PHYSICAL REVIEW D VOLUME 22, NUMBER 10 15 NOVEMBER 1980

Charged Schrödinger particle in a c-number radiation field

Tadashi Toyoda and Karl Wildermuth
Institut für Theoretische Physik der Universität Tübingen, Auf der Morgenstelle 14, D-7400 Tübingen 1, Federal Republic of Germany
(Received 27 May 1980)

We formulate a method to derive the time-dependent Schrödinger equation which describes the interaction of an electron with a c-number radiation field. To obtain this semiclassical approximation from the complete quantum-mechanical formulation we use the corresponding Schrödinger equation in its projection form, which is an extension of the Hill-Wheeler generator coordinate method. For the basis states of the radiation field we introduce a complete, but not overcomplete, subset of coherent states, which were found by von Neumann and whose proof of completeness was given by Bargmann *et al.* and Perdomov. We find that the conventional overcomplete continuous coherent states are not suitable for the projection form of the Schrödinger equation. The method also allows one to calculate quantum-mechanical correction terms systematically.

I. INTRODUCTION

Since the very beginning of the development of quantum electrodynamics and other field theories, their classical limits have been attracting much attention.[1] It is well known that coherent states are useful for studying such classical or semiclassical limits.[2]

Very recently, several authors have used coherent states to study various field theories and have shown their powerfulness. It has been shown that the use of coherent states with a path integral for elements of the S matrix gives a different treatment of the Callan-Coleman vacuum tunneling.[3] Coherent states combined with the variational method have been used to treat an isovector meson field interacting with a static source.[4] It has also been shown that coherent states can be used to describe equilibrium states of boson fields.[5]

Apart from their applications, coherent states themselves have been intensively studies since the pioneering work of Bargmann and Segal.[6] In 1971, Bargmann, Butera, Ghiraradello, Klauder, and Perelomov proved that a certain subset of coherent states form a complete, but not overcomplete, set.[7] This will be important for our considerations.

In the present paper we shall study the semiclassical treatment of a charged Schrödinger particle interacting with a c-number radiation field. We shall derive the corresponding Schrödinger equation from the fully quantized theory, in which the radiation field is also quantized, by using coherent states to describe the radiation field. To avoid trouble due to the above-mentioned overcompleteness of coherent states we shall use the complete subset of coherent states of Bargmann *et al.* and Perelomov, which we shall call VNLCS (see Sec. III). In order to introduce such a subset of coherent states (VNLCS) in a consistent manner into the quantized theory, we shall use the time-dependent Schrödinger equation written in projection form,[8] which is a generalization of the Hill-Wheeler method.[9] This method has been used to obtain a microscopic nuclear theory for low-energy phenomena from a unified point of view.[8]

As a charged Schrödinger particle we shall consider a nonrelativistic atomic electron interacting with a strong radiation field. In Sec. II we shall formulate the problem quantitatively by defining the semiclassical treatment of the system. In Sec. III we shall introduce coherent states, especially VNLCS, as the basis states for the radiation field. In Sec. IV we shall rewrite the Schrödinger equation for the system in the projection form. In Sec. V we shall introduce a classical approximation to the projection form of the Schrödinger equation to obtain the semiclassical Schrödinger equation. In Sec. VI we shall give some concluding remarks.

II. QUANTITATIVE FORMULATION OF THE PROBLEM

We shall consider the system of an atomic electron interacting with a strong radiation field in the nonrelativistic case. Such a system can be described semiclassically by the time-dependent Schrödinger equation

$$[\hat{H}_e - (e/mc)\vec{A}(\vec{r},t)\cdot(\hbar/i)\vec{\nabla}_r]\,|\psi(\vec{r},t)\rangle = i\hbar\frac{\partial}{\partial t}\,|\psi(\vec{r},t)\rangle\,, \tag{2.1}$$

where $\vec{A}(\vec{r},t)$ is the classical time-dependent vector potential and $|\psi\rangle$ is the wave function of the atomic electron. For simplicitly, we consider a one-electron atom. Then, $\hat{H}_e$ has the form

$$\hat{H}_e = -(\hbar/2m)\nabla_r{}^2 + V(r)\,, \tag{2.2}$$

where $V(r)$ is the electrostatic potential[10] in which the electron moves. We shall show under what conditions Eq. (2.1) can be a very good approximation to the fully quantized description, where the transverse radiation field $\vec{A}(\vec{r},t)$ is also quantized. The corresponding Schrödinger equation is

$$\hat{H}|\Psi\rangle = [\hat{H}_e + \hat{H}_A - (e/mc)\hat{\vec{A}}(\vec{r})\cdot\hat{\vec{p}}]|\Psi\rangle = i\hbar\frac{\partial}{\partial t}|\Psi\rangle . \quad (2.3)$$

Here, $|\Psi\rangle$ is the state vector which describes the quantum-mechanical state of the electron and the light-quanta states of the transverse radiation field. $\hat{H}_A$ is the Hamiltonian of the light quanta, which can be written as

$$\hat{H}_A = \sum_{\vec{k},\lambda} \hbar\omega_k a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda} = \sum_{\vec{k},\lambda} \hat{H}_{\vec{k}\lambda}, \quad |\vec{k}| = \omega_k/c , \quad (2.4a)$$

with the commutation relation for $a^\dagger_{\vec{k}\lambda}$ and $a_{\vec{k}\lambda}$

$$[a_{\vec{k}\lambda}, a^\dagger_{\vec{q}\eta}] = \delta_{\vec{k},\lambda;\vec{q},\eta} , \quad (2.4b)$$

where $a_{\vec{k}\lambda}$ and $a^\dagger_{\vec{k}\lambda}$ are the annihilation and the creation operators of the light quanta with wave vector $\vec{k}$ and polarization λ. The operator $\hat{\vec{A}}(\vec{r})$ has the form

$$\hat{\vec{A}}(\vec{r}) = (1/V)^{1/2} \sum_{\vec{k},\eta} c(\hbar/2\omega_k)^{1/2}(a_{\vec{k}\eta}e^{i\vec{k}\cdot\vec{r}} + a^\dagger_{\vec{k}\eta}e^{-i\vec{k}\cdot\vec{r}})\vec{\epsilon}_{\vec{k}\eta} , \quad (2.5)$$

where $\epsilon_{\vec{k}\eta}$ is the polarization vector and V is the normalization volume.

