Three-jet final states in Z^0 decay for heavy quarks

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We study the influence of quark masses on the values of average spherocity S and average thrust T for Z^0 boson decay into three jets in the standard model. We find that for quark masses (m_q) greater than ~ 10 GeV the average values $\langle S\rangle$ and $\langle 1-T\rangle$ may be significantly altered from their values for massless quarks previously obtained in the literature.

I. INTRODUCTION

Recently, the combination of the Weinberg-Salam (WS)[1] model of unified weak and electromagnetic interactions and the model of the strong interactions, quantum chromodynamics (QCD),[2] has met with many experimental successes. One of the most important predictions of QCD is that of jets[3] in e^+e^-, eN, and hadron-hadron collisions; in particular, the observation of three-jet events[4] in e^+e^- away from heavy $q\bar{q}$ resonances seems to be in accord with the predictions of QCD[5] with massless quarks.

Another source of jets is the decay of heavy particles; in this paper we will consider the decay of the neutral gauge boson, the Z, in the standard WS model into three-jet final states: $Z\to q\bar{q}g$. In particular, we will examine the influence of finite quark masses on the average values of the infrared-safe variables spherocity S (Ref. 6) and thrust T (Ref. 7). This will be of particular importance for heavy quarks such as the t or possible heavier quarks into which the Z can decay.

In Sec. II we consider the calculation of the differential decay distribution for the $Z\to q\bar{q}g$ decay in terms of the scaling variables x_i ($i=1,2,3$) (Ref. 8) for massive quarks. An important result here is that the distribution is flavor dependent since the $Q=\frac{2}{3}$ and $Q=-\frac{1}{3}$ quarks couple to the Z^0 with different vector and axial-vector coupling constants.

In Sec. III we discuss the detailed kinematics for the $Z\to q\bar{q}g$ process with massive quarks. This includes a determination of the bounds on the scaling variables x_i and the variables spherocity and thrust. This is complicated by the fact that there now exist three distinct kinematic regions depending on the correspondence between the variables x_i and T.

Our results can be found in Sec. IV and our conclusions can be found in Sec. V.

II. $Z\to qq$ + GLUON FOR MASSIVE QUARKS

The lowest-order-in-α_s contribution to the process $Z\to q\bar{q}$ +gluon ($Z\to q\bar{q}g$) comes from the two diagrams in Fig. 1. Apart from color considerations we can write the matrix element for this process directly as

$$\mathfrak{M}=-ig_s\bar{u}(p_1)\{\gamma_\nu[(\not{p}_1+\not{p}_3)-m]^{-1}\gamma_\mu(v-a\gamma_5) -\gamma_\mu(v-a\gamma_5)[(\not{p}_2+\not{p}_3)-m]^{-1}\gamma_\nu\} \times v(p_2)\epsilon_g^\nu\epsilon_Z^\mu\,. \tag{2.1}$$

Here we have defined the quark-antiquark-Z^0 coupling by

$$\mathcal{L}_{q\bar{q}Z}=\bar{\psi}_q\gamma_\mu(v-a\gamma_5)\psi_q Z^\mu\,. \tag{2.2}$$

$p_{1(2)}$ is the momentum of the (anti)quark and p_3 is the gluon momentum; the mass of the outgoing quark or antiquark is denoted by m. By simple manipulation we can rewrite Eq. (2.1) as

$$\mathfrak{M}=\frac{-2ig_s}{(2p_1\cdot p_3)(2p_2\cdot p_3)}\epsilon_g^\nu\epsilon_Z^\mu\bar{u}(p_1) \times\{(p_2\cdot p_3)\gamma_\nu[(\not{p}_1+\not{p}_3)+m]\gamma_\mu(v-a\gamma_5) -(p_1\cdot p_3)\gamma_\mu(v-a\gamma_5)[(\not{p}_2+\not{p}_3)+m]\gamma_\nu\}v(p_2)\,. \tag{2.3}$$

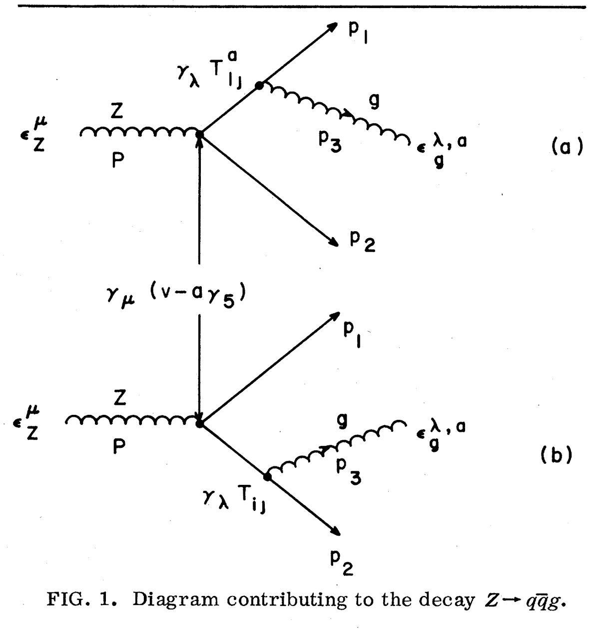

FIG. 1. Diagram contributing to the decay $Z\to q\bar{q}g$.

