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Matrix N/D method with absorption and the unitarity problem in coupled-channel Regge theory

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In a crossing-symmetric Regge theory with several coupled two-particle channels, there are many-particle absorptive effects due to crossed two-particle processes. Also, some of the partial-wave amplitudes have overlapping left-hand and right-hand cuts in the s plane. To enforce unitarity in such a theory, one needs a coupled-channel N/D method allowing arbitrary absorption parameters and overlapping cuts. These requirements lead to a nonlinear marginally singular N/D equation, which is proposed as part of a scheme to construct a coupled-channel Regge theory with exact crossing symmetry.

I. INTRODUCTION

A central problem in analytic S-matrix theory is the determination of scattering amplitudes through the principles of analyticity, crossing symmetry, and unitarity. A partial solution to this problem for the case of meson-meson scattering was proposed in Ref. 1. The scheme is based on a nonlinear integral equation for partial-wave amplitudes, continued to complex angular momentum. A solution of the equation leads to a total amplitude with exact crossing symmetry, Mandelstam analyticity, Regge behavior, and unitarity. Here unitarity is understood as being exact in the elastic region; in the inelastic region there are intermediate states of multiperipheral type arising from crossed two-particle processes, plus any additional contributions that may be introduced through an externally prescribed central spectral function. The central spectral function, which corresponds to virtual states of high mass in all three channels, is an arbitrary element of the scheme, but there is some reason to think that an accurate description of that function is not necessary if the main goal is to understand the patterns of low-energy resonances and the related Regge behavior at high energy and small momentum transfer. Perhaps the central spectral function can be parametrized in a simple way, for instance, in terms of Regge trajectories coupling to the relevant inelastic states. Such an outcome would be more likely if one could generate a larger part of the spectral functions dynamically, by including more channels explicitly in the crossing-unitarity equations. For instance, the equations for π-π scattering might be extended to include K mesons, in such a way as to preserve exact crossing symmetry. One would then have coupled integral equations for the amplitudes of transitions $\pi\pi\to\pi\pi$, $\pi\pi\to K\bar{K}$, $K\bar{K}\to K\bar{K}$, $\pi K\to\pi K$, and $\pi\bar{K}\to\pi\bar{K}$.

The aim of the present work is to treat the unitarity problem that arises in such a coupled-channel scheme. In the example mentioned there are transitions between channel 1 ($\pi\pi$) and channel (2) ($K\bar{K}$), as well as absorption from these channels due to many-particle states. A further complication is that the partial-wave amplitude for $K\bar{K}\to K\bar{K}$ has overlapping left- and right-hand cuts. Consequently, the partial-wave unitarity condition for the 2×2 transition matrix T of channels 1 and 2 in a definite isospin state has the form

$$(T_+ - T_-)/2i = T_+\rho T_- + F + \Delta_L T \ , \qquad (1.1)$$

where F is the absorption matrix (sometimes called the overlap matrix[2]), $\Delta_L T$ is the discontinuity of T over the intruding left-hand cut, and ρ is a diagonal phase-space matrix. The notation is defined more exactly in Sec. II. Stated schematically,

$$F_{ij} = \sum_{n>2} T_{+in}\rho_n T_{-nj} \ , \qquad (1.2)$$

where the sum actually includes integrations if many-particle states are involved, and ρ_n is an appropriate phase-space factor.

Since T satisfies the reality condition $[T(s) = T^*(s^*)]$ and is symmetric, the matrix F is Hermitian, but in general not real. Since $(T_+ - T_-)/2i$ and $\Delta_L T$ are real, the imaginary part of F must satisfy the constraint

$$\mathrm{Im}F = -\mathrm{Im}(T_+\rho T_-) \ . \qquad (1.3)$$

In the coupled-channel generalization of the scheme of Ref. 1, both ReF and B_L (the latter being the part of T due to its left-hand cut) are given as nonlinear functionals of the set of coupled amplitudes (continued to complex angular momentum) and the set of central spectral functions. That ReF is so expressed is a reflection of the fact that crossed two-particle processes give inelastic contributions of multiperipheral type. Thus, the dynamical problem may be considered as the problem of solving partial-wave

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dispersion relations of the form

$$T(l,s)=B_L(l,s)+\frac{1}{\pi}\int_{s_I}^{\infty}\frac{F(l,s')ds'}{s'-s}+\frac{1}{\pi}\int_{s_1}^{\infty}\frac{T(l,s'_+)\rho(s')T(l,s'_-)ds'}{s'-s}, \tag{1.4}$$

at physical squared-energy s and complex angular momentum l. One is to solve for the amplitude T, taking into account that B_L and ReF depend on the continued transition amplitudes themselves. In the single-channel theory of Ref. 1, an essential but relatively easy step in the solution of the full nonlinear system is to solve an equation like (1.4) for T when B_L and F are regarded as given functions of l and s, temporarily disregarding the fact they actually depend on T itself. The analogous problem for the many-channel case is the subject of this paper.

