Consequences of the angular momentum of the cosmic microwave background

Edward L. Wright
Department of Physics and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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A nontrivial conserved isotropic total angular momentum for the cosmic microwave background radiation in the chronometric theory of cosmology gives a distribution which is isotropic when viewed from the origin of the angular momentum measurement, but not homogeneous. Everywhere except the origin (and its antipode in a spherical universe) the distribution is not isotropic. Furthermore, at the origin the spectrum reduces to the Planck law and cannot explain the observations of Woody and Richards.

Woody and Richards[1] have observed the cosmic microwave background radiation (3-K background) and found a small but possibly significant deviation from the best fitting Planck function. Theoretical calculations generally predict a spectrum that is wider than the Planck function,[2,3] but the data of Woody and Richards are narrower than a blackbody. Jakobsen, Kon, and Segal[4] (hereafter JKS) have proposed a radical model of the background, which claims to be able to fit the Woody and Richards data. It is the purpose of this paper to show that the proposed model gives a microwave background that is not homogeneous, is not isotropic except at one (or two) points, and gives a Planck spectrum at the point(s) where it is isotropic.

JKS use the chronometric cosmology[5] which predicts that space has spherical geometry with a radius $R \sim 3\times 10^{26}$ cm, then work in units such that $\hbar = c = R = 1$. In these units, they find that the eigenstates of photon energy are $\nu = 2, 3, \ldots$, where ν is the magnitude of the photon wave vector, so $E = p/c = \hbar\nu/c = \nu$. Therefore, a photon circumnavigating the Universe accumulates a phase change of $2\pi R\nu$ rad or simply ν complete cycles. Note that $\tilde{\nu} = 6$ cm^{-1}, at the peak of the 3-K background, corresponds to $\nu = 2\pi R\tilde{\nu} \sim 10^{28}$, so all quantum numbers are enormous, justifying a classical approach to the calculation. A complete set of commuting quantum numbers[4] is formed by the energy, the total angular momentum $M^2 = m_x^2 + m_y^2 + m_z^2$, the z component of the angular momentum m_z, and the helicity λ, where the eigenvalues of M^2 are $l(l+1)$ with l an integer $1 \leq l < \nu$, the eigenvalues of m_z are integers $-l \leq m_z \leq l$, and λ is ± 1.

The microwave background for a constrained total energy and total M^2 is given by JKS as

$$F(\nu) = 4\nu \int_0^{\nu} (e^{\beta\nu + \gamma l^2} - 1)^{-1} l\, dl\,. \tag{1}$$

Here β and γ are Lagrange multipliers to be adjusted until the total energy and total angular momentum constraints are satisfied, and $F(\nu)$ is the energy density in photons with wave vector ν, averaged over the entire universe. The effect of the parameter γ is to suppress states with $l \gtrsim \gamma^{-0.5}$, and if $\gamma = 0$, all the allowable l's for a given ν are populated according to their statistical weights. Thus, the difference between the JKS spectrum and a blackbody is due to a relative absence of high-angular-momentum photons.

What are the characteristics of the high-l photons? Classically, a particle with momentum ν and angular momentum l has an impact parameter $b = l/\nu$. Quantum mechanically, a state with wave number ν, angular momentum l, and $m_z = m$ is given in flat space by

$$\psi(r, \theta, \phi) = j_l(\nu r) Y_{lm}(\theta, \phi)\,, \tag{2}$$

where Y_{lm} is a spherical hermonic and j_l is a spherical Bessel function, and for $r < b = l/\nu$ the radial wave function is not oscillatory but decays like $(r/b)^l$. Since $l \sim 10^{28}$, the probability of finding a particle with $r < b$ is negligible. Thus the high-l photons are the ones that never approach the origin of the (r, θ, ϕ) coordinate system. That the value of l for a photon depends on the origin used for its measurement follows from the classical definition $\vec{L} = \vec{r} \times \vec{p}$. While JKS take l as the total angular momentum, not just the orbital angular momentum, the difference is not significant for $l \sim 10^{28}$ and a photon spin of 1. In any case, the total angular momentum is not invariant under space translations. Quantum mechanically, the density matrix depends on l, so it does not commute with space translations. JKS recognize this, and impose spatial homogeneity by making an arbitrary translation (rotation in a spherical universe) for a given base point. This can be done for one observer, but not for two observers simultaneously. In what follows I will show that we must be at the coordinate origin, and that, therefore, the background spectrum we observe is a pure Planck function.