We now simply square $\mathfrak{M}$ and use the polarization sums

$$\sum_\lambda \epsilon_Z^\mu(\lambda)\epsilon_Z^\mu(\lambda) = -g^{\mu\mu'} + p^\mu p^{\mu'}/M_Z^2\,, \tag{2.4}$$

$$\sum_\lambda \epsilon_g^\nu(\lambda')\epsilon_g^{\nu'}(\lambda') = -g^{\nu\nu'}\,. \tag{2.5}$$

The decay rate is then given directly by (apart from color factors)

$$d\Gamma = (2\pi)^{-5}(2M_Z)^{-1}\delta^4 \times (P - p_1 - p_2 - p_3)\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3}|\mathfrak{M}|^2\,. \tag{2.6}$$

To make contact with the usual development on $e^+e^- \to q\bar{q}g$ we define the scaling variables[8]

$$x_i \equiv 2E_i/\sqrt{s}\,, \quad \sum_i x_i = 2 \tag{2.7}$$

(here, of course, $\sqrt{s} = M_Z$). The various dot products of four-momenta can then be rewritten in terms of these variables:

$$\begin{aligned} P\cdot p_i &= \tfrac{1}{2}s x_i\,,\\ 2p_1\cdot p_2 &= s(1-x_3) - 2m^2\,,\\ 2p_1\cdot p_3 &= s(1-x_2)\,,\\ 2p_2\cdot p_3 &= s(1-x_1)\,. \end{aligned} \tag{2.8}$$

We also have

$$\begin{aligned} s_{12} &\equiv (p_1+p_2)^2 = 2p_1p_2 + 2m^2 = s(1-x_3)\,,\\ s_{13} &\equiv (p_1+p_3)^2 = 2p_1p_3 + m^2 = s(1-x_2) + m^2\,,\\ s_{23} &= (p_2+p_3)^2 = 2p_2p_3 + m^2 = s(1-x_1) + m^2\,. \end{aligned} \tag{2.9}$$

The usual results for the massless case can be obtained by setting $m^2 = 0$ in the above formulas. Eliminating x_3 in favor of x_1, x_2 through the constraint Eq. (2.7) and defining $\delta \equiv 2m/\sqrt{s} = 2m/M_Z$ we find, after performing a trivial angular integration,

$$\frac{d\Gamma_{q\bar{q}g}}{dx_1dx_2} = C\left\{(v^2+a^2)(x_1^2+x_2^2)Z_1Z_2 - \frac{a^2\delta^2}{2}[2(Z_1+Z_2)+Z_1^2+Z_2^2] - \tfrac{1}{2}v^2\delta^2\left[6Z_1Z_2 - 4(Z_1+Z_2) - 2 - \frac{Z_1}{Z_2} - \frac{Z_2}{Z_1} + Z_1^2 + Z_2^2\right] + \tfrac{1}{4}(2v^2-a^2)\delta^4(2Z_1Z_2+Z_1^2+Z_2^2)\right\}, \tag{2.10}$$

where $Z_i = (1-x_i)^{-1}$ and C is an overall coefficient obtained only after performing the color summation

$$C = \left(\frac{2\alpha_s}{3\pi}\right)\frac{M_Z}{4\pi}\,. \tag{2.11}$$

Noting that the decay rate for $Z \to q\bar{q}$ for massive quarks is[9]

$$\Gamma_0 = \frac{M_Z}{4\pi}(1-\delta^2)^{1/2}[v^2(1+\tfrac{1}{2}\delta^2) + a^2(1-\delta^2)]\,, \tag{2.12}$$

we may write

$$\frac{1}{\Gamma_0}\frac{d\Gamma_{q\bar{q}g}}{dx_1dx_2} = \left(\frac{2\alpha_s}{3\pi}\right)\left\{\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)} - \frac{a^2\delta^2}{2(v^2+a^2)}[2(Z_1+Z_2)+Z_1^2+Z_2^2] - \frac{v^2\delta^2}{2(v^2+a^2)}\left[6Z_1Z_2 - 4(Z_1+Z_2) - 2 - \frac{Z_1}{Z_2} - \frac{Z_2}{Z_1} + Z_1^2 + Z_2^2\right] + \frac{(2v^2-a^2)\delta^4}{4(v^2+a^2)}[2Z_1Z_2+Z_1^2+Z_2^2]\right\} \times (1-\delta^2)^{-1/2}\left[\frac{v^2}{v^2+a^2}(1+\tfrac{1}{2}\delta^2) + \frac{a^2}{v^2+a^2}(1-\delta^2)\right]^{-1}, \tag{2.13}$$

where we have written the first term in the expansion explicitly; this is the only remaining term as $\delta \to 0$ and gives the usual result obtained for massless quarks.

