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Neutrino masses in SU(2) ⊗ U(1) theories

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We analyze $SU(2)\times U(1)$ theories, denoted by (n,m), in which there are n neutrinos belonging to isodoublets and m neutrino isosinglets. The charged-current weak interactions are described by a rectangular matrix K which we explicitly parametrize. The neutral-current neutrino interactions are described by a square matrix $P = K^+K$. This has the consequences that neutrinos may decay into three lighter ones and that neutrino oscillations involving neutral-current interactions should exist. The general formalism for the latter situation is given. Associated material on parametrization of unitary matrices and the quantum theory of Majorana particles is also briefly discussed.

I. INTRODUCTION

Recently there has been a great deal of interest in the possibility that neutrinos may in fact be massive particles. On the experimental side this is in part due to the work of Reines *et al.*[1] on neutrino oscillations. Actually the earlier experiments of Davis on solar-neutrino flux were also interpreted as evidence for neutrino oscillations. The whole subject is nicely reviewed by Bilenky and Pontecorvo.[2] On the theoretical side this interest is due to the fact that many of the symmetry groups which unify $SU(2)_L \times U(1)$ with strong interactions require massive neutrinos for self-consistency.[3]

In the present note we will discuss the question of how the weak interactions involving massive neutrinos should be parametrized. That this is a nontrivial question can be seen by referring to the Kobayashi-Maskawa (KM) parametrization[4] of weak interactions involving massive quarks. There (in addition to the quark masses) four mixing angles are needed. In a certain sense we may think of these mixing angles as representing the "kinematics" or "geometry" of the theory. Now it is immediately clear that the parametrization depends on the particular model adopted. Since almost all theories of present interest are considered to reduce to $SU(2)_L \times U(1)$ (Ref. 5) effectively at low energies it seems reasonable to work in an $SU(2)_L \times U(1)$ framework. To give a logical structure to our presentation we will demand that the theory be *natural*,[6] in a sense to be spelled out precisely. As we shall see, the lepton mixings are inevitably more complicated than the KM scheme.

We consider a natural theory to be one in which, once the particle content is specified, the Lagrangian is the most general local one consistent with proper Lorentz invariance and renormalizability. The latter requirement includes the cancellation of anomalies and hence rules out certain particle assignments. It should be stressed that no assumption about the P, C, and T symmetries is to be made at the beginning. Whether or not and to what degree these symmetries hold should emerge from the theory itself. This is a sense in which the parametrization of the theory is related to its "geometry." Note that by the initial assumption the CPT theorem[7] will hold, so CPT is automatically a good symmetry. We are aware that if $SU(2)_L \times U(1)$ is embedded in a larger unifying group G, the criterion of renormalizability for the $SU(2)_L \times U(1)$ subgroup by itself is too restrictive. Nevertheless, it seems to be the most reasonable first approach. In any event, the main part of our analysis is independent of this assumption.

Now let us discuss in a step-by-step way how the usual $SU(2)_L \times U(1)$ model of leptons can be modified to include massive neutrinos. The usual theory[5] contains a complex $SU(2)_L$ Higgs doublet with weak hypercharge $Y=1$, n two-component fermion $SU(2)_L$ doublets

$$\psi_a = \begin{pmatrix} N_a \\ E_a \end{pmatrix}_L \tag{1.1}$$

[N_a is neutral, E_a is negatively charged, and L means $(1+\gamma_5)/2$ projection] with $Y=-1$, and n two-component (right-handed) fermion $SU(2)_L$ singlets E_{aR} with $Y=-2$. n is the number of "generations" which we will allow to be arbitrary. Note that all fermion fields are two-component spinors which can be considered to be van der Waerden spinors.[8] Introducing four-component Dirac fields, while convenient for computation, is something of a "mystification" in a theory where no assumptions about P, C, and T symmetries are made *a priori*. If one wishes to get massive neutrinos without introducing any new fermion fields in the theory, it is necessary to add a complex Higgs triplet with $Y=2$. The triplet may be put into a 2×2 matrix,

$$h = \begin{pmatrix} h^{(+)} & h^{(++)} \\ h^{(0)} & -h^{(+)} \end{pmatrix}, \tag{1.2}$$

and the Yukawa terms in the weak Lagrangian look like

$$\sum_{a,b} g_{ab}\psi_a^T C^{-1}\tau_2 h\psi_b + \text{H.c.}\,, \tag{1.3}$$

where C is the charge-conjugation matrix of the Dirac theory and g_{ab} are coupling constants. Neutrino masses proportional to $g_{ab}\langle h^{(0)}\rangle$ will be generated. It is well known that the coupling of h above to the gauge fields will alter the famous relation between W and Z masses. Now one has

$$\frac{m^2(Z)}{m^2(W)} \simeq \frac{1}{\cos^2\theta_W}(1+\delta)\,, \tag{1.4}$$

where δ (assumed small) is the ratio $2\langle h^{(0)}\rangle^2/\langle\phi_0\rangle^2$, $\langle\phi_0\rangle$ being the vacuum value of the usual Higgs field. If we consider the coupling constants g_{ab} in (1.3) to be of the same order of magnitude as the coupling constants in the Yukawa terms which generate charged-lepton masses, we are then led to expect the order-of-magnitude relation

$$\frac{\text{neutrino mass}}{\text{charged-lepton mass}} = O(\delta^{1/2})\,. \tag{1.5}$$

Since neutrino masses are presumably small one might not expect an experimentally significant deviation from the Z-to-W mass ratio.

