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Rotational perturbations of Friedmann universes

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Several new analytic solutions for rotational perturbations of the Friedmann metrics are found in order to incorporate the possibility of a rotating universe. The field equations impose restrictions on the matter rotation $\omega(r,t)$ and some of the solutions for $\Omega(r,t)$, which is related to the local dragging of inertial frames, are expressed in terms of hypergeometric functions. Uniform rotation is shown to be incompatible with the present universe ($P = 0$) and with the radiation-dominated universe ($P = \rho/3$). Geodesics of the metric are studied to reveal the intrinsic nature of the rotation and to elucidate the role of Ω.

I. INTRODUCTION

The cosmological solutions of the time-dependent Einstein field equations by Friedmann[1-3] have successfully incorporated the observed large-scale expansion, homogeneity, and isotropy characteristics of the universe. At the present time, in spite of some outstanding problems such as the uncertainty of the value of the cosmological constant, the actual extent of the missing mass which would be required to render a closed universe, and the mechanism for galaxy formation, it is widely accepted that one of the Friedmann models accurately describes the present general state of the universe. Any deviations which may exist are expected to be small.

In recent years, various authors have considered some general properties of density, distortion, and rotational perturbations of Friedmann cosmologies.[4-10] However, according to Sanz,[11] there have been no exact analytic solutions of the perturbation equations published apart from his own with respect to distortion.

In this paper, rotational perturbations of Friedmann models are considered in detail in order to incorporate the possibility that the universe is endowed with a slight rotation and several exact analytic solutions are presented. To the order considered, the models are homogeneous: every rest observer in the substratum Friedmann cosmology sees himself as the center of the same distribution of small rotation. The perturbed metric in terms of the usual Robertson[2]-Walker[3] coordinates can be expressed in the form[12]

$$ds^2 = dt^2 - e^g \left(\frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) + 2r^2 \sin^2\theta e^g \, \Omega(r,t) d\phi \, dt \,, \tag{1.1}$$

where Ω is the metric rotation function which is related to the local dragging of inertial frames.

A solution is determined by the specification of an equation of state and a distribution of the angular velocity of matter $\omega(r,t) = d\phi/dt$, which is consistent with the field equations. The entire range of possibilities for ω is considered. Some of the exact solutions for $\Omega(r,t)$ are given in terms of hypergeometric functions. These, in turn, can be expressed in terms of elementary functions. Other solutions are given explicitly. To be assured that all perturbations considered are physical and not merely coordinate effects, the geodesics of the metric are studied. This, in turn, clarifies the nature of Ω as related to the dragging of local inertial frames.

It is difficult to detect rotation directly unless it is relatively large. Indeed, the only manner in which one could hope to detect transverse velocities of objects at distances of the order of hundreds of megaparsecs is by the transverse Doppler effect. Unfortunately, since this is a second-order effect, the limit which is places on the rotation is rather large: 7×10^{-11} rad yr^{-1}.[13] However, Collins and Hawking[5] have used the limits on the 24^h (dipole) component of the anisotropy of the microwave background radiation to obtain a rotation rate for closed models of less than 3×10^{-11} sec of arc/century if the microwave background was last scattered at a red-shift of 7 and less than 2×10^{-14} sec of arc/century if the last scattering was at a red-shift of 1000.

Recent observations of the large-scale anisotropy of the microwave background indicate that our galaxy is moving with a velocity of 520 ± 75 km/sec with respect to the background radiation.[14] This value is rather large from the standpoint that the peculiar velocities of all the nearby galaxies are at the level of or below 200 km/sec. It is intriguing to consider whether this could be related to a rotation of the universe.

One might also wish to look for evidence for or against a rotation of the universe by determining the net angular momentum of galaxies in volumes of the order (100 Mpc)^3. It would be interesting if such a survey were to be carried out. If the universe were to rotate, it would also be of interest with regard to the formation of galaxies since this would provide a reservoir for angular momentum.

It should be noted that since the models studied contain matter in bulk which rotates with respect to the compass of inertia they are non-Machian. In the usual interpretation of Mach's principle, the bulk matter of the universe determines the intertial frame and hence should be nonrotating with respect to it.