While the previous paragraph considered a Euclidean geometry, the result can be carried over into a spherical universe by identifying the impact parameter l/ν with the smallest classically allowed value of $\sin\chi$ in the standard χ, θ, ϕ coordinates on the three-sphere with metric

$$ds^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{3}$$

There is one forbidden region $\chi < \sin^{-1}(l/\nu)$ around the origin and a second region at the antipode $\chi > \pi - \sin^{-1}(l/\nu)$.

Now I can calculate the spectrum observed at a position χ at an angle α to the χ axis. The tangential momentum of a photon with wave number ν is $\nu\sin\alpha$, so $l = \nu \sin\alpha \sin\chi$. The observed intensity is then

$$I(\nu, \alpha, \chi) = 2\nu^3/[\exp(\beta\nu + \gamma\nu^2 \sin^2\alpha \sin^2\chi) - 1]. \tag{4}$$

Note that the average of $I(\nu, \alpha, \chi)$ over α and χ is

$$\langle I(\nu, \alpha, \chi)\rangle = \frac{2\nu^3}{\pi}\int_0^\pi \int_0^\pi \frac{\sin^2\chi \sin\alpha\, d\alpha\, d\chi}{\exp(\beta\nu + \gamma\nu^2 \sin^2\alpha \sin^2\chi) - 1}, \tag{5a}$$

which can be rewritten using $x = \cos\alpha$, $y = \cos\chi$, and $l = \nu \sin\alpha \sin\chi$ as

$$\begin{aligned}\langle I(\nu, \alpha, \chi)\rangle &= \frac{8\nu^3}{\pi}\int_0^1 \int_0^1 \frac{(1-y^2)^{1/2} dx\, dy}{\exp(\beta\nu + \gamma l^2) - 1} \\ &= \frac{8\nu^3}{\pi}\int_0^\nu \int_0^{(1-l^2/\nu^2)^{1/2}} \left|\frac{dx}{dl}\right| \frac{(1-y^2)^{1/2} dy\, dl}{\exp(\beta\nu + \gamma l^2) - 1} \\ &= \frac{8\nu}{\pi}\int_0^\nu \frac{l\, dl}{\exp(\beta\nu + \gamma l^2) - 1} \\ &\quad \times \int_0^{(1-l^2/\nu^2)^{1/2}} \frac{dy}{[1 - (l^2/\nu^2) - y^2]^{1/2}} \\ &= 4\nu \int_0^\nu \frac{l\, dl}{\exp(\beta\nu + \gamma l^2) - 1} = F(\nu),\end{aligned} \tag{5b}$$

as asserted above. The intensity $I(\nu, \alpha, \chi)$ shows a quadrupole anisotropy of

$$\frac{\Delta I}{I} = \frac{-\gamma\nu^2 \sin^2\chi \sin^2\alpha}{1 - e^{-\beta\nu}}. \tag{6}$$

Figure 1 illustrates these conclusions. The two solid curves show the minimum and maximum intensity at $\chi = \pi/2$ where the anisotropy is greatest, plotted for the values $\beta^{-1} = 3.4$ K and $\gamma/\beta^2 = 0.1$.[6] The upper curve is a Planck function, and is also the intensity observed at $\chi = 0$ or π where the distribution is isotropic. The dashed curve is $F(\nu)$. From the observed upper limit[7] to quadrupole anisotropy of $<3 \times 10^{-4}$ at $\tilde{\nu} = 1.1$ cm^{-1}, I can derive a limit on the shape of the spectrum. 1.1 cm^{-1} corresponds to $\beta\nu = 0.5$, so the observed isotropy requires $\gamma\nu^2 \sin^2\chi < 1.2 \times 10^{-4}$ when $\tilde{\nu} = 1.1$ cm^{-1}, so $\gamma\nu^2\sin^2\chi < 10^{-2}$ for all $\tilde{\nu} < 10$ cm^{-1}. For such a small $\gamma \sin^2\chi$ the spectrum $I(\nu)$ deviates insignificantly from a blackbody.