In the standard Weinberg-Salam model[1] we have

$$\begin{aligned} v &= (T_3 - 2Q\sin^2{}_W)\frac{e}{\sin 2\theta_W}\,,\\ a &= T_3\frac{e}{\sin 2\theta_W}\,, \end{aligned} \tag{2.14}$$

where T_3 and Q are the third component of the weak isospin and the electric charge of the relevant quark and $\sin^2\theta_W \simeq 0.23$. Note that, in general, two quarks with the same mass but with different charges would have different distributions since the terms of order δ^2 and δ^4 are functions of v and a. Realistically, however, since $a^2 \propto T_3^2$ is the same for both $Q = \frac{2}{3}$ and $Q = -\frac{1}{3}$ quarks, the distribution differs only in the terms proportional to $v^2/v^2 + a^2$; note that this ratio is very similar in magnitude in the two cases ($x_W = \sin^2\theta_W = 0.23$)

$$\frac{v^2}{v^2+a^2} = \begin{cases} 0.325, & Q = -\frac{1}{3}\,,\\ 0.130, & Q = \frac{2}{3}\,. \end{cases} \tag{2.15}$$

For practical purposes, as we shall see, there is very little difference between the results for $Q=\frac{2}{3}$ and $Q=-\frac{1}{3}$ quarks.

We now proceed to translate this distribution from the variables x_1 and x_2 into the well known, infrared-safe variables spherocity S (Ref. 6) and thrust T (Ref. 7).

III. KINEMATICS OF THE $Z \to qqg$ PROCESS

A discussion of the kinematics of the $Z \to q\bar{q}g$ process must begin with a discussion of the allowed ranges of the variables x_i.[8] These can be obtained by first considering the allowed ranges of the three invariants s_{12}, s_{13}, and s_{23}, which are familiar from the usual Dalitz-plot kinematic considerations

$$\begin{aligned} &4m^2 \leq s_{12} \leq s\,, \\ &m^2 \leq s_{13},\, s_{23} \leq (\sqrt{s}-m)^2\,. \end{aligned} \tag{3.1}$$

Using Eq. (2.9) these can be translated into

$$0 \leq x_3 \leq 1-\delta^2, \quad \delta \leq x_1, x_2 \leq 1\,, \tag{3.2}$$

where δ has been defined previously.

We now turn to the definitions of S (Ref. 6) and T (Ref. 7); we will consider T (thrust) first. We will define T as

$$T = 2\max \frac{\tilde{\sum}_i p_\parallel^i}{\sum_i E_i}\,, \tag{3.3}$$

where $\tilde{\sum}$ signifies a sum over all particles in a hemisphere and $p_\parallel$ is the momentum parallel to the jet axis chosen to maximize T. With our definitions of x_i, Eq. (3.3) becomes[10]

$$T = \max\{(x_1^2-\delta^2)^{1/2}, (x_2^2-\delta^2)^{1/2}, x_3\}\,. \tag{3.4}$$

This reverts to the usual expressions as $\delta \to 0$. (The sum $\sum_i E_i$ in the denominator is chosen to simplify the normalization.) For spherocity S we will use the definition

$$S = \frac{16}{\pi^2}\left(\frac{\sum_i |p_T^i|}{\sum_i E^i}\right)^2\,, \tag{3.5}$$

where p_T is the momentum transverse to the jet axis.

For the three-jet final state, S is easily calculated using the angular relations[10]

$$\begin{aligned} \cos\theta_{12} &= \frac{x_1x_2-2(1-x_3)+\delta^2}{(x_1^2-\delta^2)^{1/2}(x_2^2-\delta^2)^{1/2}}, \\ \cos\theta_{13} &= \frac{x_1x_3-2(1-x_2)}{x_3(x_1^2-\delta^2)^{1/2}}\,, \\ \cos\theta_{23} &= \frac{x_2x_3-2(1-x_1)}{x_3(x_2^2-\delta^2)^{1/2}}\,, \end{aligned} \tag{3.6}$$

where θ_{ij} is the angle between momenta p_i and p_j.