In the single-channel case, F is real and is related to the elasticity parameter η by

$$F=\mathrm{Re}F=\frac{1-\eta^2}{4\rho}. \tag{1.5}$$

The solution of (1.4) for given B_L and η is accomplished by an appropriate version of the N/D method,[3] which reduces the problem to solution of a linear Fredholm equation.

An extension of the N/D method to the many-channel case with B_L and F given was proposed in Ref. 4. I have belatedly noticed that there are some shortcomings in the proposal, however, which are corrected in the following. I had mistakenly thought that one could specify both B_L and F, but a simple argument shows that the problem is overdetermined if B_L and the entire matrix F are specified. It is enough to give B_L and ReF, for instance.

If B_L and the entire F are taken as inputs to the N/D equation of Ref. 4, the result is to produce an output amplitude T having B_L as its left-hand-cut part, but with some F' as its absorption matrix, where $F'\neq F$ in general. On the other hand, a successful incorporation of the equation in a crossing-symmetric system would require that $F=F'$. I therefore propose that the input value of ImF be adjusted by iteration so as to make $F=F'$.

Aside from the false implication that $F'=F$ automatically, there is another oversight in Ref. 4. In fact, the equation of Ref. 4 is correct only in the case of coincident thresholds of all explicit channels. When the equation is corrected, it turns out to be nonlinear and marginally singular.[5] A convenience of the usual N/D method is lost, in that one no longer has a linear Fredholm equation, but the properties essential to the scheme of Ref. 1 are retained; namely, Regge trajectories may be generated as zeros of det$D$, and the Castillejo-Dalitz-Dyson (CDD)[3] ambiguity is accounted for. Although the N/D equation looks more complicated than the equation (1.4) it replaces, its mathematical properties are much better suited to the problem at hand. As in the coupled-channel N/D theory without absorption,[6] the matter of overlapping cuts is treated simply and automatically.

I had originally hoped that the equation of Ref. 4 would be useful in phenomenology, but the new version is perhaps too awkward for that purpose, except in the approximation of coincident explicit thresholds. For phenomenology, a much simpler approach proposed by Stelbovics and Stingl[7] seems promising. In place of F the input is a matrix generalization of the function R of Chew and Mandelstam,[8] the ratio of total and elastic cross sections. In a forthcoming paper I discuss the matrix R method, considering its possible role in Regge theory, its extension to the case of overlapping cuts, and its close relation to a method of Pham and Truong[9] for solution of many-channel Omnès-Muskhelishvili equations with absorption.

Section II contains the argument that ReF and B_L determine the amplitude (up to the CDD ambiguity). In Sec. III the integral equation is derived as a necessary condition on ImD when T is a given scattering matrix. Section IV is concerned with construction of an amplitude T from a solution of the integral equation, and Sec. V deals briefly with remaining technical problems that would arise in applications of the equation.

In this paper I emphasize formal structure, and leave aside questions of convergence of integrals and existence of $i\epsilon$ limits; such matters were treated carefully in Ref. 6. I suppress the angular momentum index and in fact write the equations for $l=0$. The equations are easily extended to complex l if $T_{ij}(l,s)$ is understood as a modified partial-wave amplitude which differs from the original amplitude by a threshold factor $[p_ip_j/q_iq_j]^l$. Here q_i is the channel-i momentum, and p_i is a factor analogous to (2.28) in the first paper of Ref. 1; thus $q_i/p_i=O(1)$ at $s=\infty$. Correspondingly, the phase-space matrix element ρ_i is modified by a factor $(q_i/p_i)^{2l}$.

II. DETERMINATION OF AMPLITUDE BY B_L AND ReF

To simplify notation and emphasize essentials, the discussion is restricted to the case of two explicit channels, each containing two spinless mesons of equal mass. For the work of Secs. III and IV, the extension to n explicit channels is not entirely obvious, but should be possible with some effort. The threshold of the ith channel is s_i

$=4m_i^2$; take $s_2 \geq s_1$. The phase-space matrix $\rho(s)$ consists of the elements

$$\rho_{ij}(s)=\rho_i(s)\delta_{ij}=\theta(s-s_i)\left(\frac{s-s_i}{s_i}\right)^{1/2}\delta_{ij}, \quad (2.1)$$

where $\theta(x)$ is the unit step function. The unitarity condition has the form

$$(T_+-T_-)/2i=T_+\rho T_-+F+\Delta_L T, \quad s\geq s_1 \quad (2.2)$$

where

$$T_\pm(s)=\lim_{\epsilon\to 0+} T(s\pm i\epsilon), \quad s\geq s_1 . \quad (2.3)$$