Another way to generate neutrino masses is simply to add a number m of $Y=0$, $SU(2)_L$-singlet left-handed spinor fields ρ_{La}. Since these fields would not couple to the gauge bosons they could not contribute to any anomalies. The condition of renormalizability thus does not say anything about the number m, although on esthetic grounds one might want $m=n$. These fields can have Majorana mass terms such as

$$\sum_{a,b} g'_{ab}\rho_{La}^T C^{-1}\rho_{Lb} + \text{H.c.}\,, \tag{1.6}$$

or can develop Dirac masses in combination with the N_{aL} of (1.1) from a Yukawa term,

$$\sum_{a,b} g''_{ab}\bar{\psi}_a\tau_2\phi^* C\bar{\rho}_{Lb}^T + \text{H.c.} \tag{1.7}$$

Note that in a theory which is natural in the present sense *both* (1.6) and (1.7) must exist. One can also contemplate neutrinos which belong neither to $SU(2)_L$ singlets or doublets. This would bring into the picture unusual leptons (and quarks) and will for simplicity be neglected.

To summarize the preceding discussion, we will consider the class of natural $SU(2)_L\times U(1)$ theories labeled[9] by (n,m), n being the number of generations and m being the number of $SU(2)_L\times U(1)$ singlet fields. We call the general case, in which Higgs triplets are present, theories of type I, while theories without Higgs triplets will be called type II. Note that there are no theories with massive neutrinos which are natural under $SU(2)\times U(1)$ and also conserve lepton number. This is because (1.3) and (1.6) both violate lepton number conservation.

We shall see that, in general, theories which are natural under $SU(2)\times U(1)$ contain a fairly large number of "kinematical" parameters. This is the price that must be paid for generality. If one considers theories which are natural under some larger unifying group G in which $SU(2)\times U(1)$ is embedded, the number of kinematical parameters will be restricted, at least in an approximate sense. Nevertheless, we consider it worthwhile to initially analyze the situation just on an $SU(2)\times U(1)$ basis. The discussion of how the parameters become restricted when specific unifying groups G are assumed and of detailed applications of our results to experiments will be taken up elsewhere.

In Sec. II we discuss a convenient parametrization for unitary matrices which will be used later. As an application, the KM argument is repeated for the case of n generations. We also review the Lagrangian for a massive Majorana particle in two-component spinor language by choosing a specific representation of the ordinary Dirac matrices and dividing the Dirac equation into two pieces. In a related Appendix we discuss in a simple way the relationship between the Dirac and Majorana quantum theories.

The determination of the form of the charged- and neutral-current weak interactions in the general (n,m) theory is discussed in Sec. III. The analog of the KM matrix for the charged-current weak interaction in the lepton case is a rectangular matrix which we denote by K. In this case the neutral-current neutrino interactions are not diagonal and the associated matrix is a projection operator $P=K^\dagger K$. We also count the number of independent parameters in the rectangular matrix K for the general case and supply a specific realization which has exactly the right number.

A consequence of K being rectangular rather than square in general is that the usual formalism for neutrino oscillations[10] must be modified. This is done in Sec. IV, where it is noted that the probability factors for associated electron a going to associated electron b in a neutrino beam $I_{CC}(a\to b)$ no longer obey

$$\sum_b I_{CC}(a\to b)=1$$

in general. Two other consequences of P being nontrivial are that decay modes, "heavy" neutrino $\to$ 3 lighter neutrinos, are allowed and that oscillation phenomena for neutral-current interactions should exist. The probability factor for the latter is simply

$$\sum_b I_{CC}(a \to b)\,,$$

when K is real. The neutrino decay mode just mentioned is expected to affect neutrino oscillation phenomena only if one neutrino has a mass greater than about 2 MeV.

II. SOME PRELIMINARIES

It seems helpful to first establish some notation for unitary matrices and also to remind the reader how the Dirac equation is taken apart into two-component spinors.

Define the n^2 generators[11] A_a^b of unitary transformations as the matrices A_a^b with ij matrix elements,

$$(A_a^b)_{ij} = \delta_{bi}\delta_{aj}\,. \tag{2.1}$$

An arbitrary diagonal unitary matrix may be written as

$$\omega_0(\alpha) = \exp i \sum_{a=1}^{n} \alpha_a A_a^a\,, \quad \alpha_a = \alpha_a^*\,, \tag{2.2}$$

and an arbitrary "complex" rotation in the ab plane with parameter $\eta_{ab} = |\eta_{ab}|e^{i\theta_{ab}}$ is

$$\omega(\eta_{ab}) = \exp(\eta_{ab}A_a^b - \eta_{ab}^* A_b^a) \quad (a \neq b)\,. \tag{2.3}$$

For example the rotation in the 12 plane is

$$\omega(\eta_{12}) = \begin{bmatrix} \cos|\eta_{12}| & e^{i\theta_{12}}\sin|\eta_{12}| & 0 & \ldots \\ -e^{-i\theta_{12}}\sin|\eta_{12}| & \cos|\eta_{12}| & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \ldots & \ldots & & \end{bmatrix}. \tag{2.4}$$

The following identity is useful:

$$\omega_0(\alpha)\omega(|\eta_{ab}|e^{i\theta_{ab}})\omega_0^\dagger(\alpha) = \omega\{|\eta_{ab}|\exp[i(\alpha_a + \theta_{ab} - \alpha_b)]\}. \tag{2.5}$$

We parametrize[12] an arbitrary unitary matrix U by the product

$$U = \omega_0(\gamma)\prod_{a<b}\omega(\eta_{ab})\,, \tag{2.6}$$

where the $\omega(\eta_{ab})$ are to be written in some particular (but unspecified) order. The parametrization (2.6) is a slight modification of the "canonical coordinates of the second kind" of Pontryagin[13] and is expected to cover a *finite* neighborhood of the identity. As an illustration of the usefulness of this approach we can give a very simple yet essentially rigorous discussion of the quark weak interactions for an arbitrary number n of generations.