We find

$$S = \frac{64}{\pi^2T^2}[(1-x_1)(1-x_2)(1-x_3)-(\delta/2)^2x_3^2]\,. \tag{3.7}$$

Next, we consider the allowed ranges of S and T; these depend on the kinematic region

$$\begin{aligned} &\text{I:}\quad T=(x_1^2-\delta^2)^{1/2} \geq (x_2^2-\delta^2)^{1/2}, x_3\,, \\ &\text{II:}\quad T=(x_2^2-\delta^2)^{1/2} \geq (x_1^2-\delta^2)^{1/2}, x_3\,, \\ &\text{III:}\quad T=x_3 \geq (x_1^2-\delta^2)^{1/2}, (x_2^2-\delta^2)^{1/2}. \end{aligned} \tag{3.8}$$

The minimum value of thrust is obtained when $p_1=p_2=p_3$, or simply

$$(x_1^2-\delta^2)^{1/2}=(x_2^2-\delta^2)^{1/2}=x_3\,. \tag{3.9}$$

Using $x_1=x_2=x$ and $x_3=2-x_1-x_2$, Eq. (3.9) is satisfied when

$$\begin{aligned} &x=\tfrac{4}{3}-\tfrac{2}{3}(1-\tfrac{3}{4}\delta^2)^{1/2}, \\ &\text{so} \\ &T^{\min}=x_3=\tfrac{4}{3}(1-\tfrac{3}{4}\delta^2)^{1/2}-\tfrac{2}{3}\,. \end{aligned} \tag{3.10}$$

Note that for $\delta=0$, $T^{\min}=\frac{2}{3}$ as is well known. This result is, of course, independent of the kinematic region; $T^{\max}$, however, does depend on the kinematic region through the allowed range of the x_i:

$$\begin{aligned} &\text{I and II:}\quad T^{\max}=(1-\delta^2)^{1/2}, \\ &\text{III:}\quad T^{\max}=1-\delta^2\,. \end{aligned} \tag{3.11}$$

Note $T_{\max} \to 1$ in all three regions as $\delta \to 0$.

For spherocity S we obtain the allowed regions by the following procedure; first, to obtain $S^{\max}$ in any of the three physical regions, consider S as a function of two independent variables out of the set (x_1, x_2, x_3). This is easily accomplished by using Eq. (2.7); we have

$$\begin{aligned} &\text{I and II:}\quad S=\frac{64}{\pi^2T^2}[(1-x_1)(1-x_2)(x_1+x_2-1) \\ &\qquad\qquad -(\delta/2)^2(2-x_1-x_2)^2]\,, \\ &\text{III:}\quad S=\frac{64}{\pi^2T^2}[(1-x_3)(1-x)(x+x_3-1)-(\delta/2)^2x_3^2]\,, \end{aligned} \tag{3.12}$$

where x is either x_1 or x_2. Keeping the variable corresponding to thrust fixed in each region we simply take the partial derivative with respect to the remaining variable, set the result to zero, and solve for this remaining variable. (Simple calculation of the second derivative shows that this procedure does indeed locate $S^{\max}$ and not the minimum S value $S^{\min}$.) We find

$$\begin{aligned} &\text{I:}\quad x_2=(1-\tfrac{1}{2}x_1)\frac{(1-x_1)+\frac{1}{2}\delta^2}{(1-x_1)+\frac{1}{4}\delta^2}\,, \quad x_3=2-x_1-x_2, \\ &\text{II:}\quad \text{same as I with } x_1 \leftrightarrow x_2\,, \\ &\text{III:}\quad x_1=x_2=1-\tfrac{1}{2}x_3\,. \end{aligned} \tag{3.13}$$

Substitution of these values into Eq. (3.12) for the appropriate kinematic region gives S^{max} as a function of the x_i corresponding to thrust in that region; e.g., in region III we have

$$S^{max}=\frac{16}{\pi^2}[(1-\delta^2)-T]\ . \tag{3.14}$$

All of these formulas reduce to the original result

$$S^{max}=\frac{16}{\pi^2}(1-T) \tag{3.15}$$

as $\delta \to 0$.

There remains only to find S^{min}; for this purpose we need consider the kinematic boundaries of the three regions for nonzero δ in comparison with the usual ($\delta=0$) case. [This is because, since $S(x_i, x_j)$ is a continuous function with a maximum in the interior of the allowed region, the minimum must occur on a boundary.].