For $s\geq s_1$ the left-hand-cut discontinuity of T is assumed to have the form

$$\Delta_L T(s)=\begin{bmatrix}0 & 0\\ 0 & \theta(s_L-s)\phi(s)\end{bmatrix}, \quad s_L<s_2 . \quad (2.4)$$

This form follows from Mandelstam analyticity, and indeed from weaker assumptions.[6] Since T is assumed to be symmetric and real-analytic $[T(s)=T^T(s)=T^*(s^*)]$, the absorption matrix is Hermitian

$$F=F^\dagger , \quad (2.5)$$

and (2.2) may be written as

$$\mathrm{Im}T_+=T_+\rho T_-+F+\Delta_L T=T_-\rho T_++F^*+\Delta_L T . \quad (2.6)$$

In general, F is not real, and

$$\mathrm{Im}F=-\mathrm{Im}(T_+\rho T_-) . \quad (2.7)$$

For simplicity F is assumed to be zero for $s_1\leq s\leq s_I$, where the threshold s_I of channels coupled to the two explicit channels may be less than s_2 and even less than s_L, as it is in the case of the $\pi\pi$-$K\bar{K}$ system for which

$$s_1<s_I<s_L<s_2 ,$$
$$4m_\pi^2<16m_\pi^2<4(m_K^2-m_\pi^2)<4m_K^2 . \quad (2.8)$$

If F were to be nonzero down to $s=s_1$, a technique like that of Ref. 6, Sec. IV could be used.

It is useful to note the implication of the unitarity condition on the S matrix,

$$S=1+2i\rho^{1/2}T_+\rho^{1/2} . \quad (2.9)$$

From (2.2) it follows that

$$SS^\dagger=1-4\rho^{1/2}F\rho^{1/2} . \quad (2.10)$$

Because of the step function in (2.1) and the form of (2.4), $\Delta_L T$ does not appear in the unitarity condition on S. The matrix $H=SS^\dagger$ is Hermitian and also non-negative, in the sense that its eigenvalues are non-negative. In terms of eigenvalues λ_i and orthonormal eigenvectors ψ_i,

$$H=H^\dagger=SS^\dagger=\sum_{i=1}^{2}\psi_i\lambda_i\psi_i^\dagger . \quad (2.11)$$

Since F represents the sum (1.2) with $T_{+in}=T_{-in}^*$, one knows that F and $\rho^{1/2}F\rho^{1/2}$ are also non-negative. Hence

$$0\leq\lambda_i\leq 1, \quad 0\leq\mu_i\leq\tfrac{1}{4} , \quad (2.12)$$

where the μ_i are eigenvalues of $\rho^{1/2}F\rho^{1/2}$. Henceforth, the stronger assumption that λ_i is positive will be adopted:

$$0<\lambda_i\leq 1. \quad (2.13)$$

I presume that the case of positive λ_i is generic, at least at physical values of angular momentum.

In the appendix of Ref. 4 it is shown that any complex symmetric $n\times n$ matrix S with positive $H=SS^\dagger$ has the representation

$$S=\sum_{i=1}^{n}\psi_i\,\alpha_i\,\psi_i^T , \quad (2.14)$$

where T denotes transpose and

$$H=\sum_{i=1}^{n}\psi_i|\alpha_i|^2\psi_i^\dagger . \quad (2.15)$$

From (2.14) and (2.15) it is seen that S admits a polar decomposition

$$S=H^{1/2}\Omega , \quad (2.16)$$

$$H^{1/2}=\sum_{i=1}^{n}\psi_i|\alpha_i|\psi_i^\dagger, \quad \Omega=\sum_{i=1}^{n}\psi_i e^{i\phi_i}\psi_i^T, \quad \alpha_i=|\alpha_i|e^{i\phi_i}. \quad (2.17)$$

(The $\phi_i=\phi_i^*$ depend on the choice of phases of the ψ_i, and can be taken to be zero without restriction of generality.) The Hermitian matrix $H^{1/2}$ is the generalization of the elasticity parameter η, and the symmetric unitary matrix Ω that of the function $e^{2i\delta}$, where δ is the real phase shift.

If H (equivalently, F) is given and is nondegenerate, then $|\alpha_i|$ is known and the eigenvectors ψ_i are determined up to phase factors. Since the ϕ_i may be set equal to zero, there are then just n free real parameters, the phases of the ψ_i. Once a nondegenerate H is specified, there is relatively little freedom left in the determination of S, and this statement is independent of any consideration of analyticity. It will then not be too surprising to find that when requirements of analyticity are added, a complete specification of H, together with the left-hand-cut part B_L, leads to an overdetermined problem. In the 2×2 case there are two free parameters allowed by unitarity, but specification of the symmetric matrix B_L amounts to fixing three parameters. When H is degenerate the situation is different, since there is additional freedom in the choice of orthonormal eigenvectors ψ_i.