Let $\psi_{-1/3,L}$ be a column vector (in generation space) of bare down quarks and $\psi_{2/3,L}$ be a column of up quarks. The charged-current weak-interaction term is

$$\frac{ig}{\sqrt{2}} W_\mu^+ \bar\psi_{2/3,L}\gamma_\mu\psi_{-1/3,L} + \text{H.c.} = \frac{ig}{\sqrt{2}} W_\mu^+ \bar U_L \gamma_\mu U_L^{(2/3)\dagger}U_L^{(-1/3)}D_L + \text{H.c.}\,, \tag{2.7}$$

where D_L and U_L are the columns of physical down and up fields. The unitary matrices $U_L^{(2/3)}$ and $U_L^{(-1/3)}$ are to be considered arbitrary in a natural theory since they arise from bidiagonalization of arbitrary mass matrices in the charge $\frac{2}{3}$ and charge $-\frac{1}{3}$ sectors. The KM or generalized Cabibbo matrix

$$C = U_L^{(2/3)\dagger}U_L^{(-1/3)} \tag{2.8}$$

is then an arbitrary matrix which we parametrize by (2.6). We may take $\det C = 1$ by appropriately choosing an overall relative U_L-D_L phase. Once the quark mass matrices are in diagonal form we may still make the redefinitions

$$\begin{aligned} D_{L,R} &= \omega_0^\dagger(\alpha)D'_{L,R}\,, \\ U_{L,R} &= \omega_0(\gamma - \alpha)U'_{L,R}\,, \\ \sum_a \alpha_a &= \sum_a \gamma_a = 0 \end{aligned} \tag{2.9}$$

without changing the form of the free Lagrangian. Putting (2.9) and (2.6) into (2.7) gives the weak-interaction term

$$\frac{ig}{\sqrt{2}} W_\mu^+ \bar U'_L \gamma_\mu \omega_0(\alpha)\prod_{a<b}\omega(\eta_{ab})\omega_0^\dagger(\alpha)D'_L\,,$$

so that using the identity (2.5) we have the effective C matrix

$$C_{\text{eff}} = \prod_{a<b}\omega\{|\eta_{ab}|\exp[i(\alpha_a + \theta_{ab} - \alpha_b)]\}. \tag{2.10}$$

$(n-1)$ of the α_a's are at our disposal (noting $\sum_a \alpha_a = 0$) so they may be used to eliminate any $(n-1)$ of the phases θ_{ab} in (2.10). For $n=3$ this is of the famous KM form,[4] with two real planar rotations and one complex planar rotation. Furthermore, noting that CP invariance requires the matrix elements of C_{eff} to be real and accepting the parametrization (2.6), we see that every *remaining* phase θ_{ab} is an independent CP-violating parameter.

In what follows it will be convenient to work with a particular (γ_5 diagonal) representation of the Dirac algebra:

$$\vec\gamma = \begin{pmatrix} 0 & -i\vec\sigma \\ i\vec\sigma & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.11}$$

The matrix C which satisfies $-\gamma_\mu^T = C^{-1}\gamma_\mu C$, $C^T = -C$, $C^{-1} = C^\dagger$ is

$$C = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{2.12}$$

In this representation write the Dirac field operator ψ as

$$\psi = \begin{pmatrix} \chi \\ \sigma_2\phi^* \end{pmatrix}. \tag{2.13}$$

Then, an examination of the equations of motion of ϕ and χ show that they both transform in the same way under the group SL(2, C) (as contravariant, *dotted* spinors[8]). We shall work entirely with spinors which transform in this way. Comparison with (2.13) shows that χ (and hence also ϕ) can be considered a left-handed field of the usual type. It is interesting to rewrite the free-particle Lagrangian $\mathcal{L} = -\bar{\psi}(\gamma_\mu\partial_\mu + m)\psi$ in terms of ϕ and χ. Using the above and neglecting symmetrizations we find

$$\mathcal{L} = -i\phi^\dagger\sigma_\mu\partial_\mu\phi - i\chi^\dagger\sigma_\mu\partial_\mu\chi - m\phi^T\sigma_2\chi - m\chi^\dagger\sigma_2\phi^*, \tag{2.14}$$

where

$$\sigma_\mu = (\vec{\sigma}, -i). \tag{2.15}$$

Now make the unitary change of variables,

$$\chi = \frac{1}{\sqrt{2}}(\rho_2 + i\rho_1), \quad \phi = \frac{1}{\sqrt{2}}(\rho_2 - i\rho_1), \tag{2.16}$$

and note that the Dirac Lagrangian falls apart into two separate pieces,

$$\mathcal{L} = -\sum_{a=1}^{2}\left[i\rho_a^\dagger\sigma_\mu\partial_\mu\rho_a + \left(\frac{m}{2}\rho_a^T\sigma_2\rho_a + \text{H.c.}\right)\right]. \tag{2.17}$$

Each of these pieces by itself is a consistent SL(2, C)-invariant Lagrangian representing a Majorana particle. Note that the mass terms in (2.17) are of the same kind as the ones in (1.3) and in (1.6). Thus a Lagrangian describing a single spinor,

$$\mathcal{L} = -i\rho^\dagger\sigma_\mu\partial_\mu\rho - \left(\frac{m}{2}\rho^T\sigma_2\rho + \text{H.c.}\right), \tag{2.18}$$

is a desirable building block for the general theory. By itself it describes a Majorana particle and together with a twin brother of equal mass it describes a Dirac particle. Note that the phase of m in (2.18) can be altered at will by the redefinition $\rho = e^{i\theta/2}\rho'$ without affecting the kinetic term. However, once interaction terms are added this can no longer be done. Also note that the invariance of the ordinary Dirac equation (2.14) under the U(1) phase transformation $\psi \to e^{i\alpha}\psi$ is reflected in the invariance of (2.17) under the rotation

$$\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}. \tag{2.19}$$

In the Appendix we shall discuss the quantum theory of (2.18). Actually we should really consider (2.18) to involve quantum fields from the very beginning since $\rho^T\sigma_2\rho$ vanishes unless ρ is an anticommuting quantity.