III. COHERENT LIGHT-QUANTA STATES

In this section we shall discuss the basis states for the radiation field which are especially suited for the transition from the completely quantum-mechanical Schrödinger equation (2.3) to the semiclassical equation (2.1). The most suited basis states of the radiation field for this purpose are the so-called coherent states which were introduced first by Schrödinger.[11] Coherent states are minimum wave-packet states[12] and are labeled by complex eigenvalues, whose real and imaginary parts correspond to the expectation values of two canonically conjugate observables, respectively [see Eqs. (3.5e) and (3.5f)]. This is one of the reasons why the coherent states are extremely useful for the discussion of the classical limit of the quantum-mechanical treatment. Indeed, Klauder has shown that the classical-mechanical description can be obtained formally from the quantum theory by using the continuous representation, which is a generalization of the coherent states.[2]

The coherent states were discussed by many authors during the last 20 years.[6] Here we list the main properties of these states and refer to the literature[12] for their derivations.

With the operators $a^\dagger$ and a (for simplicity from now on we write the indices $\vec{k},\eta$ explicitly only when it is necessary for discussion), which have been introduced by Eq. (2.4b), we can define the coherent state $|\alpha\rangle$, where α is a complex number to label the state, as

$$|\alpha\rangle = \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle . \quad (3.1)$$

Here,

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|\varphi_0\rangle, \quad |\varphi_0\rangle = |\text{vacuum state}\rangle \quad (3.2)$$

is the eigenstate of the number operator

$$\hat{N} = a^\dagger a , \quad (3.3)$$

containing n light quanta of $(\vec{k},\eta)$. For our later considerations it is useful to split $a^\dagger$ and a into a sum of Hermitian operators, i.e.,

$$a = (\hat{u} + i\hat{p})/(2\hbar)^{1/2}, \quad a^\dagger = (\hat{u} - i\hat{p})/(2\hbar)^{1/2} . \quad (3.4)$$

The coherent states $|\alpha\rangle$ have the following properties:

$$\langle\alpha|\alpha\rangle = 1 , \quad (3.5a)$$

$$a|\alpha\rangle = \alpha|\alpha\rangle = (1/2\hbar)^{1/2}(u+ip)|\alpha\rangle , \quad (3.5b)$$

$$\langle\alpha|a^\dagger = \langle\alpha|\alpha^* = (1/2\hbar)^{1/2}(u-ip)\langle\alpha| , \quad (3.5c)$$

$$\langle\hat{N}\rangle = \langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = \alpha^*\alpha , \quad (3.5d)$$

$$\langle\hat{u}\rangle = \langle\alpha|\hat{u}|\alpha\rangle = (2\hbar)^{1/2}\,\mathrm{Re}(\alpha) = u , \quad (3.5e)$$

$$\langle\hat{p}\rangle + \langle\alpha|\hat{p}|\alpha\rangle = (2\hbar)^{1/2}\,\mathrm{Im}(\alpha) = p , \quad (3.5f)$$

$$|\langle u|\alpha\rangle|^2 = \text{const}\times\exp[-(u-\langle\hat{u}\rangle)^2/2\hbar] , \quad (3.5g)$$

$$|\langle p|\alpha\rangle|^2 = \text{const}\times\exp[-(p-\langle\hat{p}\rangle)^2/2\hbar] , \quad (3.5h)$$

where $\langle u|\alpha\rangle$ and $\langle p|\alpha\rangle$ are the u and p representations of the state $|\alpha\rangle$, respectively. The eigenvalues of $\hat{u}$ and $\hat{p}$ are written as u and p, respectively. Equation (3.5b) shows that the coherent state $|\alpha\rangle$ is the eigenstate of the non-Hermitian annihilation operator a with the complex eigenvalue $\alpha = (u+ip)/(2\hbar)^{1/2}$.

If this complex eigenvalue α, which labels the coherent states as already stated, runs over the whole complex plane, the coherent states become overcomplete for the Hilbert space. Such an overcomplete set of coherent states cannot be used for our considerations because they are linearly dependent (see Sec. IV and Appendix B). However, Bargmann *et al.* and Perelomov[7] proved that a subset of the overcomplete coherent states forms a complete set. This subset is given by

$$\{|\alpha\rangle:\ \alpha = (\pi)^{1/2}(l+im);\ l = 0, \pm1, \pm2, \ldots;\ m = 0, \pm1, \pm2, \ldots\} . \quad (3.6)$$

This fact was originally stated by von Neumann without proof. Therefore, these states are called von Neumann lattice coherent states (VNLCS). For later use we shall list some properties of VNLCS:

$$\langle\beta\,|\,\alpha\rangle = \exp[-\,|\alpha-\beta|^2/2 + i\,\mathrm{Im}(\beta^*\alpha)]\,,$$
$$= \exp[-(l-s)^2/2 - i\pi t l]$$
$$\times\exp[-(m-t)^2/2 + i\pi s m]\,, \qquad (3.7a)$$

with

$$\beta = (\pi)^{1/2}(s+it),\quad s,t=0,\pm1,\pm2,\ldots\,, \qquad (3.7b)$$

$$\sum_\alpha \langle\beta\,|\,\alpha\rangle = \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\langle\beta|\alpha\rangle$$
$$= \vartheta_3(-\pi t/2, e^{-\pi/2})\vartheta_3(\pi s/2, e^{-\pi/2})\,, \qquad (3.7c)$$

$$\sum_\alpha \langle\beta\,|\,\alpha\rangle\alpha = \beta\vartheta_3(-\pi t/2, e^{-\pi/2})\vartheta_3(\pi s/2, e^{-\pi/2})\,, \qquad (3.7d)$$

where

$$\vartheta_3(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{i2nz}\,.$$

The derivations of Eqs. (3.7c) and (3.7d) are given in Appendix A. It is important to note that the subset states of (3.6), which form a complete set of states, are not mutually orthogonal, as can be seen from Eq. (3.7a). On the other hand, the states $|\alpha_{\vec{k}\lambda}\rangle$, with different $\vec{k}, \lambda$, are mutually orthogonal.