To see the effect of analyticity, one may look at a simple iterative solution of the partial-wave dispersion relation (1.4). It is assumed that B_L is symmetric, $B_L = B_L^T$, and has appropriate analyticity; for instance, B_L may be a Cauchy integral over the cut $-\infty < s < s_L$, where $s_L < s_2$, as in the case of Mandelstam analyticity. Actually, it is sufficient to make $B_{L_{11}}$ and $B_{L_{12}}$ analytic in a domain having the half-line $[s_1, \infty)$ in its interior, and $B_{L_{22}}$ analytic in a similar domain minus a cut on the real axis running from s_L to the left; see Ref. 6 for a diagram and fuller explanation. Let us describe such an assumption by saying that B_L is analytic in a region $\hat{\Omega}$. After substitution of (1.3) for ImF, Eq. (1.4) is construed as an integral equation for the boundary values $T(s_\pm)$, as in Ref. 10. If B_L and ReF are sufficiently small and smooth, the convergence of an iterative sequence to a locally unique solution may be proved by applying the contraction mapping principle in an appropriate Banach space.[10] Although this argument establishes existence of a solution only for small B_L and ReF (so small that resonances probably do not occur), there is little reason to think that the situation will be different in general. Up to the CDD ambiguity[3,1] and possible ambiguities associated with the nonlinear, marginally singular character of the N/D equation of Sec. III, it is expected that B_L and ReF determine a properly analytic and unitary amplitude, at least when det$D$ does not have ghost zeros; cf. Sec. IV.

III. THE INTEGRAL EQUATION AS A NECESSARY CONDITION ON ImD WHEN T IS A PROPER AMPLITUDE

Suppose that $T(s) = T^T(s) = T^*(s^*)$ is given and satisfies (1.4), with B_L analytic in a domain $\hat{\Omega}$ as described in Sec. II. Hence, it satisfies the unitarity condition (2.2). Suppose also that $H = SS^\dagger$ is positive, so that the S matrix (2.10) has the polar form (2.16), with $H^{1/2}$ positive and Hermitian, and Ω symmetric and unitary. To derive an integral equation one seeks linear relations between $N = TD$ and D. In order to derive a linear relation from (2.16), as in the single-channel case, N must be formed from S. Accordingly, the factor $\rho^{1/2}$ on the right-hand side of T_+ in S must be removed before the analogy with the single-channel case can be completed. To that end, eliminate $H^{1/2}$ and Ω in favor of the matrices

$$\eta = \rho^{1/2} H^{1/2} \rho^{-1/2}, \quad M = \rho^{1/2} \Omega \rho^{-1/2} . \tag{3.1}$$

Since $\rho_2(s) = 0$ for $s \leq s_2$, these definitions make sense only for $s > s_2$; the region $s \leq s_2$ will be treated presently. Now (2.16) takes the form

$$1 + 2i\rho T_+ = \eta M . \tag{3.2}$$

The D matrix will be defined as a solution of the Hilbert problem[11,12] (conjugation problem) with

$$D_-(s) = M(s) D_+(s), \quad s \geq s_1 . \tag{3.3}$$

A solution of the Hilbert problem, which is known to exist under mild conditions on M, is understood[12] as a matrix $D(s)$, nonsingular[13] and meromorphic in the s plane with cut for $s \geq s_1$, such that that $D(s) = D^*(s^*)$, $D(s) \to 1$, $|s| \to \infty$, and such that (3.3) holds on the cut. Notice that the property

$$M^{-1} = M^* \tag{3.4}$$

is essential in the definition of D; otherwise, $D(s) = D^*(s^*)$ could not hold. Now the desired linear relation between N and D follows from substitution of $M = D_- D_+^{-1}$ in (3.2):

$$2iN_+ = \eta D_- - D_+ . \tag{3.5}$$

Define n by

$$\text{Im} D_+ = -\rho n . \tag{3.6}$$

In the case without absorption $n = N$. It will be shown presently that n is bounded at thresholds. Separation of real and imaginary parts in (3.5) gives

$$2\,\text{Im}N_+ = \rho^{-1}(1 - \text{Re}\eta)\text{Re}D_+ + \rho^{-1}(\text{Im}\eta)\rho n , \tag{3.7}$$

$$2\,\text{Re}N_+ = \rho^{-1}(1 + \text{Re}\eta)\rho n + \rho^{-1}(\text{Im}\eta)\text{Re}D_+ . \tag{3.8}$$

The equations (3.7) and (3.8) have been derived for $s > s_2$. If they held as well for $s > s_1$, then they would lead to the integral equation for n as derived in Ref. 4. Thus, the equation of Ref. 4 is correct in the case of coincident thresholds, $s_1 = s_2$. It is possible to give natural definitions of the coefficients in (3.7) and (3.8),

$$\rho^{-1}(1 - \text{Re}\eta), \quad \rho^{-1}(\text{Im}\eta), \quad \rho^{-1}(\text{Re}\eta)\rho \tag{3.9}$$

in the region $s_1 \leq s \leq s_2$. The second row of (3.7) or (3.8) is false for $s < s_2$, however, if those definitions are used.