III. THE (n,m) MODELS

The n neutrinos belonging to $\text{SU}(2)_L$ doublets are described by two-component spinors $\rho_1, \ldots, \rho_n$ and the m neutrinos which are $\text{SU}(2)_L$ singlets are described by two-component spinors $\rho_{n+1}, \ldots, \rho_{n+m}$. The four-component objects which go into the various interaction terms of the theory are related to these by [see (2.13)]

$$\rho_{\alpha L} = \begin{pmatrix} \rho_\alpha \\ 0 \end{pmatrix}. \tag{3.1}$$

We adopt the convention that Greek generation indices run from 1 to $n + m = N$ while Latin ones go from 1 to n. The quadratic part of the neutrino Lagrangian is a generalization of (2.18),

$$\mathcal{L} = \sum_\alpha \left[-i\rho_\alpha^\dagger\sigma_\mu\partial_\mu\rho_\alpha - \tfrac{1}{2}(\rho_\alpha^T\sigma_2 M_{\alpha\beta}\rho_\beta + \text{H.c.})\right]. \tag{3.2}$$

The matrix M in (3.2) must evidently be symmetric:

$$M_{\alpha\beta} = M_{\beta\alpha}. \tag{3.3}$$

It can be decomposed as

$$\begin{pmatrix} M_1 & D \\ D^T & M_2 \end{pmatrix}. \tag{3.4}$$

The $n \times n$ M_1 block comes from (1.3), the $m \times m$ M_2 block from (1.6), and the D piece from (1.7). Naturalness implies that M is a completely arbitrary, symmetric matrix. The most general case is type I. In type II cases, M_1 should be set to zero.

Our first task is to find a transformation which will bring the mass terms of (3.2) to a standard form which we take to be *real, diagonal*. Since we do not want to destroy the form of the first term of (3.2) we take the transformation to physical fields ν_α to be unitary:

$$\rho = U\nu, \quad UU^\dagger = U^\dagger U = 1, \tag{3.5}$$

in an obvious matrix notation. We then require

$$U^T M U = X\,,\qquad X_{\alpha\beta} = \delta_{\alpha\beta} X_\beta,\quad X_\beta = X_\beta^*\,. \tag{3.6}$$

Since U^T rather than U^{-1} appears in (3.6), there is no immediate guarantee that there exists a U fulfilling (3.5). We now show that this must be the case. First, note that the counting is right since M has $2N$ real parameters along the diagonal and $N(N-1)$ real parameters off diagonal for a total of N^2+N. This exactly matches[14] the N parameters of X plus the N^2 parameters of an arbitrary U. Since the product $M^\dagger M$ is Hermitian we can always find some unitary matrix U such that

$$U^\dagger M^\dagger M U = C = \text{real diagonal}\,.$$

Note that we can multiply U by a diagonal matrix of phases, without changing C. Taking the complex conjugate of this equation yields

$$C^* = C = U^T M^T M^{\dagger T} U^* = U^T M M^\dagger U^*\,,$$

where we have used (3.3). Defining $X \equiv U^T M U$ we then have

$$XX^\dagger = C = X^\dagger X\,.$$

From these we easily verify[15] that X must be diagonal. The freedom to multiply U by a diagonal matrix of phases shows, by using (3.6), that the elements of X can be always adjusted to be real, positive, for example. Thus we can always find a suitable U to bring (3.2) to the form

$$\mathcal{L} = \sum_\alpha \left[-i\nu_\alpha^\dagger \sigma_\mu \partial_\mu \nu_\alpha - \tfrac{1}{2}(\nu_\alpha^T \sigma_2 \nu_\alpha X_\alpha + \text{H.c.})\right]. \tag{3.7}$$

Note that because ν_α^T rather than $\nu_\alpha^\dagger$ appears in the second term we cannot make separate phase transformations on the ν_α without changing (3.7). This is different from the situation with charged-fermion fields.

We next use (3.5) and also an arbitrary $n\times n$ unitary transformation on the column of physical electron fields e_L,

$$E_L = \Omega e_L,\quad \Omega\Omega^\dagger = \Omega^\dagger\Omega = 1\,,$$

to express the charged-current weak interaction in terms of physical quantities,

$$\begin{aligned}\mathcal{L} &= ig2^{-1/2}W_\mu^- \sum_{a=1}^n \overline{E}_{aL}\gamma_\mu \rho_{aL} + \text{H.c.}\\ &= ig2^{-1/2}W_\mu^- \sum_{a,b,\alpha} \overline{e}_{bL}\gamma_\mu \Omega_{ab}^* U_{a\alpha}\nu_{\alpha L} + \text{H.c.}\end{aligned} \tag{3.8}$$

We define the rectangular matrix K, which is the analog of the KM matrix for the present case, by

$$K_{b\alpha} = \sum_{c=1}^n (\Omega^\dagger)_{bc} U_{c\alpha}\,. \tag{3.9}$$

K has N columns and n rows. It satisfies

$$KK^\dagger = 1\,. \tag{3.10}$$

However $K^\dagger K$ does not equal the unit operator. The basic charged-current weak interaction is then, in matrix notation,

$$\mathcal{L} = ig2^{-1/2}W_\mu^- \overline{e}_L \gamma_\mu K \nu_L + \text{H.c.} \tag{3.11}$$