II. SOLUTIONS OF THE FIELD EQUATIONS

To establish the perturbed metric form of Eq. (1.1), consider a general perturbation h_{ik} of the Friedmann metric $g_{ik}^{(0)}$:

$$g_{ik} = g_{ik}^{(0)} + h_{ik}, \tag{2.1}$$

where $\{x^0, x^1, x^2, x^3\} = \{t, r, \theta, \phi\}$. With the assumption of axial symmetry, coordinate transformations can be employed to yield nonvanishing components h_{00}, h_{11}, h_{22}, h_{33}, and h_{03}. The perturbations h_{11}, h_{22}, h_{33} of the diagonal components come into play in the case of distortion, which is not the subject of the present study and hence they are set to zero. In the case of rotational perturbations, the component h_{00} will be of second order relative to h_{03}. This is because the effect of rotation is to take $d\phi \to d\phi - \Omega\, dt$. Hence the only component to be considered is h_{03}. Moreover, the rotation will be considered to be sufficiently slow so that deviations from spherical symmetry can ge neglected.[15-17] Hence, the perturbed metric for slow rotation with axial symmetry can be expressed as in Eq. (1.1) with the function $\Omega(r, t)$ to be determined by the field equations.

To first order in Ω, the pressure and the density are unperturbed[6] and are expressed by the field equations as

$$8\pi P = -\frac{1}{R^2} e^{-g} - \ddot{g} - \tfrac{3}{4}\dot{g}^2 + \Lambda, \tag{2.2}$$

$$8\pi\rho = \frac{3}{R^2} e^{-g} + \tfrac{3}{4}\dot{g}^2 - \Lambda, \tag{2.3}$$

where Λ is the cosmological constant, and a dot denotes $\partial/\partial t$. This justifies the assumption of homogeneity in the rotation to first order. To first order in Ω, the Ricci tensor components which involve Ω are

$$R_{03} = -8\pi(T_{03} - \tfrac{1}{2} g_{03} T) + \Lambda g_{03}, \tag{2.4}$$

$$R_{13} = -8\pi T_{13}. \tag{2.5}$$

Using the perfect-fluid stress-energy tensor

$$T^{ik} = (\rho + P) u^i u^k - P g^{ik}, \tag{2.6}$$

with $u^1 = u^2 = 0$, $u^3 = \omega$ and Eqs. (1.1), (2.1)–(2.3), Eqs. (2.4) and (2.5) can be expressed as (a prime denotes $\partial/\partial r$)

$$\left(1 - \frac{r^2}{R^2}\right)\Omega'' + \left(\frac{4}{r} - \frac{5r}{R^2}\right)\Omega' - \left(\frac{4}{R^2} - 2\ddot{g}e^{g}\right)\Omega = -\left(\frac{4}{R^2} - 2\ddot{g}e^{g}\right)\omega(r, t), \tag{2.7}$$

$$(-\tfrac{1}{2})(\tfrac{3}{2} r^2 \dot{g}\, \Omega' + r^2 \dot{\Omega}') \sin^2\theta e^{g} = R_{13} = 0, \tag{2.8}$$

with

$$T_{13} = 0, \qquad T_{03} = (\rho + P) e^{g} r^2 \sin^2\theta (\Omega - \omega) - P r^2 \sin^2\theta\, e^{g} \Omega. \tag{2.9}$$

Generally, the temporal dependence of Ω will be determined by Eq. (2.8), and Eq. (2.7) will then yield the spatial dependence. Indeed, Eq. (2.8) is readily integrated to

$$\Omega(r, t) = A(r) e^{-(3/2)g(t)} + K(t), \tag{2.10}$$

where $K(t)$ can be set to zero without altering the physical structure.[17] From Eqs. (2.10) and (2.7),

$$\left(1 - \frac{r^2}{R^2}\right)\frac{A''}{A} + \left(\frac{4}{r} - \frac{5r}{R^2}\right)\frac{A'}{A} = \left(\frac{4}{R^2} - 2\ddot{g}e^{g}\right)\left(1 - \frac{e^{(3/2)g}\omega(r, t)}{A(r)}\right). \tag{2.11}$$

For a given cosmological background, $g(t)$ is known and the input of the field equations via Eq. (2.11) places restrictions on the possible forms for ω. Indeed, since the left side of the equation is a function of r alone, the right side must be either a function of r alone or a function of t alone (and hence set equal to a constant). From this it follows that $\omega(r, t)$ has to be a separable function. Let $\omega(r, t) \equiv a(r) b(t)$ and consider the possibilities for a and b with Eq. (2.11).