To relate the second row of N to D for $s < s_2$ it is sufficient to go back to the definition of N and apply unitarity. It will turn out that for $s < s_2$ only $\mathrm{Im}N_{+2j}$, not $\mathrm{Re}N_{+2j}$, must be expressed in terms of D to obtain an integral equation. For $s < s_2$,

$$\mathrm{Im}N_{+2j} = \mathrm{Im}T_{+21}\mathrm{Re}D_{+1j} + \mathrm{Im}T_{+22}\mathrm{Re}D_{+2j} - \mathrm{Re}T_{+21}\rho_1 n_{1j} \ . \tag{3.10}$$

Here the coefficients involving T may all be expressed in terms of D and F if unitarity and symmetry of T are invoked. The first row of T, hence $T_{+21} = T_{+12}$, may be obtained from (3.5) if the extended definition of η given below is applied; see Eq. (3.17). Then $\mathrm{Im}T_{+22}$ is obtained from unitarity as

$$\mathrm{Im}T_{+22} = \rho_1 T_{+21}T_{-12} + F_{22} + \theta(s_L - s)\phi, \tag{3.11}$$

where ϕ is the left-hand-cut discontinuity defined in (2.4).

Before giving the explicit formulas for the coefficients in (3.10), let us proceed with the derivation of the integral equation. The extended definitions of matrices (3.1) and (3.9) are required. The principle of the definition is that the matrices should obey relations which they do obey for $s > s_2$, and which are well-defined for $s < s_2$. For instance, from (2.10), (2.11), and (3.1) it follows that

$$\eta^2 = 1 - 4\rho F \ . \tag{3.12}$$

Since η^2 is zero below its diagonal for $s < s_2$, it is easy to compute η in that region by substituting a general superdiagonal form in (3.12). The result is

$$\eta = \begin{bmatrix} \eta_1 & (F_{12}/F_{11})(\eta_1 - 1) \\ 0 & 1 \end{bmatrix}, \quad s_1 \leqslant s \leqslant s_2 \tag{3.13}$$

$$\eta_1 = (1 - 4\rho_1 F_{11})^{1/2} \ . \tag{3.14}$$

Having determined η, one can then define M in accord with (3.2) as

$$M = \eta^{-1}(1 + 2i\rho T_+) \ . \tag{3.15}$$

This definition is satisfactory if and only if $M^{-1} = M^*$ holds. In fact, $M^{-1} = M^*$ follows from the unitarity condition on T, as may be proved through a computation aided by the following identity:

$$T_+\rho F^* - F\rho T_+ = \mathrm{Im}F \ . \tag{3.16}$$

To verify that (3.16) follows from unitarity, multiply (2.2) on the right by $1 + 2i\rho T_+$, and then use (2.2) again to simplify the left-hand side of the resulting equation.

With the definition (3.15) of M, the first row of N_+ (equivalently, T_+) is given in terms of $D_\pm$ and η [in its extended definition (3.13)], for $s_1 \leqslant s \leqslant s_2$:

$$2i\rho_1 N_{+1j} = [\eta D_- - D_+]_{1j} \ . \tag{3.17}$$

Next, the extended definition of $A = \frac{1}{2}\rho^{-1}(1 - \eta)$ may be fixed by the equation A satisfies for $s > s_2$,

$$\tfrac{1}{2}A(1 + \eta) = F \ . \tag{3.18}$$

The result for $s_1 \leqslant s \leqslant s_2$ is

$$\{\tfrac{1}{2}\rho^{-1}(1 - \mathrm{Re}\eta)\} = \mathrm{Re}A = \begin{bmatrix} (1-\eta_1)/2\rho_1 & (\mathrm{Re}F_{12}/F_{11})(1-\eta_1)/2\rho_1 \\ (\mathrm{Re}F_{12}/F_{11})(1-\eta_1)/2\rho_1 & (\mathrm{Re}F_{12}{}^2/F_{11}{}^2)(1-\eta_1)/2\rho_1 + \det F/F_{11} \end{bmatrix}. \tag{3.19}$$

The curly brackets in the left-hand side of (3.19) will be retained as a reminder that separate factors within the brackets are not defined. With the above definitions all matrices are continuous at $s = s_2$, as is verified by calculating $H^{1/2}$ in detail as a function of F for $s > s_2$. This continuity, and certain nice cancellations that occur later, provide the justification for the otherwise arbitrary definitions. Similarly, an explicit computation of M above and below s_2 shows that n in (3.6) is continuous for $s \geqslant s_1$. Clearly, M has a superdiagonal form for $s < s_2$, so that $\mathrm{Im}D_{+2j} = 0$ for $s < s_2$.