Because K is in general (unless $m=0$) rectangular rather than square, the neutral-current (Z gauge field) interactions of the neutrino require a matrix for their parametrization. The appropriate term in $\mathcal{L}$ is

$$\begin{aligned}\mathcal{L} &= \frac{ig'}{2\sin\theta_W} Z_\mu \sum_{a=1}^n \overline{\rho}_{aL}\gamma_\mu \rho_{aL}\\ &= \frac{ig'}{2\sin\theta_W} Z_\mu \sum_{a,\alpha,\beta} \overline{\nu}_{\alpha L}\gamma_\mu U_{a\alpha}^* U_{a\beta}\nu_{\beta L}\,.\end{aligned} \tag{3.12}$$

The parametrizing matrix here is

$$P_{\alpha\beta} = \sum_{a=1}^n U_{\alpha a}^\dagger U_{a\beta}\,. \tag{3.13}$$

It is a square N-dimensional matrix. From (3.9) we note that

$$\sum_{a=1}^n \Omega_{ba} K_{a\alpha} = U_{b\alpha}$$

so that P may be simply expressed as

$$P = K^\dagger K\,. \tag{3.14}$$

Thus once the charged-current interactions are specified by K, no new parameters have to be introduced to describe the neutral-current interactions. In matrix notation the neutral-current neutrino interaction is finally

$$\mathcal{L} = \frac{ig'}{2\sin\theta_W} Z_\mu \overline{\nu}_L \gamma_\mu P \nu_L\,. \tag{3.15}$$

The fact that $P\neq 1$ is a statement that the Glashow-Iliopoulas-Maiani (GIM) mechanism is unnatural (unless $m=0$) for lepton theories with massive neutrinos. It has the physical consequences, to be discussed later, that heavier neutrinos can decay into three lighter ones and that neutral-current interactions should also show oscillation effects as a neutrino beam evolves. Note that P is Hermitian,

$$P = P^\dagger\,, \tag{3.16}$$

and, using (3.10), is seen to be a projection operator,

$$P^2 = P\,. \tag{3.17}$$

It is desirable to develop a parametrization for the rectangular matrix K. Putting indices into (3.10) gives

$$\sum_{\alpha=1}^{N} K_{a\alpha} K^*_{b\alpha} = \delta_{ab} ,$$

so the rows of K form a set of n (complex) orthonormal vectors, each with N entries. Before applying any restrictions we see that K is described by $2n(m+n)$ real parameters. From this should be subtracted n for the normalizations of the rows and $n(n-1)$ for their orthogonalizations. This gives a total of $n(n+2m)$ real parameters. Now we still have the freedom to multiply the n electron fields by arbitrary phases as in (2.9), for example. However, the neutrino fields cannot be multiplied by phases, since (3.7) would not then be invariant. So altogether the general (n, m) models of type I are described by a rectangular matrix K, with

$$n(n+2m-1) \tag{3.18}$$

real parameters. One's first thought about giving an explicit representation of K might be simply to use a matrix such as (2.6) with the $\omega_0(\gamma)$ factor deleted and to truncate the last m rows. However, this would involve more than the number of independent parameters (3.18). An easy way to proceed using the elementary (complex) planar rotations (2.3) is as follows. Define the basis vectors

$$e^{(\alpha)}_{\beta} = \delta_{\alpha\beta} . \tag{3.19}$$

Then the first row of K can be represented by the transpose of

$$X^{(1)} = \prod_{\alpha=2}^{N} \omega(\eta_{1\alpha}) e^{(1)} , \tag{3.20}$$

which has $2(n+m-1)$ real parameters. The second row is taken to be the transpose of

$$X^{(2)} = \prod_{\alpha=2}^{N} \omega(\eta_{1\alpha}) \prod_{\beta=3}^{N} \omega(\eta_{2\beta}) e^{(2)} . \tag{3.21}$$

This adds $2(n+m-2)$ real parameters. Similarly the third row is taken to be the transpose of

$$X^{(3)} = \prod_{\alpha=2}^{N} \omega(\eta_{1\alpha}) \prod_{\beta=3}^{N} \omega(\eta_{2\beta}) \prod_{\gamma=4}^{N} \omega(\eta_{3\gamma}) e^{(3)} \tag{3.22}$$

and so on. These vectors form an orthonormal set as required: $X^{(a)\dagger} X^{(b)} = \delta^{ab}$. For example,

$$X^{(1)\dagger} X^{(2)} = e^{(1)\dagger} \prod_{\beta=3}^{N} \omega(\eta_{2\beta}) e^{(2)} = e^{(1)\dagger} e^{(2)} = 0 , \tag{3.23}$$

since $\omega(\eta_{2\beta}) e^{(1)} = e^{(1)}$, for $\beta > 2$. Refer to (2.4) and note that K is now being parametrized by a set of

$$\sum_{a=1}^{n} (n+m-a) = \tfrac{1}{2} n(2m+n-1)$$

complex parameters $|\eta_{ab}| e^{i\theta_{ab}}$. Thus, exactly half of the parameters—the θ_{ab}'s—may be interpreted as CP-violating phases. The description of K given above is practical rather than just formal since it merely involves the multiplication of matrices with nontrivial 2×2 subblocks.

Finally let us consider the theories of type II in which no Higgs triplets are present. The mass matrix (3.4) now has an $n \times n$ block of zeros in the upper left corner:

$$M = \begin{bmatrix} 0 & D \\ D^T & M_2 \end{bmatrix} . \tag{3.24}$$

It is easy to verify that an arbitrary matrix of this type has an $(n-m)$-dimensional null space. Since the rank of a matrix is preserved[16] under the transformation (3.6) we conclude that X_β in (3.6) has $(n-m)$ zeros, which we take to be the first $(n-m)$ entries. Then, there is the additional possibility of making a $U(n-m)$ transformation, thereby deleting $(n-m)^2$ parameters on the ν_α without affecting (3.7). Thus, in our parametrization of K it would be reasonable to delete $\omega(\eta_{ab})$ factors associated with this subgroup and also eliminate $(n-m)$ other phases $\theta_{\alpha\beta}$. Note that the counting for (n, m) II theories with $n \le m$ is the same as for (n, m) I theories. Clearly, for $m=0$ all (n, m) theories give square K matrices.