Let us first consider the case $\delta=0$; here all results are symmetric under $x_1 \leftrightarrow x_2 \leftrightarrow x_3$ so we define the two independent variables as x and T (the remaining variable is given by $\bar{x}=2-x-T$). The allowed kinematic region is shown as the shaded area in Fig. 2. The uppermost boundary results from the definition that T be greater than x ($x \leq T$) while the right-hand boundary results from the constraint $T \leq 1$. Similarly the left-hand boundary comes from the constraint $T \geq \frac{2}{3}$ and the lower boundary from the energy-momentum constraint $\sum_i x_i=2$. Since $x+T=2-\bar{x}$ and $\bar{x} \leq 1$, obviously $x+T \geq 1$ or $x \geq 1-T$; this forms the lower boundary. To find S^{min} we merely check which of the boundaries give physically meaningful results. We find immediately that S^{min} must occur when $x=T$ (the upper boundary) yielding the familiar result (when $\delta=0$)

$$S^{min}=\frac{64}{\pi^2 T^2}(1-T)^2(2T-1)\ . \tag{3.16}$$

We now must ask how these results change when $\delta \neq 0$. As might be expected this depends on the kinematic region of interest. We will find, however, that S^{min} always occurs for the upper boundary independent of the kinematic region.

Figure 3 shows region II for $\delta=0.2$ with x_1 and T as the independent variables. This figure differs only slightly from the massless case shown in the previous figure. The upper and lower limits on T have changed from 1 and $\frac{2}{3}$ to $(1-\delta^2)^{1/2}$ and $\frac{4}{3}(1-\frac{3}{4}\delta^2)^{1/2}-\frac{2}{3}$; these give the right-hand and left-hand boundaries, respectively. There are two lower boundaries here, the first being $x_1 \geq \delta$ and the second resulting from $x_1+x_2 \geq 1+\delta^2$ since max $(x_3)=1-\delta^2$. Obviously the upper limit here is given by $x_1=x_2$ and it is this boundary which gives S^{min}. We find

$$S^{min}=\frac{64}{\pi^2 T^2}[(1-x)^2(2x-1)-\delta^2(1-x)^2]\ , \tag{3.17}$$

where

$$x=x_1=x_2=(T^2+\delta^2)^{1/2}. \tag{3.18}$$

Because of $x_1 \leftrightarrow x_2$ symmetry the same result pertains in region I.

Figure 4 shows the allowed area of region III for $\delta=0.2$ and with T and x_1 being the independent variables. Here, again, the left- and right-hand boundaries are given by T^{min} and T^{max}, respectively:

$$T^{min}=\tfrac{4}{3}(1-\tfrac{3}{4}\delta^2)^{1/2}-\tfrac{2}{3}\ ,$$
$$T^{max}=1-\delta^2\ . \tag{3.19}$$

The lower boundary results from $\delta \leq x_1$ as before

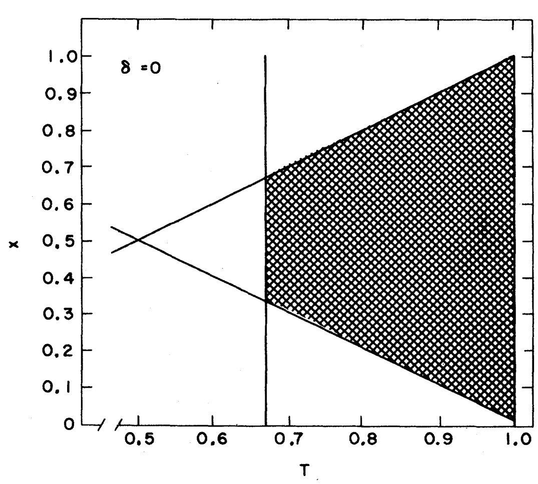

FIG. 2. Allowed kinematic region in the x-T plane for $\delta=0$.

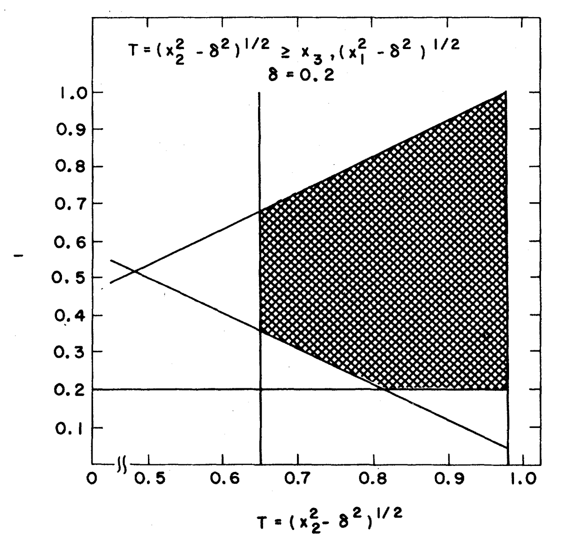

FIG. 3. Allowed kinematic region in the x_1-T plane for $\delta=0.2$ with $T=(x_2^2-\delta^2)^{1/2}$.