With the properties of the coherent states $|\alpha\rangle$ given in Eqs. (3.5) we can now discuss their physical qualities by considering the expectation value of the radiation field operator $\hat{\vec{A}}(\vec{r})$. For the moment we consider only coherent states belonging to a given wave vector $\vec{k}$ and polarization λ. That means the coherent states with $(\vec{k}', \lambda') \neq (\vec{k}, \lambda)$ have the complex eigenvalues $\alpha_{\vec{k}'\lambda'} = 0$ (i.e., these modes are in their ground states). In the interaction picture, with respect to the Hamiltonian $\hat{H}_A$ given in Eq. (2.4), the radiation field operator has the form

$$\hat{\vec{A}}(\vec{r},t) = V^{-1/2}\sum_{\vec{k},\eta} c(\hbar/2\omega_k)^{1/2}$$
$$\times(a_{\vec{k}\eta}e^{i\vec{k}\cdot\vec{r}-i\omega_k t} + a^\dagger_{\vec{k}\eta}e^{-i\vec{k}\cdot\vec{r}+i\omega_k t})\vec{\epsilon}_{\vec{k}\eta}\,, \qquad (3.8a)$$

where

$$|\vec{k}| = \omega_k/c\,.$$

Using the formulas (3.5), its expectation value can be obtained:

$$\langle\alpha\,|\hat{\vec{A}}\,|\,\alpha\rangle = c(\hbar/2\omega_k V)^{1/2}(\alpha_{\vec{k}\eta}e^{i\vec{k}\cdot\vec{r}-i\omega_k t} + \mathrm{c.c.})\,\vec{\epsilon}_{\vec{k}\eta}$$
$$= \vec{A}^{(0)}_{\vec{k}\eta}\cos(\vec{k}\cdot\vec{r} - \omega_k t + \phi_{\vec{k}\eta})\,. \qquad (3.8b)$$

Thus, the above coherent state $|\alpha\rangle$ corresponds to the classical radiation field $A^{(0)}_{\vec{k}\eta}\cos(\vec{k}\cdot\vec{r} - \omega_k t + \phi_{\vec{k}\eta})$. However, it remains to be examined whether the above quantum-mechanical expectation value can be approximately substituted by the corresponding classical value in any functions of A_k. For this, it must be shown that the quantum-mechanical root-mean-square deviations from the expectation values of the radiation field are relatively small. By using the formulas (3.5) we can obtain for these deviations

$$\Delta u_{\mathrm{rel}} = \Delta u(\langle\hat{u}\rangle^2 + \langle\hat{p}\rangle^2)^{-1/2}$$
$$= [\langle(\hat{u} - \langle\hat{u}\rangle)^2\rangle]^{1/2}(\langle\hat{u}\rangle^2 + \langle\hat{p}\rangle^2)^{-1/2}$$
$$= 2^{-1}(|\alpha|^2)^{-1/2} = 2^{-1}(\langle\hat{N}\rangle)^{-1/2} \qquad (3.9a)$$

and

$$\Delta p_{\mathrm{rel}} = \Delta p(\langle\hat{u}\rangle^2 + \langle\hat{p}\rangle^2)^{-1/2}$$
$$= [\langle(\hat{p} - \langle\hat{p}\rangle)^2\rangle]^{1/2}(\langle\hat{u}\rangle^2 + \langle\hat{p}\rangle^2)^{-1/2}$$
$$= 2^{-1}(\langle\hat{N}\rangle)^{-1/2}\,, \qquad (3.9b)$$

where $\langle\hat{N}\rangle$ is the number operator of the light quanta in the normalization volume V [see Eq. (2.5)]. Similarly, we obtain

$$\Delta N_{\mathrm{rel}} = \Delta N/\langle\hat{N}\rangle = (\langle\hat{N}\rangle)^{-1/2}\,. \qquad (3.9c)$$

Hence, we can see that if the expectation values of the light-quanta number operator for the coherent states are very large, the classical approximation can be a good approximate substitute to the quantum-mechanical description.

To obtain an impression for the magnitude of these deviations for macroscopic dimensions, let us consider the radiation field that corresponds to the classical field of the wavelength 1 cm having the volume of 10^3 cm^3 and the energy density 1 $\mathrm{erg/cm}^3$—the belonging absolute value of the field strength is 1 G. For such a state $\langle\hat{N}\rangle \approx \mathrm{energy}/\hbar\omega$ becomes about 5×10^{18} and therefore Δu_{rel}, Δp_{rel}, and ΔN_{rel} become about 5×10^{-10}, which is indeed very small. It should be noted that ΔN itself is proportional to $(\langle\hat{N}\rangle)^{1/2}$ and therefore also becomes large in the classical case, i.e., only the relative deviations Δu_{rel}, Δp_{rel}, and ΔN_{rel} become small in this limit case.

Up to now we have discussed the physical properties of the single-mode radiation-field coherent state specified by the complex $\alpha = \alpha_{\vec{k}\eta}$. We can straightforwardly generalize the above discussion to arbitrary radiation fields by taking into account all modes. If we form the direct product of single-mode coherent states with a set of $\alpha_{\vec{k}\eta}$, which we write as $\{\alpha\}$,

$$|\{\alpha\}\rangle = \prod_{\vec{k},\eta}|\alpha_{\vec{k}\eta}\rangle\,, \qquad (3.10)$$

then such a state gives the expectation value

$$\langle\{\alpha\}|\hat{\vec{A}}(\vec{r},t)|\{\alpha\}\rangle = V^{-1/2}\sum_{\vec{k},\eta} c(\hbar/2\omega_k)^{1/2}(\alpha_{\vec{k}\eta}e^{i\vec{k}\cdot\vec{r}-i\omega_k t}+\text{c.c.})\vec{\epsilon}_{\vec{k}\eta}\langle\{\alpha\}|\{\alpha\}\rangle$$

$$= \sum_{\vec{k},\eta} A^{(0)}_{\vec{k}\eta}\cos(\vec{k}\cdot\vec{r}-\omega_k t+\phi_{\vec{k}\eta}). \qquad (3.11)$$

Here we have used Eqs. (3.5b), (3.5c), and (3.8b). Thus, by choosing appropriate $\{\alpha\}$ we can construct a coherent state $|\{\alpha\}\rangle$ which corresponds to a given classical field, which is expressed in general as a linear superposition of electromagnetic plane waves.

One interesting fact should be noted: To construct a quantum-mechanical wave-packet state for a free massive particle one has to superpose linearly plane-wave states with different wave vectors $\vec{k}$, as is well known. On the contrary, to construct a quantum-mechanical state of the radiation field which corresponds to a classical electromagnetic wave packet one has to make a direct product of many single-mode states and not a superposition of such states.

IV. REFORMULATION OF THE SCHRÖDINGER EQUATION FOR THE ELECTRON-LIGHT-QUANTA SYSTEM

In this section we shall reformulate the electron–light-quanta Schrödinger equation using the projection equation method[8] with the VNLCS for the light quanta as basis states. With respect to the atomic electron we remain in the usual x representation, i.e., we use as basis states the continuous electron coordinate eigenstates. First we write down the Schrödinger equation (2.3) in the mixed picture, which is the interaction picture with regard to the radiation field and is the Schrödinger picture in the x representation with regard to the atomic electron:

$$[\hat{H}_e+\hat{H}_{\text{int}}(t)]|\Psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle \qquad (4.1a)$$

with

$$\hat{H}_{\text{int}}(t) = -(e/mc)\exp(i\hat{H}_A t/\hbar)\hat{\vec{A}}\exp(-i\hat{H}_A t/\hbar)\cdot\hat{\vec{p}}_e. \qquad (4.1b)$$

The Schrödinger equation (3.1a) can now be rewritten in the projection form

$$\left\langle\delta\Psi(t)\right|\left[\hat{H}_e+\hat{H}_{\text{int}}(t)-i\hbar\frac{\partial}{\partial t}\right]\left|\Psi(t)\right\rangle = 0. \qquad (4.2)$$

If $\langle\delta\psi(t)|$ represents an arbitrary variation at every time t in the whole Hilbert space, then Eq. (4.2) is equivalent to Eq. (4.1a).