Let us now illustrate the counting in some simple cases. The usual three-generation model with massless neutrinos is, in our notation, called (3, 0), type II. By the above we see that there are no mixing parameters since a full arbitrary SU(3) transformation can be made on the neutrino fields.

Equation (3.18) shows that the (2, 0) I case is described by one complex parameter as opposed to a real one for the GIM model. The (3, 0) I case is described by three complex parameters as opposed to two real and one complex in the KM case. Thus CP violation begins at the two- rather than the three-generation level for $(n, 0)$ I models.

The most popular model[3] would seem to be (3, 3). Either the type I or type II case is described by 12 independent *complex* parameters. In an attempt to reduce the number of parameters one might consider, for example, (3, 1) I, described by 12 *real* parameters, or (3, 1) II, described by $12-4=8$ real parameters. In any case it is clear that realistic theories which are natural under $SU(2)_L \times U(1)$ involve a large number of kinematical parameters. A larger unifying group G will tend to reduce this number, so a reasonable approach might be to see what constraints follow (as approximation or exactly) when $SU(2)_L \times U(1)$ is embedded in G. This question will be taken up elsewhere.

IV. NEUTRINO OSCILLATIONS

This topic was both pioneered and reviewed by Bilenky and Pontecorvo.[2] It has been the subject of much recent literature.[10] We would like to stress that the theory for the (n, m) case when $m \neq 0$ has some significant differences from the usual one.

In the first place, it is now possible to have the decay modes[17]

$$\nu_\alpha \to \nu_\beta + \nu_\gamma + \nu_\delta, \tag{4.1}$$

if energetically allowed. Equation (3.15) shows that they will be mediated by a four-fermion interaction proportional to

$$G_F(\bar{\nu}_L P \gamma_\mu \nu_L)^2. \tag{4.2}$$

Assuming (in the most "optimistic" case) the matrix elements of P to be of order of magnitude unity, the lifetime of a heavy neutrino ν' may be roughly estimated as

$$\tau(\nu' \to 3\nu) \approx \left|\frac{m(\mu)}{m(\nu')}\right|^5 \tau(\mu \to e\nu\bar{\nu}). \tag{4.3}$$

In order for this to have any observable effects, τ should be less than the order of 10^3 sec (sun-earth travel time). With (4.3) this implies that we should only expect physical effects for a "heavy" neutrino mass $m(\nu')$ satisfying

$$m(\nu') \gtrsim 2 \text{ MeV}. \tag{4.4}$$

We shall now set up the neutrino oscillation formalism for the (n, m) theories. Consider a beam of neutrinos produced in association with an electron of type a. From (3.11) we observe that the neutrino state produced will be a mixture of physical states given by

$$\sum_\alpha K_{a\alpha}|\nu_\alpha\rangle. \tag{4.5}$$

In the course of time this will evolve to the state

$$\sum_\alpha K_{a\alpha} \exp[-i(E_\alpha - i\Gamma_\alpha/2)t]\,|\nu_\alpha\rangle, \tag{4.6}$$

where E_α is the energy eigenvalue of $|\nu_\alpha\rangle$ and Γ_α its laboratory-frame width. The usual approximation is $E_\alpha \approx E + m_\alpha{}^2/2E$. The probability amplitude for having a neutrino interaction with emission of an associated electron of type b is proportional to the overlap of (4.6) with the state $\sum_\beta \langle \nu_\beta | K^*_{b\beta}$ and is then

$$\text{amp}(a \to b) \propto \sum_\alpha K_{a\alpha} K^*_{b\alpha} \exp[-i(E_\alpha - i\Gamma_\alpha/2)t]. \tag{4.7}$$

This yields a probability *factor*

$$I_{\text{CC}}(a \to b) = \sum_{\alpha,\alpha'} K_{a\alpha} K^*_{a\alpha'} K_{b\alpha'} K^*_{b\alpha} \times \exp[i(E'_\alpha - E_\alpha)t - \tfrac{1}{2}(\Gamma_\alpha + \Gamma_{\alpha'})t]. \tag{4.8}$$

In the usual case, K is a square matrix and the Γ_α are zero. Then the unitarity of K implies that $\sum_b I_{\text{CC}}(a \to b) = 1$. Here even when the Γ_α are set to zero we have

$$\sum_b I_{\text{CC}}(a \to b) < 1 \tag{4.9}$$

unless $m = 0$. The physical reason for this is that the set of neutrinos which couple to the electrons do not span the complete set.

Another different phenomenon in a natural $\text{SU}(2)_L \times \text{U}(1)$ theory is the possibility of oscillations for neutral-current reactions. Using the field expansion (A4) of the Appendix and the neutral-current interaction (3.15), we find that the matrix element for a physical neutrino of type α to produce one of type β by Z_μ exchange with hadrons is proportional to

$$T_{\beta\alpha} \equiv [i(\text{Im}P_{\beta\alpha})\bar{u}\gamma_\mu u + (\text{Re}P_{\beta\alpha})\bar{u}\gamma_\mu\gamma_5 u]S_\mu, \tag{4.10}$$

where $\bar{u}$ and u are ordinary Dirac spinors, and S_μ is a kinematical factor. In a neutrino beam produced at $t = 0$ in association with an electron of type a, the spin-dependent amplitude for producing a type β neutrino off hadrons is then proportional to

$$\sum_\alpha K_{a\alpha} \exp[-i(E_\alpha - i\Gamma_\alpha/2)t]\,T^*_{\beta\alpha}. \tag{4.11}$$