For the integral equation one needs a Cauchy representation of D in terms of $\mathrm{Im}D_+ = -\rho n$, and the definition of a function B to replace the input function $B = B_L$ used in N/D theory without absorption. According to Ref. 6, a general representation of D is

$$D(s) = 1 + \sum_{i=1}^{n_c} \frac{C_i}{s - \sigma_i} - \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\rho(s')n(s')ds}{s' - s} \ , \tag{3.20}$$

where the Castillejo-Dalitz-Dyson (CDD) poles are at arbitrary distinct real points $\sigma_i < s_1$ and the CDD residue matrices are real and singular: $\det C_i = 0$. Let us for the moment suppose that the amplitude T is such that CDD poles are not present. The definition of B will be that of Ref. 4, altered by putting some new terms, suggested by (3.10), in the second row:

$$B=B^{(1)}+B^{(2)}\ , \tag{3.21}$$

$$B^{(1)}(s)=B_L(s)+\frac{1}{\pi}\int_{s_I}^{\infty}\frac{ds}{s'-s}\{\tfrac{1}{2}\rho^{-1}(s')[1-\mathrm{Re}\eta(s')]\},$$

$$B^{(2)}(s)=\begin{bmatrix}0 & 0\\ \frac{1}{\pi}\int_{s_1}^{s_2}\frac{\mathrm{Im}T_{+21}(s')}{s'-s}ds' & \frac{1}{\pi}\int_{s_1}^{s_2}\frac{\mathrm{Im}T_{+22}(s')-\theta(s_L-s')\phi(s')}{s'-s}ds'\end{bmatrix}.$$

The integral equation is derived by evaluating the auxiliary function Λ:

$$\Lambda(s)=[T(s)-B(s)]D(s)+\frac{1}{\pi}\int_{s_1}^{\infty}\frac{\mathrm{Re}B_+(s')\mathrm{Im}D_+(s')ds'}{s'-s}\ . \tag{3.22}$$

The procedure is first to find the value of Λ implied by the expressions (3.7) and (3.10) for $\mathrm{Im}N_+$, and then to take the real part of (3.22) on the cut. When the representation (3.20) is introduced, the result is an integral equation for $n(s)$. By (1.4) the difference $T-B$ is analytic except for a cut at $s\geq s_1$, and therefore the same may be said of Λ. With the goal of writing Λ as a Cauchy integral over its cut, $\mathrm{Im}\Lambda_+$ is evaluated for $s>s_2$ by means of (3.7) and (3.21):

$$\mathrm{Im}\Lambda_+=\mathrm{Im}N_+-\mathrm{Im}B_+\mathrm{Re}D_+-\mathrm{Re}B_+\mathrm{Im}D_++\mathrm{Re}B_+\mathrm{Im}D_+=\tfrac{1}{2}\rho^{-1}(\mathrm{Im}\eta)\rho n,\quad s\geq s_2\ . \tag{3.23}$$

For $s<s_2$, the first and second rows of $\mathrm{Im}\Lambda_+$ are treated differently. For the first row the required value of $\mathrm{Im}N_{+1j}$ is obtained from (3.17) and (3.13):

$$\mathrm{Im}N_{+1j}=\frac{(1-\eta_1)}{2\rho_1}[\mathrm{Re}D_{+1j}+(\mathrm{Re}F_{12}/F_{11})\mathrm{Re}D_{+2j}],\quad s_1\leq s\leq s_2\ . \tag{3.24}$$

This is equal to $(\mathrm{Im}B_+\mathrm{Re}D_+)_{1j}$, however, so that

$$\mathrm{Im}\Lambda_{+1j}=0,\quad s_1\leq s\leq s_2. \tag{3.25}$$

With account taken of (3.10) the second row of $\mathrm{Im}\Lambda_+$ is given by

$$\begin{aligned}\mathrm{Im}\Lambda_{+2j}&=\mathrm{Im}\,T_{+21}\mathrm{Re}D_{+1j}+\mathrm{Im}\,T_{+22}\mathrm{Re}D_{+2j}-\mathrm{Re}\,T_{+21}\rho_1 n_{1j}-[\{\tfrac{1}{2}\rho^{-1}(1-\mathrm{Re}\eta)\}\mathrm{Re}D_+]_{2j}-\theta(s_L-s)\phi\,\mathrm{Re}D_{+2j}\\&\quad-\mathrm{Im}\,T_{+21}\mathrm{Re}D_{+1j}-\mathrm{Im}\,T_{+22}\mathrm{Re}D_{+2j}+\theta(s_L-s)\phi\,\mathrm{Re}D_{+2j}\\&=-\mathrm{Re}\,T_{+21}\rho_1 n_{1j}-[\{\tfrac{1}{2}\rho^{-1}(1-\mathrm{Re}\eta)\}\mathrm{Re}D_+]_{2j}\ .\end{aligned} \tag{3.26}$$