(i) For $a(r) = A(r)$, Eq. (2.11), separates into

$$\left(1 - \frac{r^2}{R^2}\right)\frac{A''}{A} + \left(\frac{4}{r} - \frac{5r}{R^2}\right)\frac{A'}{A} = s_0, \tag{2.12}$$

$$\left(\frac{4}{R^2} - 2\ddot{g}e^{g}\right)(1 - b(t) e^{(3/2)g}) = s_0, \tag{2.13}$$

where s_0 is a separation constant, and for a given cosmological background $b(t)$ is found without quadrature from Eq. (2.13). Equation (2.13) determines $A(r)$.

(ii) For $\omega(r, t) = \Omega(r, t) = A(r) e^{-(3/2)g}$, which corresponds to "perfect dragging," Eq. (2.13) determines the separation parameter to be zero and Eq. (2.12) again determines $A(r)$.

(iii) For the right-hand side of Eq. (2.11) to be a function of r alone, $\omega = a(r)e^{-(3/2)g}$ and the equation of state must be chosen to yield $\ddot{g}\, e^{g} = C_0$ (constant). In this case,

$$\left(1 - \frac{r^2}{R^2}\right)A'' + \left(\frac{4}{r} - \frac{5r}{R^2}\right)A' + D_0 A = D_0 a(r), \tag{2.14}$$

where

$$D_0 \equiv -\frac{4}{R^2} + 2C_0 .$$

These are the only possibilities which are compatible with Eqs. (2.7) and (2.8).

For case (i), the homogeneous hypergeometric equation (2.12) which is to be solved can be expressed in standard form by defining a new variable $z \equiv r^2/|R^2|$ (R finite):

$$z(1-z)A_{zz} + (\tfrac{5}{2} - 3z)A_z - \frac{s_0 R^2}{4} A = 0 . \tag{2.15}$$

(Initially, we consider $z \leq 1$, which is the case for closed Friedmann models.) The Gauss hypergeometric function F is defined by[18,19]

$$z(1-z)F_{zz} + [\gamma - (1+\alpha+\beta)z]F_z - \alpha\beta F = 0, \tag{2.16}$$

and the general solution is given by

$$F = A_0 F(\alpha,\beta;\gamma;z) + B_0 z^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; z), \tag{2.17}$$

where A_0 and B_0 are arbitrary constants, $\gamma \neq 0, 1, 2, \ldots$, and

$$F(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{k!\,(\gamma)_k} z^k , \tag{2.18}$$

where

$$(\lambda)_0 = 1, \quad (\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+k-1), \quad k = 1, 2, \ldots .$$

The radius of convergence of Eq. (2.21) is unity and if one of α, β, $\gamma-\alpha$, $\gamma-\beta$ is a negative integer, then the series of Eq. (2.18) terminates. $F(\alpha,\beta;\gamma;z)$ can also be given as

$$F(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta; \gamma; z). \tag{2.19}$$

In the problem at hand, $\gamma = \frac{5}{2}$ and hence the general solution is

$$A(r) = A_0 \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k z^k}{k!\,(\frac{5}{2})_k} + B_0 z^{-3/2} \sum_{k=0}^{\infty} \frac{(\alpha-\frac{3}{2})_k(\beta-\frac{3}{2})_k z^k}{k!\,(-\frac{1}{2})_k} . \tag{2.20}$$

Since the second term is not regular at $z=0$, we take $B_0 = 0$. If one of the integer conditions is satisfied, then A can be expressed in terms of polynomials by using Eq. (2.18) or

$$A(r) = A_0(1-z)^{(5/2)-\alpha-\beta} \sum_{k=0}^{\infty} \frac{(\frac{5}{2}-\alpha)_k(\frac{5}{2}-\beta)_k z^k}{k!\,(\frac{5}{2})_k} , \tag{2.21}$$

where Eq. (2.19) has been used. Some explicit solutions are as follows.

(a) $F(2, 0, \frac{5}{2}, z) = 1$. In this case, $\beta = 0$ and hence $s_0 R^2 = 0$. $A(r) = A_0$, which yields $\Omega(r,t) = A_0 e^{-(3/2)g}$, and so it is in fact a function of t alone. This, however, can be removed by a coordinate transformation and so does not correspond to a physical rotation.