The probability factor for instigating a particular hadronic neutral-current reaction is the magnitude of (4.11) squared summed over spins and also over the unobserved neutrinos β. This result can be very much simplified if it is assumed that K, and hence P, are purely real. Then the probability factor for a neutral-current reaction is simply

$$I_{\text{NC}}(a \to \nu\text{'s}) = \sum_{\alpha,\alpha',\beta} K_{a\alpha} K_{a\alpha'} P_{\beta\alpha} P_{\beta\alpha'} \times \exp[i(E_{\alpha'} - E_\alpha)t - \tfrac{1}{2}(\Gamma_\alpha + \Gamma_{\alpha'})t]. \tag{4.12}$$

Using (3.17), (3.16), and (3.14) we note

$$\sum_\beta P_{\beta\alpha} P_{\beta\alpha'} = P_{\alpha\alpha'} = \sum_b K_{b\alpha'} K_{b\alpha}.$$

Comparing (4.12) with (4.8) finally gives

$$I_{\text{NC}}(a \to \nu\text{'s}) = \sum_b I_{\text{CC}}(a \to b), \tag{4.13}$$

which holds for real K. In the usual case the right-hand side of (4.13) is just unity so no oscillation is possible. In general, however, oscillation effects are expected. We may check this for the simplest (one-generation) case $(n,m)=(1,1)$. Then the matrix K (assumed real) is simply

$$K=(\cos\theta \quad \sin\theta)\,,$$

where θ is some angle. Substituting this into (4.8) and (4.13) gives (neglecting neutrino decay)

$$I_{CC}(1\to 1)=I_{NC}(1\to \nu\text{'s})=1-\sin^2 2\theta \sin^2\left[\frac{(E_1-E_2)t}{2}\right],$$

which clearly shows the oscillations. Here the ratio of charged-current to neutral-current events is constant, but this feature does not hold in general.

The formalism developed here can be applied to many situations of interest. This will be considered elsewhere. In the present paper we have pointed out the existence of possible new neutrino oscillation effects and provided a general framework for their analysis.

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APPENDIX

The quantum theory of (2.18) was discussed a long time ago by Case.[18] Here we would just like to point out that we can get it in a simple way by noting that the ordinary quantization of the Dirac equation gives it to us *twice*. With conventional notations, the Dirac field is expanded (in a box of volume V) as

$$\psi=\frac{1}{\sqrt{V}}\sum_{\vec{p},r}\left(\frac{m}{E_p}\right)^{1/2}[e^{ip\cdot x}a_r(\vec{p})u^{(r)}(\vec{p}) + e^{-ip\cdot x}b_r^\dagger(\vec{p})\,v^{(r)}(\vec{p})]\,, \qquad (\text{A}1)$$

with $E_p=(\vec{p}^{\,2}+m^2)^{1/2}$ and the phase conventions

$$v^{(r)}(\vec{p})=C\bar{u}^{T\,(r)}(\vec{p})\,. \qquad (\text{A}2)$$

Using (2.13) we note

$$\begin{pmatrix}\chi\\0\end{pmatrix}=\frac{1+\gamma_5}{2}\,\psi,\quad \begin{pmatrix}\phi\\0\end{pmatrix}=\frac{1+\gamma_5}{2}\,C\bar{\psi}^T\,. \qquad (\text{A}3)$$

Focusing on ρ_2 for definiteness we have

$$\begin{aligned}\begin{pmatrix}\rho_2\\0\end{pmatrix}&=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}\chi\\0\end{pmatrix}+\begin{pmatrix}\phi\\0\end{pmatrix}\right]\\ &=\left(\frac{1+\gamma_5}{2}\right)\sum_{\vec{p},r}\left(\frac{m}{E_pV}\right)^{1/2}[e^{ip\cdot x}u^{(r)}(\vec{p})A_r(\vec{p}) + e^{-ip\cdot x}v^{(r)}(\vec{p})A_r^\dagger(\vec{p})]\,,\\ A_r(\vec{p})&=\frac{1}{\sqrt{2}}\,[a_r(\vec{p})+b_r(\vec{p})]\,.\end{aligned} \qquad (\text{A}4)$$

(A similar expression holds for ρ_1.) Note that the anticommutation relations for the a's and b's imply

$$[A_r(\vec{p}),A_{r'}^\dagger(\vec{p}\,')]_+=\delta_{rr'}\delta_{\vec{p},\vec{p}\,'}\,,\ \text{etc.} \qquad (\text{A}5)$$

Thus we expect that relabeling $\rho_2\to\rho$ in (A4) gives the proper quantum theory of (2.18). It is seen from (A4) that a convenient set of wave functions for this theory are the left-handed projections of the usual massive Dirac wave functions u and v. To convince ourselves that (A4) is in fact correct we examine the energy operator, essentially following Case. The canonical procedure applied to (2.18) gives the equation of motion

$$i\sigma_\mu\partial_\mu\rho=-m\sigma_2\rho^*\,, \qquad (\text{A}6)$$

and the energy-density operator

$$\mathcal{H}=\frac{i}{2}\,\rho^\dagger\frac{\partial}{\partial t}\,\rho+\text{H.c.} \qquad (\text{A}7)$$

Equation (A6) was used in obtaining (A7). Note that (A6) and its conjugate imply that ρ satisfies the Klein-Gordon equation

$$(-\Box+|m|^2)\rho=0\,. \qquad (\text{A}8)$$

Substituting (A4) into (A7) gives, after some standard manipulations,

$$\begin{aligned}H=\int d^3x\,\mathcal{H}&=m\sum_{\vec{p},r,r'}[A_{r'}^\dagger(\vec{p})A_r(\vec{p})u_L^{r'\dagger}(\vec{p})u_L^r(\vec{p}) + A_r^\dagger(\vec{p})A_{r'}(\vec{p})\,v_L^{r'\dagger}(\vec{p})\,v_L^r(\vec{p})]\\ &\quad+\text{zero-point energy}\,.\end{aligned} \qquad (\text{A}9)$$

