right|_{p_c} = 0, \quad \left.\frac{\partial\beta}{\partial p^*}\right|_{p_c} = 0 .$$

For $\mathrm{Im}\, p \geq 0$ we find

$$p_c = -p_c^* = \frac{i}{2h\mu^2} . \tag{3.18}$$

The contribution of this saddle gives

$$\mathrm{Im}\, E_G^{(2)}(-h) = \frac{\mu\sqrt{2}}{32\pi h}\, \mathbf{C}^{(2)}(p_c, p_c^*) \exp\left(-\frac{2}{3h}\right)[1 + O(h)] . \tag{3.19}$$

This should be compared with the known value[10]

$$\mathrm{Im}\, E_G^{(2)}(-h) = 4\mu h^{-1} \exp\left(-\frac{2}{3h}\right)[1 + O(h)] .$$

Then

$$\sqrt{2}\, \mathbf{C}(p_c, p_c^*) = 2^7 \pi . \tag{3.20}$$

For the one-dimensional anharmonic oscillator we compute $Z^{(1)}$ by the same lines, now using Eq. (3.15), and we find

$$E_G^{(1)}(-h) = -\frac{i\mu^3}{4\pi} \int_0^\infty dp\, e^{-\beta(p,-p)}\, \mathbf{C}^{(1)}(p) + F^{(1)}(h) . \tag{3.21}$$

The saddle point (3.18) also dominates the $h \to 0^+$ behavior of this integral. We find in this limit

$$\mathrm{Im}\, E_G^{(1)}(-h) \underset{h\to 0^+}{\sim} \frac{\mu\sqrt{2}}{16\sqrt{\pi h}}\, \mathbf{C}^{(1)}(p_c) \exp\left(-\frac{2}{3h}\right)[1 + O(h)] . \tag{3.22}$$

The imaginary part of the energy for a one-dimensional anharmonic oscillator with coupling $-h/2$ is, for $h \to 0^+$ (Ref. 10),

$$\mathrm{Im}\, E_G^{(1)}(-h) = 2\mu(\pi h)^{-1/2} \exp\left(-\frac{2}{3h}\right)[1 + O(h)] .$$

We get then

$$\sqrt{2}\, \mathbf{C}^{(1)}(p_c) = 2^5 . \tag{3.23}$$

The ground-state energies are finally given by Eqs. (1.7) and (1.8) where

$$D^{(1)}(p) = \frac{\sqrt{2}}{2^5}\, \mathbf{C}^{(1)}(p) , \tag{3.24}$$

$$D^{(2)}(p, p^*) = \frac{\sqrt{2}}{2^7 \pi}\, \mathbf{C}^{(2)}(p, p^*) . \tag{3.25}$$

IV. FINAL REMARKS

The functional integration procedure we use here and in I can be easily worked out for any other canonical variables constructed by inverse spectral techniques. For example a nonsymmetric N-dimensional anharmonic oscillator[11] can be treated by the present methods. The use of the CSV associated with the AKNS system[5,12] enables one to compute functional integrals where the action is a linear combination of terms such as

$$S_1 = \int qr\,dx\,, \quad S_2 = \int \left(q\,\frac{\partial r}{\partial x} - \frac{\partial q}{\partial x}\,r\right) dx\,,$$

$$S_3 = \int \left(\frac{\partial r}{\partial x}\,\frac{\partial q}{\partial x} + q^2 r^2\right) dx\,,$$

and terms with high-order derivatives.

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*Laboratoire Associé au CNRS.

[1]H. J. de Vega, Phys. Rev. D 21, 395 (1980). Referred to as I in the text.
[2]V. E. Zajarov and L. D. Faddeev, Funkts. Anal. Ego Pri 5, 18 (1971) [Funct. Anal. Appl. 5, 280 (1972)].
[3]L. A. Tajtadzhyan and L. D. Faddeev, Teor. Mat. Fiz. 21, 160 (1974) [Theor. Math. Phys. (USSR) 21, 1046 (1975)].
[4]V. E. Zajarov and S. V. Manakov, Teor. Mat. Fiz. 19, 332 (1974) [Theor. Math. Phys. (USSR) 19, 551 (1975)].
[5]S. K. Dodd and R. K. Bullough, Phys. Scr. 20, 514 (1979).
[6]M. Stone and J. Reeve, Phys. Rev. D 18, 4746 (1978).
[7]S. Graffi and V. Grecchi, Lett. Math. Phys. 2, 335 (1978).
[8]See, for example, H. J. de Vega, J. L. Gervais, and S. Sakita, Nucl. Phys. B139, 20 (1978).
[9]*Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (National Bureau of Standards, Washington, D. C., 1965).
[10]T. Banks, C. M. Bender, and T. T. Wu, Phys. Rev. D 8, 3346 (1973); 8, 3366 (1973).
[11]See Sec. 4 of H. J. de Vega, Commun. Math. Phys. 70, 29 (1979).
[12]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. 30, 1262 (1973); 31, 125 (1973); Stud. Appl. Math. LIII, 249 (1974).