Klein-Gordon equation and rotating black holes

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The Klein-Gordon equation for a scalar field of mass μ is analyzed in the geometry of a rotating black hole. It is shown that in the limit $\mu M \ll 1$, i.e., particle Compton wavelength much larger than the size of the black hole, the scalar field is unstable with an e-folding time of $\tau = (a/M)^{-1} 24 (\mu M)^{-8} \mu^{-1}$.

I. INTRODUCTION

Black holes have by now become commonplace in the literature of physics and astrophysics. But before this could happen an important theoretical question had to be answered. If a black hole is perturbed in some small way, will the perturbation oscillate and damp out? Or will it grow exponentially until it can no longer be considered a perturbation and hence demonstrate the instability of the black hole? This problem was studied first by Regge and Wheeler[1] in the case of nonrotating black holes and later by others[2-5] and notably Press and Teukolsky[6] who resolved many of the questions concerning rotating black holes.

Most questions concerning stability have now been answered satisfactorily by a combination of analytical and numerical techniques. Nonrotating black holes are stable to all scalar, electromagnetic, and gravitational perturbations.[1-3] Rotating black holes are a bit more complicated; any of the above types of perturbations are clearly stable as long as the azimuthal index m is not positive[4-6] (the conventions and notation are described below). But if a perturbing wave is sent in from infinity with $m>0$ and frequency ω less than some critical value $\omega_c = am/2Mr_+$, then the wave is amplified upon reflection off the hole—this is called super-radiant scattering. Some analytical results have been obtained, but it has not yet been proven that this amplification cannot lead to any instability. But numerical work for massless scalar,[4] electromagnetic,[7] and gravitational[6] fields implies that these perturbations are indeed stable, although a black hole with $a=M$ is in some sense marginally unstable.[8]

It has now been recognized that a massive field around a rotating black hole may indeed be unstable.[9,10] Imagine a wave packet of the massive field in a distant circular orbit. The gravitational force binds the field and keeps it from escaping or radiating away to infinity. But at the event horizon some of the field goes down the black hole, and if the frequency of the field is in the super-radiant region then the field is amplified. Hence the field is amplified at the event horizon while being bound away from infinity. Consequently, the massive field grows exponentially and is unstable.

In this paper we use analytical methods developed by Starobinskii[11,12] to solve the Klein-Gordon equation in the limit when both the mass of the field and the frequency of the perturbation are much less than M^{-1}. For large values of r the radial part of the Klein-Gordon equation looks much like the Schrödinger equation for the electron in a hydrogen atom and can be solved in terms of confluent hypergeometic functions. Close to the event horizon the radial equation can be solved in terms of hypergeometic functions. For a small mass and frequency a region can be found where the valid regions for the analytic solutions overlap, and the solutions can be matched together.

The fastest growing instability which we find is the analog of the $2p$ state and has an e-folding time of $\tau = 24(a/M)^{-1}(\mu M)^{-8}\mu^{-1}$.

II. KLEIN-GORDON EQUATION

The metric of a rotating black hole is

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr}{\Sigma}\sin^2\theta\, dt\, d\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r}{\Sigma}\sin^2\theta\right)\sin^2\theta\, d\varphi^2 \quad (1)$$

with

$$\Sigma = r^2 + a^2\cos^2\theta \quad (2)$$

and

$$\Delta = r^2 - 2Mr + a^2 \equiv (r - r_+)(r - r_-) \quad (3)$$

in the standard Boyer-Lindquist[13] coordinate system. The quantity M is the mass of the black hole and a ($\leq M$) is the Kerr angular momentum parameter; the quantity r_+ is the radial coordinate of the event horizon.

A classical, massive scalar field obeys the wave equation

$$\nabla^a \nabla_a \psi = \mu^2 \psi \,, \tag{4}$$

where $\mu = \mathfrak{M}G/\hbar c$ for a particle of mass $\mathfrak{M}$. For a pion mass $\mu \approx (1.4\times 10^{-13}\text{ cm})^{-1} \approx (1.9\times 10^{15}\text{ g})^{-1}$.