The great advantage of the Schrödinger equation written in the projection form is its flexibility to allow various possibilities of basis states. Thus we can introduce a desirable physical ansatz into the formalism from the very beginning in a quite natural manner. That is, we can introduce a superposition of basis states with linear variational amplitudes, which can be adapted to the physical conditions, i.e., the boundary conditions, of the system considered. It is important to note that those states do not have to be mutually orthogonal but only have to be linearly independent.

The nondegenerate stationary solutions of Eq. (4.2) are mutually orthogonal, whatever basis states are chosen. This is so because if the basis states form a complete set of states in the belonging Hilbert space, the exact solution of the Schrödinger equation (4.1a) can be obtained from Eq. (4.2). This orthogonality of the solutions of Eq. (4.2) also remains valid when one constructs approximate solutions of Eq. (4.1a) by restricting the basis states to be used in Eq. (4.2). For details see Ref. 8.

After these general remarks about the Schrödinger equation written in the projection form, let us now return to the discussion of the electron–light-quanta system. Since our purpose is to obtain a classical description of the radiation field as an approximation to the quantized field, the coherent states are suitable as basis states for the radiation field. As already mentioned in Sec. III, in order to use the projection form of the Schrödinger equation we introduce the VNLCS as basis states for the radiation field:

$$|\{\alpha\}\rangle = \prod_{\vec{k},\eta}|\alpha_{\vec{k}\eta}\rangle, \qquad (4.3a)$$

where $|\alpha_{\vec{k}\eta}\rangle$ is the VNLCS for the mode $(\vec{k},\eta)$ with

$$\alpha_{\vec{k}\eta} = (\pi)^{1/2}(l_{\vec{k}\eta}+im_{\vec{k}\eta}), \quad l_{\vec{k}\eta}, m_{\vec{k}\eta} = \text{integers}. \qquad (4.3b)$$

For the electron we use the coordinate eigenstate $|\vec{r}'\rangle = \delta(\vec{r}-\vec{r}')$ with the orthonormal relation

$$\langle\vec{r}''|\vec{r}'\rangle = \int\delta(\vec{r}''-\vec{r})\delta(\vec{r}-\vec{r}')d\vec{r}$$

$$= \delta(\vec{r}''-\vec{r}'). \qquad (4.4)$$

Then, the basis states for the whole system are the direct products

$$|\{\alpha\},\vec{r}'\rangle = |\{\alpha\}\rangle\otimes|\vec{r}'\rangle, \qquad (4.5)$$

which span the whole Hilbert space. Using these basis states we now make the following generalized

Hill-Wheeler ansatz[8] for the states of the whole system:

$$|\Psi(t)\rangle=\sum_{\{\alpha\}}\int d\vec{r}'\,|\{\alpha\},\vec{r}'\rangle f(\{\alpha\},\vec{r}';t)\,, \tag{4.6}$$

where the notation is defined as

$$\sum_{\{\alpha\}}\equiv\sum_{\alpha_{\vec{k}_1\eta_1}}\sum_{\alpha_{\vec{k}_2\eta_2}}\cdots=\prod_{\vec{k},\eta}\Big(\sum_{\alpha_{k\eta}}\Big)=\prod_{\vec{k},\eta}\Big(\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\Big), \tag{4.7a}$$

with

$$\alpha_{\vec{k}\eta}=(\pi)^{1/2}(l+im)\,. \tag{4.7b}$$

The time-dependent linear amplitude $f(\{\alpha\},\vec{r}';t)$, which has to be varied arbitrarily at every time t and at every point $\vec{r}'$ for every multiple $\{\alpha\}$, is a continuous function of the parameter coordinate $\vec{r}'$ and a discrete function of the discrete values of the multiple $\{\alpha\}$.

If we introduce the ansatz Eq. (4.6) into the projection equation (4.2), we then obtain after integrating over $\vec{r}'$

$$\sum_{\{\alpha\}}\int d\vec{r}'\langle\{\beta\},\vec{r}''\,|[\hat{H}_e+\hat{H}_{\rm int}(t)]\,|\{\alpha\},\vec{r}'\rangle f(\{\alpha\},\vec{r}';t)=i\hbar\frac{\partial}{\partial t}\sum_{\{\alpha\}}\int d\vec{r}'\langle\{\beta\},\vec{r}''\,|\{\alpha\},\vec{r}'\rangle f(\{\alpha\},\vec{r}';t)\,. \tag{4.8}$$

Using Eqs. (2.2), (4.1b), and (4.7a) we can write down the three kernels appearing in Eq. (4.8) explicitly:

$$\langle\{\beta\},\vec{r}''\,|\{\alpha\},\vec{r}'\rangle=\langle\{\beta\}|\{\alpha\}\rangle\langle\vec{r}''\,|\vec{r}'\rangle=\langle\{\beta\}|\{\alpha\}\rangle\delta(\vec{r}''-\vec{r}')\,, \tag{4.9}$$

$$\langle\{\beta\},\vec{r}''\,|\hat{H}_e|\{\alpha\},\vec{r}'\rangle=\langle\{\beta\}\,|\{\alpha\}\rangle\langle\vec{r}''\,|\hat{H}_e|\vec{r}'\rangle=\langle\{\beta\}\,|\{\alpha\}\rangle\delta(\vec{r}''-\vec{r}')[(-\hbar^2/2m)\vec{\nabla}_{r'}{}^2+V(\vec{r}')]\,, \tag{4.10}$$