The Cauchy representation of Λ is then

$$\begin{aligned}\Lambda_{ij}(s)&=\frac{1}{\pi}\int_{s_2}^{\infty}\frac{ds'}{s'-s}[\tfrac{1}{2}\rho^{-1}(s')\,\mathrm{Im}\eta(s')\rho(s')n(s')]_{ij}\\&\quad-\frac{\delta_{i2}}{\pi}\int_{s_1}^{s_2}\frac{ds'}{s'-s}\mathrm{Re}\,T_{+21}(s')\rho_1(s')n_{1j}(s')\\&\quad-\frac{\delta_{i2}}{\pi}\int_{s_1}^{s_2}\frac{ds'}{s'-s}\times(\{\tfrac{1}{2}\rho^{-1}(s')[1-\mathrm{Re}\eta(s')]\}\mathrm{Re}D_+(s'))_{2j}\ .\end{aligned} \tag{3.27}$$

Now in taking the real part of (3.22) on the cut to get the integral equation, one takes $s>s_1$ for the first row and $s>s_2$ for the second, since the second row of the unknown $n(s)$ is defined and integrated only in the region $s>s_2$. In evaluating the first row, (3.8) is used; it is easy to check by (3.17) and (3.13) that (3.8) holds for $s\geq s_1$ with the extended definition of η. Thus for $s\geq s_1$,

$$\begin{aligned}\mathrm{Re}N_{+1j}&=[\tfrac{1}{2}\rho^{-1}(1+\mathrm{Re}\eta)\rho n+\tfrac{1}{2}\rho^{-1}(\mathrm{Im}\eta)\,\mathrm{Re}D]_{1j}\\&=\left[\mathrm{Re}\Lambda_++\mathrm{Re}B_+\,\mathrm{Re}D_++\{\tfrac{1}{2}\rho^{-1}(1-\mathrm{Re}\eta)\}\rho n+\frac{P}{\pi}\int_{s_1}^{\infty}\frac{\mathrm{Re}B_+(s')\rho(s')n(s')ds'}{s'-s}\right]_{1j}.\end{aligned} \tag{3.28}$$

When the Cauchy representations of Λ and D are substituted, Eq. (3.28) leads to the first row of the matrix integral equation for n, stated in (3.40) and (3.41) below. The evaluation of $\mathrm{Re}N_{+2j}$ for $s\geq s_2$ is formally the same as (3.28), except for the presence of new terms in the second row of $\mathrm{Re}\Lambda_+$. Those terms are best understood by rewriting the last term in (3.27) as

$$-\delta_{i2}\Psi_{2j}(s)-\frac{\delta_{i2}}{\pi}\int_{s_1}^{\infty}ds'\left[\frac{\Psi(s)-\Psi(s')}{s-s'}\rho(s')n(s')\right]_{2j}, \tag{3.29}$$

where

$$\Psi(s)=\frac{1}{\pi}\int_{s_1}^{s_2}\frac{ds'}{s'-s}\{\tfrac{1}{2}\rho^{-1}(s')[1-\mathrm{Re}\eta(s')]\}. \tag{3.30}$$

The expression (3.29) cancels a part of the integral in $B^{(1)}$ of (3.21) wherever the latter appears in the integral equation. Alternatively, it cancels a part of $B^{(2)}$. By leaving $B^{(1)}$ intact and letting the latter cancellation occur, the most illuminating form of the equation is obtained.

Let us see how the cancellation works by evaluating more explicitly the absorptive parts $\mathrm{Im}\,T_{+21}$ and $\mathrm{Im}\,T_{+22}$ in $B^{(2)}$. For $s_1\le s\le s_2$, (3.17) and (3.13) give the first row of T_+ as

$$T_{+11}=\frac{1}{2i\rho_1}\left[\eta_1\frac{\det D_-}{\det D_+}-1\right], \tag{3.31}$$

$$T_{+12}=\frac{\Phi}{\det D_+}-\frac{F_{12}(1-\eta_1)}{2i\rho_1F_{11}}, \tag{3.32}$$

where

$$\Phi=\Phi^*=\eta_1[n_{12}\,\mathrm{Re}D_{11}-n_{11}\,\mathrm{Re}D_{12}]. \tag{3.33}$$