Equation (4) is separable in the Kerr geometry[14] so we make the assumption that

$$\psi = e^{-i\omega t + i m\varphi} S(\theta) R(r)\,. \tag{5}$$

The separate equations for $S(\theta)$ and $R(r)$ are

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left[a^2(\omega^2-\mu^2)\cos^2\theta - \frac{m^2}{\sin^2\theta} + \lambda\right] S = 0 \tag{6}$$

and

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR}{dr}\right) + \left[\omega^2(r^2+a^2)^2 - 4aMrm\omega + a^2m^2 - \Delta(\mu^2r^2 + a^2\omega^2 + \lambda)\right] R = 0\,. \tag{7}$$

The quantity λ is the separation constant, to be found as an eigenvalue of Eq. (6), and the eigenfunctions $S(\theta)$ are the spheroidal harmonics.[15]

We are interested in solutions to Eq. (7) with boundary conditions of an outgoing wave at infinity and a downgoing wave at the event horizon. Such a solution corresponds to a particular mode of free oscillation of the scalar field. If a particular mode is stable then its frequency ω is complex with a negative imaginary part: The field is radiated away and the amplitude dies off exponentially. However, if $\mathrm{Im}(\omega)$ is positive then, despite the radiation, the amplitude of the field grows exponentially and the mode is consequently unstable.

If $\omega M \ll 1$ and $\mu M \ll 1$ then, as first noticed by Starobinskii,[11] Eq. (7) is amenable to analytic methods. We henceforth assume these inequalities to hold. Also in this limit $S(\theta)$ becomes a spherical harmonic and $\lambda \approx l(l+1)$.

For values of $r \gg M$, but not necessarily large when compared with l/ω, Eq. (7) is approximately

$$\frac{d^2}{dr^2}(rR) + \left[\omega^2 - \mu^2 + \frac{2M\mu^2}{r} - \frac{l(l+1)}{r^2}\right] rR = 0\,. \tag{8}$$

Useful definitions are

$$k^2 \equiv \mu^2 - \omega^2\,, \tag{9}$$

$$\nu \equiv M\mu^2/k\,, \tag{10}$$

and

$$x = 2kr\,. \tag{11}$$

Then Eq. (8) becomes

$$\frac{d^2(xR)}{dx^2} + \left[-\frac{1}{4} + \frac{\nu}{x} - \frac{l(l+1)}{x^2}\right] xR = 0\,. \tag{12}$$

This is the same equation as that which governs an electron in the hydrogen atom. For large values of x the two independent solutions of R go like

$$R(x) \sim x^{\pm\nu} e^{\mp x/2}\,. \tag{13}$$

Now for an unstable mode x is in the lower right quadrant of the complex plane, and an outgoing wave corresponds to the top signs in Eq. (13). For simplicity, we continue the calculation with this assumption of instability and in the end verify consistency. The solution of Eq. (12) for R with the correct boundary condition at infinity is just

$$R(x) = x^l e^{-x/2} U(l+1-\nu, 2l+2, x)\,, \tag{14}$$

where U is one of the confluent hypergeometric functions in the notation of Abramowitz and Stegun.[15]

The bound states of the hydrogen atom corresponds to integral values of ν such that $\nu = l+1+n$ where n is non-negative. And ν is then the principal quantum number. We expect a free oscillation to look much like a bound state; the difference is the boundary condition at small values of x—the electron wave function in the hydrogen atom must be regular at the origin, but in our problem the inner boundary condition must correspond to radiation down the black hole. All of this leads us to guess that for slowly growing instabilities the complex eigenvalues ω have corresponding complex ν, Eq. (10), which satisfy

$$\nu - l - 1 \equiv n + \delta\nu \tag{15}$$

for some integer n and small complex number $\delta\nu$.