$$\begin{aligned}\langle\{\beta\},\vec{r}''\,|\hat{H}_{\rm int}(t)\,|\{\alpha\},\vec{r}'\rangle&=V^{-1/2}\sum_{\vec{k},\eta}c(\hbar/2\omega_k)^{1/2}(\langle\{\beta\}\,|a_{\vec{k}\eta}|\{\alpha\}\rangle e^{i\vec{k}\cdot\vec{r}'-i\omega_k t}\\&\qquad+\langle\{\beta\}|a^\dagger_{\vec{k}\eta}|\{\alpha\}\rangle e^{-i\vec{k}\cdot\vec{r}'+i\omega_k t})\vec{\epsilon}_{\vec{k}\eta}\langle\vec{r}''\,|\hat{\vec{p}}_e|\vec{r}'\rangle(-e/mc)\\&=V^{-1/2}\sum_{\vec{k},\eta}c(\hbar/2\omega_k)^{1/2}(\alpha_{\vec{k}\eta}e^{i\vec{k}\cdot\vec{r}-i\omega_k t}+\beta^*_{\vec{k}\eta}\,e^{-i\vec{k}\cdot\vec{r}'+i\omega_k t})\vec{\epsilon}_{\vec{k}\eta}\\&\qquad\times(-e/mc)\langle\{\beta\}\,|\{\alpha\}\rangle\delta(\vec{r}''-\vec{r}')(-i\hbar)\vec{\nabla}_{r'}\\&\equiv\langle\{\beta\}|\{\alpha\}\rangle\vec{A}(\{\beta\},\{\alpha\},\vec{r}';t)(-e/mc)\cdot\delta(\vec{r}''-\vec{r}')(-i\hbar)\vec{\nabla}_{r'}\,,\end{aligned} \tag{4.11}$$

where $\vec{A}(\{\beta\},\{\alpha\},\vec{r}';t)$ is defined as

$$\vec{A}(\{\beta\},\{\alpha\},\vec{r}';t)=V^{-1/2}\sum_{\vec{k},\eta}c(\hbar/2\omega_k)^{1/2}(\alpha_{\vec{k}\eta}e^{i\vec{k}\cdot\vec{r}'-i\omega_k t}+\beta^*_{\vec{k}\eta}\,e^{-i\vec{k}\cdot\vec{r}'+i\omega_k t})\vec{\epsilon}_{\vec{k}\eta}\,. \tag{4.12}$$

Introducing Eqs. (4.9), (4.10), and (4.11) into Eq. (4.8) and integrating over r' we obtain

$$\sum_{\{\alpha\}}\langle\{\beta\}\,|\{\alpha\}\rangle\Big[-(\hbar^2/2m)\nabla_{r''}{}^2+V(r'')+(-e/mc)\vec{A}(\{\beta\},\{\alpha\},\vec{r}'';t)\cdot\frac{\hbar}{i}\vec{\nabla}_{r''}-i\hbar\frac{\partial}{\partial t}\Big]f(\{\alpha\},\vec{r}'';t)=0\,. \tag{4.13}$$

Since the basis states (4.5) are complete, Eq. (4.13) is equivalent to the Schrödinger equation (4.1a). That means until now we have not introduced any approximation.

V. CLASSICAL APPROXIMATION TO THE SCHRÖDINGER EQUATION

In the previous section we have obtained the Schrödinger equation in the projection form (4.13) using the coherent states given by Eq. (4.3a) for the radiation field. In the projection equation our coherent-states basis effectively resulted in the kernel, which is an inner product of two coherent states. In this section we shall develop a systematic way of obtaining a classical approximation to the projection equation (4.13) based on the properties of this kernel.

First we assume that the amplitude $f(\{\alpha\},\vec{r}';t)$ given by Eq. (4.6) varies very slowly with regard to the change of the variable $\{\alpha\}$ over the range of order $(\pi)^{1/2}$. This means if the square root of the corresponding light-quanta-number expectation value $\sum\langle\hat{N}_{\vec{k}\lambda}\rangle=\sum\alpha^*_{\vec{k}\lambda}\alpha_{\vec{k}\lambda}$ varies by a few units $f(\{\alpha\},\vec{r}';t)$ practically does not change [see Eq. (3.5d)]. In the classical limit region this is certainly the case (see the discussion in the second part of Sec. III). For the amplitudes which satisfy this assumption we can introduce a classical approximation into the Schrödinger equation (4.13) written in the projection form by using the property of the kernel.

The essential feature of the projection equation

(4.13) for the present purpose can be represented by the sum

$$\sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle \vec{A}(\{\beta\},\{\alpha\}) f(\{\alpha\}), \tag{5.1}$$

where the electron variables have been dropped, since we are interested in the classical approximation to the radiation field and the electron state is not subjected to such an approximation. The kernel $\langle\{\beta\}\,|\{\alpha\}\rangle$ is just the product of single-mode kernels $\langle\beta_{\vec{k}\eta}|\alpha_{\vec{k}\eta}\rangle$, which have Gaussian forms centered at $\alpha_{\vec{k}\eta}=\beta_{\vec{k}\eta}$. The widths of these Gaussian forms are of the order of $(\pi)^{1/2}$. Thus the kernel $\langle\{\beta\}|\{\alpha\}\rangle$ has a strongly localized peak at the point $\{\alpha\}=\{\beta\}$ in the $\{\alpha\}$ space. Therefore, for the microscopically slow varying $f(\{\alpha\})$, the above sum can be approximated as

$$\begin{aligned}\sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle \vec{A}(\{\beta\},\{\alpha\}) f(\{\alpha\}) &= \sum_{\alpha_{\vec{k}\eta}} \cdots \sum_{\alpha_{\vec{k}'\eta'}} \langle\beta_{\vec{k}\eta}|\alpha_{\vec{k}\eta}\rangle\cdots\langle\beta_{\vec{k}'\eta'}|\alpha_{\vec{k}'\eta'}\rangle\cdots \vec{A}(\{\beta\},\{\alpha\}) f(\{\alpha\}) \\ &\approx f(\{\beta\}) \sum_{\alpha_{\vec{k}\eta}} \cdots \sum_{\alpha_{\vec{k}'\eta'}} \cdots \langle\beta_{\vec{k}\eta}|\alpha_{\vec{k}\eta}\rangle\cdots\langle\beta_{\vec{k}'\eta'}|\alpha_{\vec{k}'\eta'}\rangle\cdots \vec{A}(\{\beta\},\{\alpha\}) \\ &\approx f(\{\beta\}) \sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle \vec{A}(\{\beta\},\{\alpha\}).\end{aligned} \tag{5.2}$$

It should be noted here that the correction to this approximation can be calculated by representing $f(\{\alpha\})$ in the form of the Taylor expansion around $\{\alpha\}=\{\beta\}$.