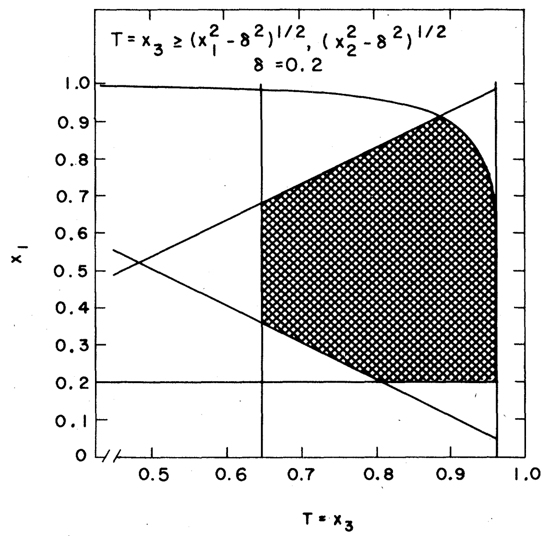

FIG. 4. Allowed kinematic region in the x_1-T plane for $\delta = 0.2$ with $T = x_3$.

and $x_1 + x_3 \geq 1$ since $\max(x_2) = 1$; the upper boundary in this case differs from that found in regions I and II. One of the boundaries is the usual one resulting from $p_3 \geq p_1, p_2$:

$$x_1 \leq (T^2 + \delta^2)^{1/2}. \tag{3.20}$$

The additional boundary results from requiring S to be positive or from requiring the angles in Eq. (3.6) to be real. In regions I and II (as well as in the massless case) this condition is trivially satisfied by the constraint $x_{1,2} \leq 1$; here, however, it implies an additional constraint on the physical region. For fixed T, the constraint $S \geq 0$ implies

$$x_1 \leq 1 - \frac{T}{2} + \frac{T}{2}\left(1 - \frac{\delta^2}{1-T}\right)^{1/2}, \tag{3.21}$$

which, as is easily seen, reduces to $x_1 \leq 1$ for $\delta = 0$. Thus the value of S in region III is given by

$$S^{\min} = \frac{64}{\pi^2 T^2}[(1-T)(1-x)(T+x-1) - (\delta/2)T^2], \tag{3.22}$$

where

$$x = \min[\text{Eq. (3.20), Eq. (3.21)}] \tag{3.23}$$

and $x_2 = 2 - x - x_3$.

Once the boundaries are known we must next translate the differential cross section as a function of x_1 and x_2 into a function of the variable S and T as well as determine the Jacobian for the transformation

$$\frac{d\Gamma_{q\bar{q}g}}{dx_1 dx_2} \to \frac{d\Gamma_{q\bar{q}g}}{dS\,dT}. \tag{3.24}$$

This essentially involves calculating $\partial S/\partial x_{1,2}$ and $\partial T/\partial x_{1,2}$ as functions of S and T for the three kinematic regions. We find

$$\text{I:}\quad dx_1 dx_2 = \frac{\pi^2 T^3 dT\,dS}{64(T^2+\delta^2)^{1/2}} \left|\{(1-x_1)[2(1-x_2^{\pm}) - x_1] + \tfrac{1}{2}\delta^2(2 - x_1 - x_2^{\pm})\}^{-1}\right| = J_{12}^{\pm} dT\,dS, \tag{3.25}$$

with $x_2^{\pm}$ given by a solution of the quadratic equation

$$x_2^2[(1-x_1) + (\delta/2)^2] - x_2[(2-x_1)(1-x_1) + 2(\delta/2)^2(2-x_1)] + \left[\frac{\pi^2 T^2 S}{64} + (1-x_1)^2 + (\delta/2)^2(2-x_1)^2\right] = 0. \tag{3.26}$$

This equation is simply obtained by rewriting the equation for S in this region as a quadratic in x_2 (x_1, S, T). Note that there are, in general, two different Jacobians, $J_{12}^{\pm}$, depending on whether $x_2 = x_2^+$ or x_2^-:

II: Same as region I with $x_1 \leftrightarrow x_2$ (3.27)

$$\text{III:}\quad dx_1 dx_2 = \frac{\pi^2 T dT\,dS}{64(1-T)}\left[1 - (1-T)^{-1}\left(\frac{\pi^2 S}{16} + \delta^2\right)\right]^{1/2} = J_3 dT\,dS.$$

(Note that in each of these regions as $\delta \to 0$ we obtain the standard result.) For this region we have

$$x_2^{\pm} = 1 - \frac{T}{2} \pm \frac{T}{2}\left[1 - (1-T)^{-1}\left(\frac{\pi^2 S}{16} + \delta^2\right)\right]^{1/2} \tag{3.28}$$

and x_1 is given by

$$x_1^{\pm} = 2 - x_3 - x_2^{\pm}. \tag{3.29}$$

We are now ready to rewrite the decay distribution Eq. (2.13) in terms of S and T. Let us define

$$\frac{1}{\Gamma_0}\frac{d\Gamma_{q\bar{q}g}}{dx_1 dx_2} = \frac{2\alpha_S}{3\pi} D(x_1, x_2). \tag{3.30}$$