For small values of x the independent confluent hypergeometric functions go like a constant and like $x^{-(2l+1)}$. And in particular for small values of $\delta\nu$, $R(r)$ is approximated by

$$\begin{aligned} R(r) &= (2kr)^l e^{-kr} U(-n-\delta\nu, 2l+2, 2kr) \\ &\sim (-1)^n \frac{(2l+1+n)!}{(2l+1)!}(2kr)^l + \cdots \\ &\quad + (-1)^{n+1}\delta\nu(2l)!\,n!\,(2kr)^{-l-1} + \cdots\,. \end{aligned} \tag{16}$$

When $kr \ll 1$ but $(kr)^{2l+1} \sim \delta\nu$ then these two terms in the expansion are comparable and yet dominate all others.

Equation (7) can also be solved analytically when $r \ll \max(l/\omega, l/\mu)$. In this case it is approximately

$$z(z+1)\frac{d}{dz}\left[z(z+1)\frac{dR}{dz}\right] + \left[P^2 - l(l+1)z(z+1)\right]R = 0\,, \tag{17}$$

where

$$P=(am-2Mr_+\omega)/(r_+-r_-) \tag{18}$$

and

$$z=(r-r_+)/(r_+-r_-)\,. \tag{19}$$

The solution for R is

$$R(z)=\left(\frac{z}{z+1}\right)^{iP}G(-l,l+1;1-2iP;z+1)\,, \tag{20}$$

where G is any solution to the hypergeometric equation. In the notation of Erdélyi *et al.*[16] two independent hypergeometric functions are

$$U_3=(-z)^lF(-l,\,-l-2iP;\,-2l;\,-z^{-1}) \sim(-z)^l\sim\left(\frac{-r}{r_+-r_-}\right)^l, \tag{21}$$

where the approximation is valid for $r/M\gg \max(P,l)$ and

$$U_4=(-z)^{-l-1}F(l+1,l+1-2iP;2l+1;\,-z^{-1}) \sim(-z)^{-l-1}\sim\left(\frac{-r}{r_+-r_-}\right)^{-l-1}. \tag{22}$$

The functions U_3 and U_4 are linear combinations of two other solutions of the hypergeometric equation,

$$\left(\frac{z}{z+1}\right)^{iP}U_3=\frac{\Gamma(2iP)\Gamma(l+1)(-1)^l}{\Gamma(-l+2iP)\Gamma(2l+1)}\left(\frac{z}{z+1}\right)^{iP}U_1 +\frac{\Gamma(-2iP)\Gamma(l+1)(-1)^l}{\Gamma(-l-2iP)\Gamma(2l+1)}\left(\frac{z}{z+1}\right)^{iP}e^{2\pi P}\,U_5 \tag{23}$$

and

$$\left(\frac{z}{z+1}\right)^{iP}U_4=\frac{\Gamma(2iP)\Gamma(2l+2)}{\Gamma(l+1)\Gamma(l+1+2iP)}\left(\frac{z}{z+1}\right)^{iP}U_1 +\frac{\Gamma(-2iP)\Gamma(2l+2)}{\Gamma(l+1-2iP)\Gamma(l+1)}\left(\frac{z}{z+1}\right)^{iP}e^{2\pi P}U_5\,. \tag{24}$$

The functions $[z/(z+1)]^{iP}U_1$ and $[z/(z+1)]^{iP}e^{2\pi P}U_5$ are the ingoing and outgoing solutions, respectively, of R near the event horizon where $z\to 0$.

As long as ωM and μM are much less than l then there exists a value of r, r_0, such that $r_0\gg M\max(P,l)$ so that the expansion (16) is applicable and yet r_0 is small enough that $r_0 \ll \min(l/\omega, l/\mu)$ and the expansions (21) and (22) are also applicable. Hence the solution $R(r)$ which is outgoing at infinity can be matched via Eqs. (16) and (21)–(24) to $[z/(z+1)]^{iP}U_1$ and $[z/(z+1)]^{iP}e^{2\pi P}U_5$ which are ingoing and outgoing at the event horizon.