(b) $F(4, -2, \frac{5}{2}, z) = 1 - \frac{16}{5} z + \frac{16}{7} z^2$. Here $\alpha = 4$, $\beta = -2$, $s_0 R^2 = -32$, and so

$$\Omega(r,t) = A_0(1 - \tfrac{16}{5} z + \tfrac{16}{7} z^2) e^{-(3/2)g} .$$

(c) For $\alpha = -\frac{1}{2}$, $\beta = \frac{5}{2}$, $s_0 R^2 = -5$,

$$F(\alpha,\beta,\beta,z) = (1-z)^{1/2}$$

and

$$\Omega(r,t) = A_0(1-z)^{1/2} e^{-(3/2)g} .$$

For open models $R^2 < 0$, and the solutions we have given in terms of hypergeometric functions satisfy the field equations with z replaced by $-z$ and s_0 replaced by $-s_0$. However, these solutions are convergent only for $z < 1$. For solutions regular outside $z = 1$, one must consider solutions found in the neighborhood of the remaining two singular points of the differential equation, namely 1 and ∞.[19]

For critically open models, $R = \infty$ and the space-dependent part of the equation becomes

$$\frac{A''}{A} + \frac{4}{r}\frac{A'}{A} = s_0 . \tag{2.22}$$

This can be solved immediately to give

$$A(r) = \frac{1}{r^2}\left[s_0(\alpha e^{\sqrt{s_0}\, r} + \beta e^{-\sqrt{s_0}\, r}) - \frac{\sqrt{s_0}}{r}(\alpha e^{\sqrt{s_0}\, r} - \beta e^{-\sqrt{s_0}\, r})\right], \tag{2.23}$$

where s_0, α, and β are constants. For case (ii) the equation to be solved for $A(r)$ for finite R is

$$z(1-z)A_{zz} + (\tfrac{5}{2} - 3z)A_z = 0 . \tag{2.24}$$

In addition to $A =$ constant as in case (a), this has the following solutions:

$$A(z) = E_0(1-z)^{1/2}\left(\frac{2}{3z^{3/2}} + \frac{4}{3z^{1/2}}\right) + E_1, \quad \text{when } R^2 > 0 \tag{2.25}$$

and

$$A(z)=E_0(1+z)^{1/2}\left(-\frac{2}{3z^{3/2}}+\frac{4}{3z^{1/2}}\right)+E_1, \quad \text{when } R^2<0$$

where E_0 and E_1 are arbitrary constants. For $R=\infty$, from Eq. (2.12) with $s_0=0$,

$$A(r)=\frac{k_0}{r^3}+k_1. \tag{2.26}$$

E_1 and k_1 can be chosen to be zero in Eqs. (2.25) and (2.26).

For case (iii), Eq. (2.14) with finite R is also a Gauss hypergeometric equation but now with an inhomogeneous term $D_0 a(r)$. As before, $a(r)$ embodies the differential rotation of the model universe. In principle, it could be determined by the physical conditions of the universe. Thus far, only the case $a(r)=\text{const}$ has been solved. This represents a uniformly rotating model. For this case, Eq. (2.14) can be transformed into a homogeneous form by the appropriate substitution $A\to A+\text{const}$.

For $R=\infty$, the equation to be solved for $A(r)$ becomes

$$A''+\frac{4}{r}A'+2C_0A=2C_0a(r). \tag{2.27}$$

Again, only the case of uniform rotation, $a(r)=a_0$ (const), has been found. For this case, Eq. (2.27) can be written as

$$\bar{A}''+\frac{4}{r}\bar{A}'+2C_0\bar{A}=0, \tag{2.28}$$

where $\bar{A}\equiv A-a_0$. The solution for Eq. (2.28) is the same as that of Eq. (2.23) with s_0 replaced by $-2C_0$ for $C_0\neq 0$. When $C_0=0$, the solution is given in Eq. (2.26). Note that the solutions in Eqs. (2.23), (2.25), and (2.26) are not regular at the origin. Hence, these solutions could, at best, be used in regions away from the origin and joined continuously to other solutions which are regular at the origin.[20]

At this point, we recall that type-(iii) models are only compatible with equations of state which yield $\ddot{g}e^{g}=C_0$. This is readily integrated with the substitution $y=e^{g(t)}$ and yields

$$t+C_1=\frac{1}{C^{1/2}}\ln\left|2(CX)^{1/2}+2Cy-2C_0\right|, \quad C>0$$

where $X\equiv Cy^2-2C_0y$,

$$t+C_1=\frac{1}{C^{1/2}}\sinh^{-1}\left(\frac{2Cy-2C_0}{(-4C_0^2)^{1/2}}\right), \quad C>0, \quad C_0^2<0$$

$$t+C_1=-\frac{1}{\sqrt{-C}}\sin^{-1}\left(\frac{2Cy-2C_0}{2C_0}\right),$$

$$C<0, \quad C_0^2>0, \quad |Cy-C_0|<C_0$$

where C and C_1 are constants of integration.