Applying this approximation to the left-hand side of the projection equation (4.13), we obtain

$$\begin{aligned}&\sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle \left[-(\hbar^2/2m)\nabla_{r'}{}^2 + V(r') + (-e/mc)\vec{A}(\{\beta\},\{\alpha\},\vec{r}';t)\cdot(-i\hbar)\vec{\nabla}_{r'} - i\hbar\frac{\partial}{\partial t}\right] f(\{\alpha\},\vec{r}';t) \\ &\approx \left[-(\hbar^2/2m)\nabla_{r'}{}^2 + V(r') - i\hbar\frac{\partial}{\partial t}\right] f(\{\beta\},r';t) \sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle \\ &\qquad + [(-e/mc)(-i\hbar)\vec{\nabla}_{r'} f(\{\beta\},\vec{r}';t)] \sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle \vec{A}(\{\beta\},\{\alpha\},\vec{r}';t).\end{aligned} \tag{5.3}$$

In order to calculate the two summations on the right-hand side of Eq. (5.3) let us generalize the formulas (3.7c) and (3.7d) to the many-mode case:

$$\begin{aligned}\sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle &= \prod_{\vec{k},\eta} \sum_{\alpha_{\vec{k}\eta}} \langle\beta_{\vec{k}\eta}|\alpha_{\vec{k}\eta}\rangle \\ &= \prod_{\vec{k},\eta} \vartheta_3(-\pi t_{\vec{k}\eta}/2, e^{-\pi/2})\vartheta_3(\pi s_{\vec{k}\eta}/2, e^{-\pi/2}) \\ &\equiv C(\{\beta\}),\end{aligned} \tag{5.4}$$

where

$$\beta_{\vec{k},\eta} \equiv (\pi)^{1/2}(s_{\vec{k}\eta} + i t_{\vec{k}\eta}) \tag{5.5}$$

and

$$\begin{aligned}\sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle \alpha_{\vec{p}\xi} &= \left[\prod_{\vec{k},\eta\neq\vec{p},\xi} \sum_{\alpha_{\vec{k}\eta}} \langle\beta_{\vec{k}\eta}|\alpha_{\vec{k}\eta}\rangle\right] \sum_{\alpha_{\vec{p}\xi}} \langle\beta_{\vec{p}\xi}|\alpha_{\vec{p}\xi}\rangle \alpha_{\vec{p}\xi} \\ &= \beta_{\vec{p}\xi} \prod_{\vec{k},\eta} \vartheta_3(-\pi t_{\vec{k}\eta}/2, e^{-\pi/2})\vartheta_3(\pi s_{\vec{k}\eta}/2, e^{-\pi/2}) \\ &= \beta_{\vec{p}\xi} C(\{\beta\}).\end{aligned} \tag{5.6}$$

Using Eq. (5.6), the summation in the second term on the right-hand side of Eq. (5.3) can be calculated as

$$\begin{aligned}\sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle \vec{A}(\{\beta\},\{\alpha\},\vec{r}';t) &= V^{-1/2} \sum_{\vec{k},\eta} c(\hbar/2\omega_k)^{1/2} \left(e^{i\vec{k}\cdot\vec{r}'-i\omega_k t} \sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle \alpha_{\vec{k}\eta} + e^{-i\vec{k}\cdot\vec{r}'+i\omega_k t} \beta^*_{\vec{k}\eta} \sum_{\{\alpha\}} \langle\{\beta\}\,|\{\alpha\}\rangle\right) \vec{\epsilon}_{\vec{k}\eta} \\ &= V^{-1/2} \sum_{\vec{k},\eta} c(\hbar/2\omega_k)^{1/2} (\beta_{\vec{k}\eta} e^{i\vec{k}\cdot\vec{r}'-i\omega_k t} + \beta^*_{\vec{k}\eta} e^{-i\vec{k}\cdot\vec{r}'+i\omega_k t}) \vec{\epsilon}_{\vec{k}\eta} C(\{\beta\}) \\ &= \vec{A}(\{\beta\},\{\beta\},\vec{r}';t) C(\{\beta\}).\end{aligned} \tag{5.7}$$

Introducing Eqs. (5.3), (5.4), and (5.7) into Eq. (4.13) we obtain the classical approximation to the projection equation (4.13):

$$\left[-(\hbar^2/2m)\nabla_{r'}^{\,2}+V(r')+(-e/mc)\vec{A}(\{\beta\},\{\beta\},\vec{r}';t)\cdot(-i\hbar)\vec{\nabla}_{r'}-i\hbar\frac{\partial}{\partial t}\right]f(\{\beta\},\vec{r}';t)C(\{\beta\})=0\,. \tag{5.8}$$

This equation must be fulfilled for all values of the multiple $\{\beta\}$. This comes from the approximation Eq. (5.2) and means physically that the reaction of the electron on the radiation field is neglected, because coherent states of different $\{\beta\}$ values are not coupled with each other. Because the operator on the right-hand side of Eq. (5.8) is a self-adjoint operator, the conservation law

$$\int d\vec{r}''\,|f(\{\beta\},\vec{r}'';t)|^2=\text{const in time} \tag{5.9}$$

is valid.

To connect the above equation [(5.8)] to the corresponding classical description we must introduce another requirement for $f(\{\beta\},\vec{r}';t)$. So far, $f(\{\beta\},\vec{r}';t)$ has been required to be only microscopically slow varying. Now we further assume that $f(\{\beta\},\vec{r}';t)$ is a macroscopically well-localized wave packet in the $\{\beta\}$ space, for which the absolute spreads $\Delta\beta_{\vec{k}\eta}$ around $\beta_{\vec{k}\eta}=\beta_{\vec{k}\eta}^{\rm cl}$ are much larger than $(\pi)^{1/2}$—see the discussion to Eq. (3.9c)—but their relative spreads $\Delta\beta_{\vec{k}\eta}/|\beta_{\vec{k}\eta}^{\rm cl}|$ are much smaller than 1. The second condition demands that the root-mean-square deviations of $\vec{A}$ from $\langle\vec{A}\rangle$ are very small. In the classical limit certainly this must be the case.

As already mentioned, Eq. (5.8) must be fulfilled for all values of the multiple $\{\beta\}$. However, the second requirement on $f(\{\beta\},\vec{r}';t)$ means that it is a macroscopically well-localized wave packet in the $\{\beta\}$ space around the point $\{\beta\}=\{\beta^{\rm cl}\}$. Therefore, we have to consider only the case $\{\beta\}\approx\{\beta^{\rm cl}\}$. Because $\Delta\beta_{\vec{k}\eta}/|\beta_{\vec{k}\eta}^{\rm cl}|$ is much smaller than one, for $\{\beta\}\approx\{\beta^{\rm cl}\}$ we may approximate $A(\{\beta\},\{\beta\},\vec{r}';t)$ by $A(\{\beta^{\rm cl}\},\{\beta^{\rm cl}\},\vec{r}';t)$. Then, from Eq. (5.8) we obtain for $\{\beta\}\approx\{\beta^{\rm cl}\}$

$$\left[-(\hbar^2/2m)\nabla_{r'}^{\,2}+V(r')+(-e/mc)\vec{A}(\{\beta^{\rm cl}\},\{\beta^{\rm cl}\},\vec{r}',t)\cdot(-i\hbar)\vec{\nabla}_{r'}-i\hbar\frac{\partial}{\partial t}\right]f(\{\beta\},\vec{r}';t)C(\{\beta\})=0\,. \tag{5.10}$$