We then can write [by $x_1 \leftrightarrow x_2$ symmetry of $D(x_1, x_2)$]

$$\frac{1}{\Gamma_0}\frac{d\Gamma_{q\bar{q}g}}{dS\,dT} = \left(\frac{2\alpha_S}{3\pi}\right)\{2[J_{12}^+ D(x_1^+, x_2^+) + J_{12}^- D(x_1^-, x_2^-)]_{\text{I and II}} + J_3[D(x_1^+, x_2^+) + D(x_1^-, x_2^-)]_{\text{III}}\}, \tag{3.31}$$

with the appropriate $x_2^{\pm}$ in the relevant kinematic region. The subscripts I, II, and III indicate the phase-space region to be integrated over with the appropriate integration ranges for S and T.

The differential decay formula Eq. (3.31) is now a function of only infrared-safe variables. We can now proceed to calculate $\langle S\rangle$ and $\langle 1-T\rangle$ for the $Z \to q\bar{q}g$ process.

IV. RESULTS

To calculate $\langle 1-T\rangle$ and $\langle S\rangle$ we merely perform the integrations for the following integrals over the appropriate ranges:

$$\langle 1-T\rangle = \int \frac{1}{\Gamma_0}\frac{d\Gamma_{q\bar{q}g}}{dSdT}(1-T)\,dSdT\,,$$
$$\langle S\rangle = \int \frac{1}{\Gamma_0}\frac{d\Gamma_{q\bar{q}g}}{dSdT}\,S\,dSdT\,. \tag{4.1}$$

Unfortunately neither of these integrals can be evaluated analytically and we must turn to a numerical evaluation. Figure 5 shows $\langle 1-T\rangle$ as a function of δ for $Q=\frac{2}{3}$ ($T_3=\frac{1}{2}$) and $Q=-\frac{1}{3}$ ($T_3=-\frac{1}{2}$) quarks in units of $2\alpha_s/3\pi$. Note that both curves are quite similar for $\delta \lesssim 0.4$, although the weak couplings v and a are different for the two values of the quark electric charges.

We see from this figure that for a given value of $\sqrt{s}$ (fixed α_s) the thrust value decreases for increasing δ, with the structure of the final state appearing more and more three-jet-like. For $\delta \geqslant 0.6$, however, phase-space considerations shrink the allowed kinematic region and, hence, decrease the decay rate until we reach a pure $\langle 1-T\rangle = 1$ (in units of $2\alpha_s/3\pi$). At $\sqrt{s}=M_Z \simeq 90$ GeV we expect $\alpha_s = 0.10$–0.15 and hence for all δ independent of the quark charge we find

$$\langle 1-T\rangle \leqslant 0.14 \sim 0.21. \tag{4.2}$$

Figure 6 shows $\langle S\rangle$ for $Q=\frac{2}{3}$ and Fig. 7 for $Q=-\frac{1}{3}$ as a decreasing function of δ. Note that the two curves are very similar. Unlike $\langle 1-T\rangle$, which initially increases as a function of δ before

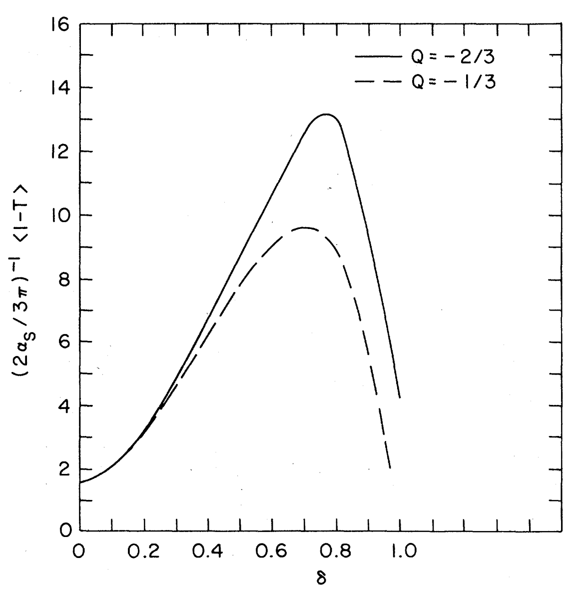

FIG. 5. Plot of $\langle 1-T\rangle$ vs δ for $Q=-\frac{1}{3}$ and $Q=\frac{2}{3}$.