It is interesting to note that for the equation of state corresponding to incoherent dust as well as that corresponding to radiation, the g function does not satisfy $\ddot{g}e^{g}=\text{const}$. Hence for these cases, which we expect to approximate the present and early stages of the universe, respectively, case (iii) rotations, which include uniform rotation, are not allowed by Einstein's field equations. Note also that for the special value $2\ddot{g}e^{g}=4/R^2$, $A(r)$ is given by Eqs. (2.25) and (2.26), irrespective of the value of ω. However, this case corresponds to the physically uninteresting equation of state $P=-\rho$.

III. GEODESICS OF THE PERTURBED METRIC

It is of interest to consider the motion of test particles in the perturbed metric with regard to the possibility that the rotation might not be an intrinsic characteristic of the solutions but rather a coordinate effect. This is particularly the case with regard to the choice $\omega=\Omega$. It might at first appear that if the angular velocity of the matter is precisely coupled to that of the frame dragging, the rotation cannot be intrinsic. Geodesics will be considered both for test particles whose initial conditions match the motion of the fluid and for those which are given an additional radial velocity. For the latter, if there were no intrinsic rotation, the test particle would intercept fluid elements at different radial positions, all of which occupied the same azimuthal angle as that of the test particle when it was released.

The equations of motion are

$$\frac{du^i}{ds}+\Gamma^i_{jk}u^ju^k=0 \tag{3.1}$$

and the Christoffel symbols are computed from the metric in Eq. (1.1). To first order in Ω,

$$\begin{aligned}
\frac{du^1}{ds}&=-\dot{g}u^1u^0-\frac{r}{R^2(1-r^2/R^2)}(u^1)^2,\\
\frac{du^2}{ds}&=0,\\
\frac{du^3}{ds}&=(\Omega\dot{g}+\dot{\Omega})(u^0)^2+\left(\frac{2\Omega}{r}+\Omega'\right)u^0u^1\\
&\quad-\frac{\dot{g}}{2}\frac{e^{g}\Omega}{(1-r^2/R^2)}(u^1)^2-\dot{g}u^0u^3-\frac{2}{r}u^1u^3.
\end{aligned} \tag{3.2}$$

To simplify the computation, attention is restricted to test particles with small velocities and hence only those terms which are linear in a velocity are retained. Thus,

$$u^0 = 1\,, \tag{3.3}$$

$$\frac{du^1}{dt} = -\dot{g}\,u^1\,, \tag{3.4}$$

$$\frac{du^2}{dt} = 0\,, \tag{3.5}$$

$$\frac{du^3}{dt} = \Omega\dot{g} + \dot{\Omega} - \dot{g}\,u^3\,. \tag{3.6}$$

The integral of Eq. (3.4) is

$$u^1 = k_0 e^{-g}\,, \tag{3.7}$$

which can be used to find $r = r(t)$. From this, Eq. (3.6) can be integrated as

$$u^3 = \Omega + k_1 e^{-g} - k_0 e^{-g} F(t)\,, \tag{3.8}$$

where

$$F(t) \equiv \int \Omega' \, dt = \int A'(r) e^{-(3/2)g} dt\,,$$

and where the final integral is evaluated using $r = r(t)$ from Eq. (3.7).

At $t = 0$, the test particle is given an initial angular velocity equal to that of the fluid:

$$u^3(t=0) = \omega(r, 0) = \Omega(t=0) + k_1 e^{-g(0)} - k_0 e^{-g(0)} F(0)\,. \tag{3.9}$$

If the metric is normalized so that $g(0) = 0$, k_0 is the initial test particle velocity in the radial direction. Equation (3.9) then determines k_1 in terms of known quantities. From Eq. (3.8), we find that at the time $\bar{t}$ when the test particle has reached $r = \bar{r}$, it will have shifted in azimuth by

$$(\Delta\phi)_{\text{particle}} = \int_0^{\bar{t}} \Omega(r, t) dt + \int_0^{\bar{t}} k_1 e^{-g(t)} dt - k_0 \int_0^{\bar{t}} e^{-g(t)} F dt\,, \tag{3.10}$$

where $r = r(t)$ is found from Eq. (3.7).