Now the factor in the first term $u_L^{r'\dagger}(\vec{p})u_L^r(\vec{p})$ can be rewritten as

$$u^{r'\dagger}(\vec{p})u^r(\vec{p})-u_R^{r'\dagger}(\vec{p})u_R^r(\vec{p})=\delta^{r'r}E_p/m-v_L^{r\dagger}(\vec{p})\,v_L^{r'}(\vec{p})\,.$$

Putting this back into (A9) gives immediately

$$H=\sum_{\vec{p},r}E_pA_r^\dagger(\vec{p})A_r(\vec{p})\,, \qquad (\text{A}10)$$

so it is legitimate to consider $A_r^\dagger(\vec{p})\,|0\rangle$ as a one-particle state, for example.

[1]F. Reines, H. W. Sobel, and E. Pasierb, University of California, Irvine, report, 1980 (unpublished).

[2]S. M. Bilenky and B. Pontecorvo, Phys. Rep. 41C, 225 (1978).

[3]For example, M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, edited by P. Van Nieuwenhuizen and D. Freedman (North-Holland, Amsterdam, 1979), p. 315; E. Witten, Phys. Lett. 91B, 81 (1980); M. Magg and C. Welterich, CERN Report No. TH2829 (unpublished); R. Barbieri, CERN Report No. TH2850, 1980 (unpublished); R. N. Mohapatra and R. E. Marshak, Phys. Rev. Lett. 44, 1316 (1980); 44, 1643(E) (1980).

[4]M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973).

[5]S. L. Glashow, Nucl. Phys. 22, 579 (1961); S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8)*, edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367.

[6]See, for example, S. Weinberg, Rev. Mod. Phys. 46, 255 (1974). One should of course be aware that naturalness is basically an esthetic criterion.

[7]See, for example, G. Luders, Ann. Phys. (N.Y.) 2, 1 (1957).

[8]See, for example, P. Roman, *Theory of Elementary Particles* (North-Holland, Amsterdam, 1961).

[9]In a previous paper we used this notation to describe a theory in which lepton-number conservation was assumed to hold. Such an assumption, though probably reasonable in some approximation, is not natural in the present sense. See J. Schechter and J. W. F. Valle, Phys. Rev. D 21, 309 (1980). An earlier discussion of some theories of this type was given by J. Donoghue, Phys. Rev. D 18, 1632 (1978). Lepton mixing matrices assuming lepton-number conservation have been discussed, for example, by K. Sato and M. Kobayashi, Prog. Theor. Phys. 58, 1775 (1977); B. Lee, S. Pakvasa, H. Sugawara, and R. Shrock, Phys. Rev. Lett. 38, 937 (1977); E. Kolb and T. Goldman, Caltech Report No. 68-799, 1979 (unpublished).

[10]The general theory of neutrino oscillations is reviewed in Ref. 2. Very recent discussions include A. De Rújula, M. Lusiguoli, L. Maiani, S. T. Petrov, and R. Petronzio, Nucl. Phys. B168, 54 (1980); V. Barger, K. Whisnat, D. Cline, and R. J. N. Phillips, Phys. Lett. 93B, 194 (1980); University of Wisconsin Report No. C00-881-146, 1980 (unpublished); V. Barger, P. Langacker, J. P. Leveille, and S. Pakvasa, University of Wisconsin Report No. C00-881-149, 1980 (unpublished). In the last report, which we received after the work described in the present paper was completed, the possibility of oscillation effects in neutral-current reactions was also discussed.

[11]This is the notation of S. Okubo, Prog. Theor. Phys. 28, 24 (1962).

[12]J. Schechter and J. W. F. Valle, Ref. 9 above.

[13]See Lev S. Pontryagin, *Topological Groups*, 2nd ed. (Gordon and Breach, New York, 1966), p. 296. We would like to thank N. Mukunda for pointing this out to us.

[14]An alternative approach, suggested to us by A. P. Balachandran, involves showing the uniqueness of the diagonalizing U by verifying that the little group of X under the action UXU^T is trivial for general X.

[15]First note that C should be considered to be an *arbitrary* real, diagonal matrix since we do not want to restrict M for a natural theory. The diagonal matrix $C^{1/2}$ is nonsingular. Setting $X=C^{1/2}Y$ we find from $XX^\dagger=C$ that $YY^\dagger=1$ and then from $X^\dagger X=C$ that $[C,Y]=0$. The latter equation explicitly reads $(C_\alpha-C_\beta)Y_{\alpha\beta}=0$ (no sum), where $C_{\alpha\beta}=C_\alpha\delta_{\alpha\beta}$. Thus the off-diagonal elements of Y vanish so $X=C^{1/2}Y$ is also diagonal, as stated. Note that if $C_\alpha=C_\beta$, for some α and β, we have a situation similar to that in the ordinary Dirac case (2.17) and a resulting larger invariance as in (2.19). The latter, which corresponds to lepton-number conservation is clearly broken for the general situation.

[16]This follows from a standard result in matrix theory. See, for example, G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, 3rd ed. (Macmillan, New York, 1965), p. 220.

[17]The higher-order decays $\nu' \to \nu\gamma$ are discussed in Ref. 2.

[18]K. M. Case, Phys. Rev. 107, 307 (1957). The massless situation has been more recently discussed by C. Ryan and S. Okubo, Suppl. Nuovo Cimento, Series I, 2, 234 (1964). A detailed discussion of the massive case will be given by P. Mannheim (private communication).