In particular, if after such a matching the coefficient of U_5 vanishes then the solution corresponds to a free oscillation of the scalar field. This coefficient vanishes if

$$\delta\nu=2iP[2k(r_+-r_-)]^{2l+1}\frac{(2l+1+n)!}{n!} \times\left[\frac{l!}{(2l)!\,(2l+1)!}\right]^2\prod_{j=1}^{l}(j^2+4P^2)\,. \tag{25}$$

The relationships between $\delta\nu, n,$ and $\omega\equiv\sigma+i\gamma$ are

$$\mu^2-\sigma^2=\mu^2\left(\frac{\mu M}{l+1+n}\right)^2, \tag{26}$$

so that $\sigma\approx\mu$, and

$$i\gamma=\frac{\delta\nu}{M}\left(\frac{\mu M}{l+1+n}\right)^3. \tag{27}$$

Finally, the imaginary part of the frequency is determined by Eqs. (25) and (27),

$$\gamma=\mu(\mu M)^{4l+4}(am/M-2\mu r_+) \times\frac{2^{4l+2}(2l+1+n)!}{(l+1+n)^{2l+4}n!}\left[\frac{l!}{(2l)!\,(2l+1)!}\right]^2 \times\prod_{j=1}^{l}[j^2(1-a^2/M^2)+(am/M-2\mu r_+)^2]\,. \tag{28}$$

For $m>0$, γ is positive and the mode is unstable, and for $m\leqslant 0$, γ is negative and the mode is stable as discussed in the Introduction.

III. CONCLUSIONS

We have demonstrated conclusively that rotating black holes are unstable to perturbations of massive scalar fields. But are the growth times short enough to be of any astrophysical significance? From Eq. (28) the fastest growing mode corresponds to $l=1$, $m=1$, and $\nu=2$, the analog of the $2p$ state of the hydrogen atom. In this case

$$\gamma=\mu\left(\frac{a}{M}\right)\frac{(\mu M)^8}{24}\,. \tag{29}$$

For a pion field around a solar-mass black hole, $\mu M\sim 10^{18}\gg 1$ so our entire analysis is inappropriate. However, the life history of any evaporating black hole most likely passes through an era during which the instability plays a crucial role.[17] When the decreasing mass of a black hole is significantly less than 2×10^{15} g (μ^{-1} for a pion field) then the approximations of this paper are valid. The growth time of the instability is

$$\tau\equiv\gamma^{-1}=24\left(\frac{a}{M}\right)^{-1}(\mu M)^{-8}\mu^{-1} =(10^{-22}\ \text{sec})\left(\frac{\mu}{\mu_\pi}\right)(\mu M)^{-8}\left(\frac{a}{M}\right)^{-1}, \tag{30}$$

which is much less than the evaporation time scale[18]

$$\tau_{evap} \approx (10^{17}\text{ sec})\left(\frac{M}{2\times 10^{15}\text{ g}}\right)^3 \qquad (31)$$

as long as μM is not as small as 10^{-5} and, of course, the angular momentum parameter a must be comparable to M.

But for the instability to be truly effective, τ should also be short when compared with the lifetime of the π^0, $\tau_\pi \sim 10^{-16}$ sec. It is not definitive from our analysis that τ will be less than τ_π for a value of μM which is small enough to trust our approximation.

However, it seems most likely that if an evaporating black hole is rotating, then when its mass drops below 2×10^{15} g this instability sets in and quickly removes the angular momentum and leaves behind a nonrotating black hole. It is expected then that the final burst of radiation from an evaporating black hole will be that which is characteristic of a nonrotating black hole.[19]

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