Here, the operator $[\cdots]$ no longer depends on $\{\beta\}$. Therefore, for $f(\{\beta\},\vec{r}';t)$ we can make the following ansatz:

$$f(\{\beta\},\vec{r}';t)=g(\{\beta\}-\{\beta^{\rm cl}\})\psi'(\{\beta^{\rm cl}\},\vec{r}',t)\,, \tag{5.11}$$

where $g(\{\beta\}-\{\beta^{\rm cl}\})$ is nonvanishing only for $\{\beta\}\approx\{\beta^{\rm cl}\}$, reflecting the macroscopic requirement on $f(\{\beta\},\vec{r}';t)$. We can also simply drop the nonvanishing factor $C(\{\beta\})$ from Eq. (5.10). In order to connect the above introduced ψ' to the electron wave function we must introduce the proper normalization. From Eq. (4.6), using the same approximation as Eq. (5.2), we get

$$1=\langle\Psi(t)|\Psi(t)\rangle\approx\sum_{\{\beta\}}\int d\vec{r}''\,|f(\{\beta\},r';t)|^2C(\{\beta\})\,. \tag{5.12}$$

Introducing the ansatz (5.11), this can be written as

$$1\approx N(\{\beta^{\rm cl}\})\int d\vec{r}'\,|\psi'(\{\beta^{\rm cl}\},\vec{r}';t)|^2\,, \tag{5.13}$$

where

$$N(\{\beta^{\rm cl}\})=\sum_{\{\beta\}}|g(\{\beta\}-\{\beta^{\rm cl}\})|^2C(\{\beta\})\,. \tag{5.14}$$

Using this $N(\{\beta^{\rm cl}\})$ we can define the properly normalized electron wave function ψ:

$$\psi(\{\beta^{\rm cl}\},\vec{r}',t)=[N(\{\beta^{\rm cl}\})]^{-1/2}\psi'(\{\beta^{\rm cl}\},\vec{r}',t)\,. \tag{5.15}$$

From Eqs. (5.10), (5.11), and (5.15) it is obvious that this ψ obeys the equation

$$\left[-(\hbar^2/2m)\nabla_{r'}^{\,2}+V(r')+(-e/mc)\vec{A}(\{\beta^{\rm cl}\},\{\beta^{\rm cl}\},\vec{r}';t)\cdot(-i\hbar)\vec{\nabla}_{r'}-i\hbar\frac{\partial}{\partial t}\right]\psi(\{\beta^{\rm cl}\},\vec{r}',t)=0\,, \tag{5.16}$$

which is the Schrödinger equation for the electron wave function ψ with the classical radiation field $\vec{A}(\{\beta^{\rm cl}\},\{\beta^{\rm cl}\},\vec{r}_A;t)$.

Now let us show that $\vec{A}(\{\beta^{\rm cl}\},\{\beta^{\rm cl}\},\vec{r}_A;t)$ is nothing else but the expectation value of the radiation field operator at the position $\vec{r}_A$ for the wave packet in the $\{\beta\}$ space, which is given by Eqs. (4.6) and (5.11). Using Eqs. (4.6), (4.11), (5.4), (5.11), (5.14), and (5.15) and introducing the same approximations that are used to obtain Eqs. (5.2) and (5.10), we can readily show

$$\langle \hat{\vec{A}}(\vec{r}_A,t)\rangle \approx \vec{A}(\{\beta^{cl}\},\{\beta^{cl}\},\vec{r}_A;t)N(\{\beta^{cl}\}) \times\int d\vec{r}'\,|\psi'(\{\beta^{cl}\},\vec{r}',t)|^2 \approx \vec{A}(\{\beta^{cl}\},\{\beta^{cl}\},\vec{r}_A;t)\int d\vec{r}'\,|\psi(\{\beta^{cl}\},\vec{r}',t)|^2 \approx \vec{A}(\{\beta^{cl}\},\{\beta^{cl}\},\vec{r}_A;t)\,. \tag{5.17}$$

Now we have completed the derivation of the semiclassical time-dependent Schrödinger equation, because $A(\{\beta^{cl}\},\{\beta^{cl}\},\vec{r}_A,t)$ describes the classical radiation field, as can be seen from Eq. (5.17), and ψ is the proper normalized electron wave function due to Eq. (5.15).

It should be noted that the classical approximation scheme discussed here has been carried out for a group of wave-packet states for the radiation field in the $\{\beta\}$ space and not for a specific wave-packet state. Every wave-packet state which belongs to this group is characterized to be microscopically slow varying and macroscopically well localized in the $\{\beta\}$ space.

VI. CONCLUDING REMARKS

One essential feature of our approach is the use of the projection form of the Schrödinger equation (4.13), because then one can straightforwardly introduce many kinds of basis states which are appropriate to the problem. This can be done in any kind of particle quantum mechanics and relativistic or nonrelativistic quantum field theories. It is important to note that these basis states have to be linearly independent.[13] The nondegenerate solutions of the Schrödinger equation (4.13) are then automatically orthogonal. This remains true even if one uses only a subspace of the complete Hilbert space.

Another essential feature of our approach is the use of the VNLCS, which represents a nonorthogonal complete set of coherent states, as basis states for the description of the radiation field. If we use the conventional overcomplete continuous coherent states, then the linear dependences between them destroy the application of the projection form of the Schrödinger equation (see Appendix B).

Next, we shall derive from the fully quantized theory the Abraham-Lorentz equation with radiation damping of the charged Schrödinger particle including quantum corrections. As already mentioned in Sec. V, for this purpose one has to expand $f(\{\alpha\},r;t)$ into a Taylor series around $\{\beta\}$ [see Eqs. (5.2) and (5.3)].

ACKNOWLEDGMENTS

We wish to thank Professor W. Dittrich and Dr. B. Englert for bringing Ref. 7 to our attention. We also wish to thank Dr. R. Sartor, Dr. K. Shimizu, and Dr. E. J. Kannelopoulos for helpful discussions. This work was supported by Deutsche Forschungsgemeinschaft.