However, the fluid element at $r = \bar{r}$, which was at the same azimuthal position as the test particle at $t = 0$, will have shifted by

$$(\Delta\phi)_{\text{element}} = \int_0^{\bar{t}} \omega(\bar{r}, t) dt\,, \tag{3.11}$$

which, in all but the most exceptional circumstances [e.g., $\Omega = \Omega(t)$, which also implies $\omega = \omega(t)$], will differ from $(\Delta\phi)_{\text{particle}}$. Thus the test particle will deviate from the radial path that it would follow in the event that the rotation was not intrinsic. It should be noted that this is the case even under the special restriction $\Omega = \omega$. It is thus natural to ask what is the precise significance of Ω as a dragging of inertial frames. To answer this, consider a test particle that has precisely the same initial conditions as the matter of the cosmological model, viz, $u^1 = u^2 = 0$, $u^3 = \omega$ at $t = 0$. From Eqs. (3.4) and (3.5),

$$\frac{du^1}{dt} = \frac{du^2}{dt} = 0\,,$$

and hence u^1 and u^2 remain zero for $t > 0$. From Eq. (3.6),

$$u^3 = \Omega + k_1 e^{-g}$$

and from the boundary condition $u^3 = \omega \,|_{t=0}$,

$$k_1 = \omega_0 - \Omega_0$$

and hence

$$u^3 = \Omega + (\omega_0 - \Omega_0) e^{-g}\,.$$

This determines the angular velocity of the test particle for subsequent times. Clearly, with the test particle determining the local inertial reference frame, it is seen that Ω, while playing a role in that determination, is not in general the actual measure of the angular velocity of the local inertial frame. However, the remaining arbitrary constant in Eqs. (2.20), (2.25), and (2.26) can be chosen to make $\omega_0 = \Omega_0$. Then $\Omega(r, t)$ is precisely the measure of dragging of the local inertial frames.

IV. SUMMARY AND CONCLUDING REMARKS

Perturbations in the form of differential rotations of Friedmann cosmologies have been analyzed and the restrictions which the field equations impose upon the angular velocity of matter ω have been found. Various solutions have been expressed in terms of the Gauss hypergeometric function, which in turn could be expressed in terms of elementary functions. To first order in the metric rotation function Ω, the field equations reduce to Eqs. (2.7) and (2.8) for $\Omega(r, t)$ in addition to the unperturbed equations for pressure and density. Equations (2.7) and (2.8) restrict the possibilities for ω and imply that $\Omega(r, t) = A(r) e^{-g/2}$. Thus, rotational perturbations decay for expanding models where $g(t)$ is an increasing function of time.

It has been shown that for closed Friedmann models with incoherent dust or radiation ($P = 0$, $P = \rho/3$), a certain class of solutions which includes uniform (i.e., nondifferential) rotation as a special case is incompatible with the Einstein field equations. The motion of test particles which characterize the local "compass of inertia" have been considered to ascertain the intrinsic character of the rotation and to elucidate the significance of Ω. Although Ω plays a role in the "dragging" of local inertial frames, it is not the angular velocity of these frames except for the special case when it coincides with the angular velocity of the matter ω. It is noted that even in this special case the rotation is still intrinsic provided it is differential. However, it is noted that the arbitrary constants in $A(r)$ can be chosen such that $\Omega(r,t)$ is precisely the dragging of local inertial frames for all solutions.

The geodesic equations show that when a test particle is initially comoving with the matter, it remains comoving. The intrinsic rotation is demonstrated by noting that for a test particle in radial motion, the change in azimuthal position is not one which would follow the path of elements along the line of original radial velocity.

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[20]Note that these solutions are meaningful only if ω and Ω are sufficiently small over the range considered so that higher-order terms are justifiably neglected. In Eqs. (2.23), (2.25), and (2.26), $A(r)$ becomes large for small r at which point one must include higher-order terms in the perturbed metric.