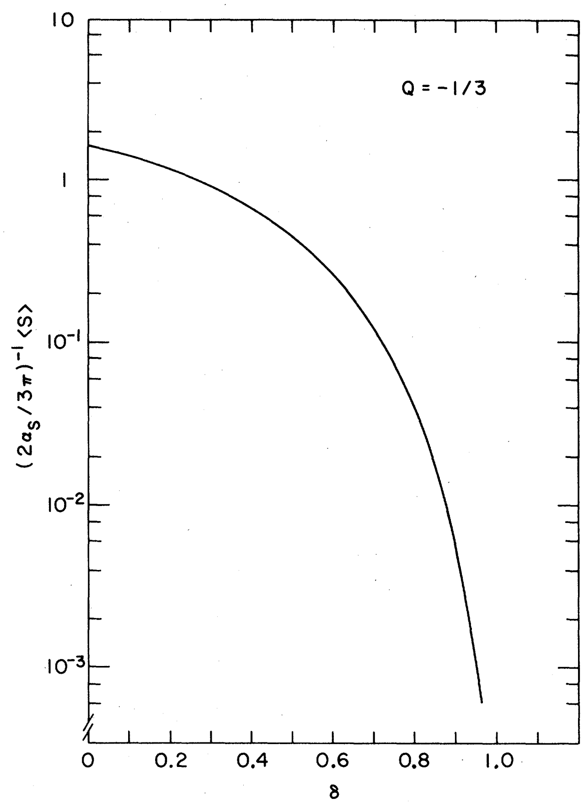

FIG. 6. Plot of $\langle S\rangle$ for $Q=\frac{2}{3}$ as a function of δ.

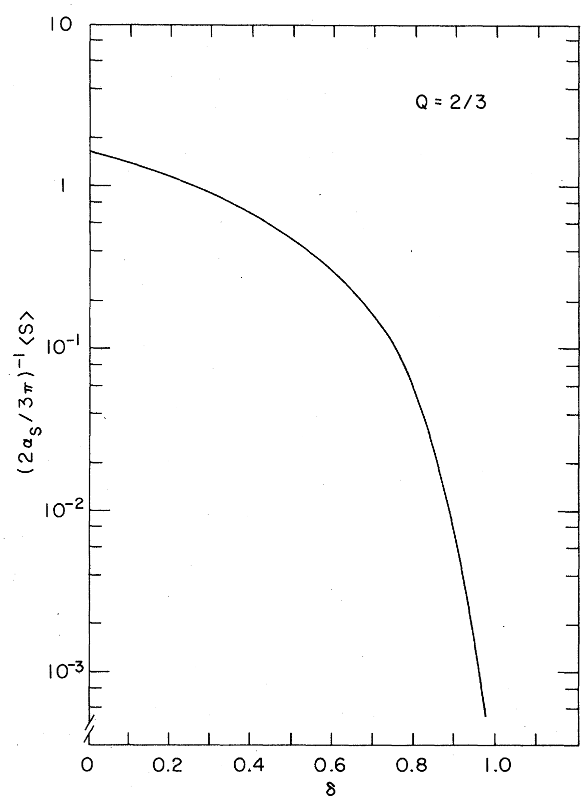

FIG. 7. Plot of $\langle S\rangle$ for $Q=-\frac{1}{3}$ as a function of δ.

phase-space effects drive it to unity, $\langle S\rangle$ decreases rapidly as a function of δ; this results from both the rapid decrease of available space as $\delta\to 1$ and the rapid decrease of S^{max} with increasing δ.

These figures show that as δ increases, the values of $\langle S\rangle$ and $\langle 1-T\rangle$ become substantially different from their values at $\delta=0$.[5] This is particularly important for decays such as $Z\to t\bar{t}g$ since the t quark is heavier than 15–16 GeV and possibly greater than 18 GeV (if it exists at all).

V. CONCLUSIONS

In this paper we have analyzed the influence of finite quark masses on the average values of spherocity and thrust for the decay $Z\to q\bar{q}g$. This involves recalculating the values of the variables S and T in terms of the x_i in each of the kinematic regions (for finite quark masses) which are usually indistinguishable in the limit of vanishing quark masses. This also requires a detailed analysis of the range of these variables and of the Jacobian necessary to transform from dx_1dx_2 to the $dSdT$ basis for performing the integration.

We have found that $\langle S\rangle$ and $\langle 1-T\rangle$ are both strongly dependent on δ $(=2m_q/\sqrt{s})$ and for quarks heavier than ~10 GeV will have distributions significantly different from the naive expectations given by the usual formulas (the $\delta=0$ limit). This result is of special importance for the t quark and any additional quarks heavier than the t into which the Z can decay; the three-jet decay associated with these heavy quarks will have quite different distributions.

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