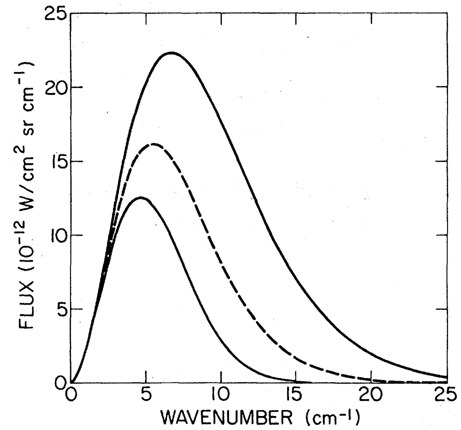

FIG. 1. Solid curves are maximum and minimum intensities showing the anisotropy at $\chi = \pi/2$ when $\beta^{-1} = 3.4$ K and $\gamma/\beta^2 = 0.1$. Dashed curve shows the average intensity $F(\nu)$.

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APPENDIX: TWO-DIMENSIONAL ANALOGY

Near $x = y = 0$, M_z acts like a generator of rotations, while M_y/R and $-M_x/R$ generate translations, so one can identify them as $P_x = M_y/R$ and $P_y = -M_x/R$. The invariant Laplacian is $M^2/R^2 = P_x^2 + P_y^2 + M_z^2/R^2$, where the last term corrects for the curvature of the two-sphere. The eigenfunctions of the energy are $Y_{lm}(\theta, \phi)$, where $\phi = \tan^{-1}(y/x)$ and $\theta = \sin^{-1}[(x^2+y^2)^{1/2}/R]$. The analog of the Planck function comes from the occupation number

$$n(l) = (e^{\beta E(l)} - 1)^{-1},$$

while the JKS model would introduce an extra

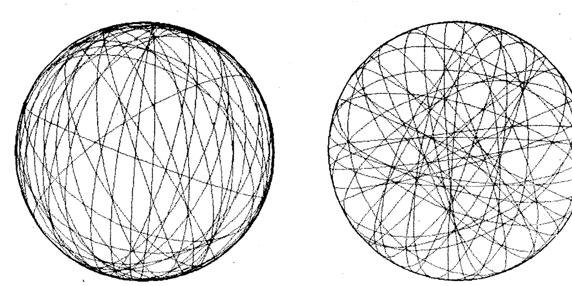

FIG. 2. Two views of a two-dimensional analogy with $\gamma l^2 = 4$, showing a large anisotropy at the equator and a large inhomogeneity. The equatorial view is a projection on the xz plane while the polar view is a projection on the xy plane. The top of the equatorial view is the center of the polar view.

term,

$$n(l, m) = (e^{\beta E(l) + \gamma m^2} - 1)^{-1}.$$

If $\gamma = 0$, the distribution is homogeneous because

$$\sum_m |Y_{lm}(\theta, \phi)|^2 = \frac{2l+1}{4\pi}$$

independent of θ and ϕ. However, by suppressing the high-m states for a given l, the JKS model is not homogeneous.

Classically, the γ term suppresses rays with impact parameters $\sin\theta_{min} > \gamma^{-1/2} l^{-1}$. Figure 2 illustrates a random sample of 44 rays drawn for a case with $\gamma l^2 = 4$. Note that the flux is very anisotropic on the equator and very inhomogeneous. The figure corresponds to $\tilde{\nu} = 15$ cm^{-1} in Fig. 1, where the anisotropy is >50:1.

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[1]D. P. Woody and P. L. Richards, Phys. Rev. Lett. 42, 925 (1979).
[2]E. L. Wright, Astrophys. J. 232, 348 (1979).
[3]A. F. Illarionov and R. A. Syunyaev, Astron. Zh. 51, 1162 (1974) [Sov. Astron. 18, 691 (1975)].
[4]H. P. Jakobsen, M. Kon, and I. E. Segal, Phys. Rev. Lett. 42, 1788 (1979).
[5]I. E. Segal, *Mathematical Cosmology and Extragalactic Astronomy* (Academic, New York, 1976).
[6]The value $\beta = 2.68$ K given by JKS is dimensionally incorrect, and $\beta^{-1} = 2.68$ K does not fit the data.
[7]G. F. Smoot, M. V. Gorenstein, and R. A. Muller, Phys. Rev. Lett. 39, 898 (1977).