APPENDIX A

Equation (3.7c) is obtained as follows:

$$\begin{aligned}\sum_\alpha \langle\beta|\alpha\rangle &= \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\exp[-\pi(l-s)^2/2-i\pi tl]\exp[-\pi(m-t)^2/2+i\pi sm]\\ &=\sum_{l=-\infty}^{\infty}\exp[-\pi l^2/2-i\pi t(l+s)]\sum_{m=-\infty}^{\infty}\exp[-\pi m^2/2+i\pi s(m+t)]\\ &=e^{-i\pi st}\vartheta_3(-\pi t/2,e^{-\pi/2})e^{i\pi ts}\vartheta_3(\pi s/2,e^{-\pi/2})\\ &=\vartheta_3(-\pi t/2,e^{-\pi/2})\vartheta_3(\pi s/2,e^{\pi/2})\,,\end{aligned} \tag{A1}$$

where the ϑ_3 function is defined as

$$\vartheta_3(z,q)=\sum_{n=-\infty}^{\infty}q^{n^2}e^{i2nz}\,.$$

The second sum, Eq. (3.7d), is

$$\begin{aligned}\sum_\alpha\langle\beta|\alpha\rangle\alpha &= \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}(\pi)^{1/2}(l+im)\exp[-\pi(l-s)^2/2-i\pi tl]\exp[\pi(m-t)^2/2+i\pi sm]\\ &=(\pi)^{1/2}\sum_{l=-\infty}^{\infty}l\exp[-\pi(l-s)^2/2-i\pi tl]\sum_{m=-\infty}^{\infty}\exp[-\pi(m-t)^2/2+i\pi sm]\\ &\quad+i(\pi)^{1/2}\sum_{l=-\infty}^{\infty}\exp[-\pi(l-s)^2/2-i\pi tl]\sum_{m=-\infty}^{\infty}m\exp[-\pi(m-t)^2/2+i\pi sm]\,.\end{aligned} \tag{A2}$$

The two summations which contain the linear factors l and m can be calculated as follows:

$$\sum_{l=-\infty}^{\infty} l\exp[-\pi(l-s)^2/2-i\pi tl]=\sum_{l=-\infty}^{\infty}(l+s)\exp[-\pi l^2/2-i\pi t(l+s)]$$
$$=\sum_{l=-\infty}^{\infty} l\exp[-\pi l^2/2-i\pi t(l+s)]+se^{-i\pi ts}\sum_{l=-\infty}^{\infty}\exp(-\pi l^2/2-i\pi tl)\,. \tag{A3}$$

Since tl is an integer, the first term vanishes. Then,

$$\sum_{l=-\infty}^{\infty} l\exp[-\pi(l-s)^2/2-i\pi tl]=se^{-its}\vartheta_3(-\pi t/2,e^{-\pi/2})\,. \tag{A4}$$

Similarly, we can obtain

$$\sum_{m=-\infty}^{\infty} m\exp[-\pi(m-t)^2/2+i\pi sm]=te^{i\pi st}\vartheta_3(\pi s/2,e^{-\pi/2})\,. \tag{A5}$$

Using Eqs. (A1), (A4), and (A5) we find from Eq. (A2)

$$\sum_{\alpha}\langle\beta|\alpha\rangle\,\alpha=(\pi)^{1/2}se^{-\pi ts}\vartheta_3(-\pi t/2,e^{-\pi/2})e^{i\pi ts}\vartheta_3(\pi s/2,e^{-\pi/2})$$
$$+i(\pi)^{1/2}e^{-i\pi ts}\vartheta_3(-\pi t/2,e^{-\pi/2})te^{i\pi st}\vartheta_3(\pi s/2,e^{-\pi/2})$$
$$=(\pi)^{1/2}(s+it)\vartheta_3(-\pi t/2,e^{-\pi/2})\vartheta_3(\pi s/2,e^{-\pi/2}) \tag{A6}$$
$$=\beta\sum_{\alpha}\langle\beta|\alpha\rangle\,. \tag{A7}$$

APPENDIX B

If we use the conventional overcomplete continuous coherent states, we encounter the integral

$$\int d^2\alpha\langle\beta|\alpha\rangle\alpha=(2\pi)^{-1}\int_{-\infty}^{\infty}d(\mathrm{Re}\alpha)\int_{-\infty}^{\infty}d(\mathrm{Im}\alpha)\exp[-|\beta-\alpha|^2/2+i\,\mathrm{Im}(\beta^*\alpha)]\alpha \tag{B1}$$

instead of the discrete summation appearing in Eq. (5.1) for the VNLCS. Unfortunately, the above integral vanishes:

$$\int d^2\alpha\langle\beta|\alpha\rangle\alpha=(2\pi)^{-1}\int_{-\infty}^{\infty}ds\int_{-\infty}^{\infty}dt\exp[-(s-u)^2/2-iws]\exp[-(t-w)^2/2+iut](s+it)$$
$$=2\pi[(u-iw)+i(w+iu)]\exp[-(w^2+u^2)]=0\,, \tag{B2}$$

where we have set $\alpha=s+it$ and $\beta=u+iw$. Obviously, this leads to an unphysical approximation. In contrast to this, if we adopt the VNLCS, we get the nonvanishing discrete summations (5.4) and (5.6).

[1]See, for example, R. Jackiw, Rev. Mod. Phys. 49, 681 (1977); A. O. Barut, *Foundations of Radiation Theory and Quantum Electrodynamics* (Plenum, New York, 1980).

[2]See, for example, J. R. Klauder, in *Path Integrals*, edited by G. J. Papadopoulos and J. T. Devreese (Plenum, New York, 1978), p. 5; see also references cited therein; J. R. Klauder, Acta Phys. Austriaca Suppl. XVIII, 1 (1977).

[3]C. L. Hammer *et al.*, Phys. Rev. D 18, 373 (1978); 19, 667 (1979).

[4]M. Bolsterli, Phys. Rev. D 13, 1727 (1976); 16, 1749 (1977).

[5]T. Toyoda, Phys. Lett 71A, 399 (1979); 74A, 167 (1979).

[6]See, for example, Chap. 7 of Ref. 12; V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961); A. O. Barut and L. Ghiraradello, Commun. Math. Phys. 21, 41 (1971); M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. D 20, 1321 (1979).

[7]V. Bargmann *et al.*, Rep. Math. Phys. 2, 221 (1971); A. M. Perelomov, Theor. Math. Phys. (USSR) 6, 156 (1971).

[8]K. Wildermuth and V. C. Tang, *A Unified Theory of the Nucleus* (Vieweg, Braunschweig, 1977).

[9]D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1953); J. J. Griffin and J. A. Wheeler, *ibid.* 108, 311 (1957).

[10]The fact that the longitudinal electrostatic potential $V(r)$ can always be treated as a c number is shown, for example, in J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, Massachusetts, 1967).

[11]E. Schrödinger, Naturwissenschaften 14, 664 (1926).

[12]See, for example, J. R. Klauder and E. C. G. Sudarshan, *Quantum Optics* (Benjamin, New York, 1967).

[13]See Secs. 8.3 and 9.2c of Ref. 8.