Unitarity and symmetry of T then yield

$$\mathrm{Im}\,T_{+22}-\theta(s_L-s)\phi=\rho_1|T_{+12}|^2+F_{22}$$

$$=\frac{\Phi^2}{|\det D_+|^2}+\frac{(1-\eta_1)\Phi}{\rho_1F_{11}}\mathrm{Im}\left(\frac{F_{12}^*}{\det D_+}\right)+\frac{(1-\eta_1)^2|F_{12}|^2}{4\rho_1{}^2F_{11}}+F_{22}. \tag{3.34}$$

Subtracting $\mathrm{Im}\Psi_+=\{\frac{1}{2}\rho^{-1}(1-\mathrm{Re}\eta)\}$ where appropriate, one has the following expressions for functions a, b, and c which appear in the integral equation:

$$a(s)=\mathrm{Im}\,T_{+21}-\mathrm{Im}\Psi_{+21}=\Phi\,\mathrm{Im}(\det D_+)^{-1}, \tag{3.35}$$

$$b(s)=\mathrm{Im}\,T_{+22}-\theta(s_L-s)\phi-\mathrm{Im}\Psi_{+22}=\frac{\Phi^2}{|\det D_+|^2}+\frac{(1-\eta_1)}{\rho_1F_{11}}\left[\Phi\,\mathrm{Im}\left(\frac{F_{12}^*}{\det D_+}\right)+\frac{(\mathrm{Im}F_{12})^2}{F_{11}}\right], \tag{3.36}$$

$$c(s)=\mathrm{Re}\,T_{+21}=\Phi\,\mathrm{Re}(\det D_+)^{-1}-(\mathrm{Im}F_{12}/2\rho_1F_{11})(1-\eta_1). \tag{3.37}$$

Define

$$C=\mathrm{Re}B^{(1)}+C^{(1)}, \tag{3.38}$$

where $B^{(1)}$ is given in (3.23) and

$$C^{(1)}=\begin{bmatrix}0 & 0\\ \frac{1}{\pi}\int_{s_1}^{s_2}\frac{a(s')ds'}{s'-s} & \frac{1}{\pi}\int_{s_1}^{s_2}\frac{b(s')ds'}{s'-s}\end{bmatrix}. \tag{3.39}$$

The integral equation, including CDD terms which will be derived presently, has the following form for $s>s_2$:

$$\rho^{-1}(s)\,\mathrm{Re}\eta(s)\rho(s)n(s)=C(s)-\tfrac{1}{2}\rho^{-1}(s)\,\mathrm{Im}\eta(s)+\sum_{i=1}^{n_c}\frac{R_i+\mathrm{Re}B(s)}{s-\sigma_i}C_i$$
$$+\frac{P}{\pi}\int_{s_1}^{s_2}\frac{ds'}{s'-s}[C(s)-C(s')-\tfrac{1}{2}\rho^{-1}(s)\,\mathrm{Im}\eta(s)-\theta(s'-s_2)\tfrac{1}{2}\rho^{-1}(s')\,\mathrm{Im}\eta(s')]\rho(s')n(s')$$
$$-\begin{bmatrix}0&0\\1&0\end{bmatrix}\frac{1}{\pi}\int_{s_1}^{s_2}\frac{ds'}{s'-s}c(s')\rho(s')n(s'),\quad s>s_2. \tag{3.40}$$

For $s_1<s<s_2$, only the first row of $n(s)$ appears on the left-hand side and the equation takes the form

$$\eta_1(s)n_{1j}(s)=\Bigg[\mathrm{Re}B^{(1)}(s)-\tfrac{1}{2}\rho^{-1}(s)\,\mathrm{Im}\eta(s)+\sum_{i=1}^{n_c}\frac{R_i+\mathrm{Re}B(s)}{s-\sigma_i}C_i$$
$$+\frac{P}{\pi}\int_{s_1}^{\infty}\frac{ds'}{s'-s}\Big(\mathrm{Re}B^{(1)}(s)-\mathrm{Re}B^{(1)}(s')-\tfrac{1}{2}\rho^{-1}(s)\,\mathrm{Im}\eta(s)$$
$$-\theta(s'-s_2)\tfrac{1}{2}\rho^{-1}(s')\,\mathrm{Im}\eta(s')\Big)\rho(s')n(s')\Bigg]_{1j},\quad s_1\le s<s_2. \tag{3.41}$$

Equations (3.40) and (3.41) constitute a nonlinear system of integral equations obeyed by the matrix $n(s)$ when the associated amplitude T satisfies the unitarity condition and is properly analytic and symmetric. The functions a and b which appear in C, and c which appears in the last term of (3.40), are nonlinear functions of n and F given by (3.35)–(3.37) and (3.33).

The origin of the CDD terms is seen by observing that CDD poles in the D matrix (3.20) produce poles in Λ of (3.22), so that the following term must be added to the expression (3.27) for Λ:

$$\sum_{i=1}^{n_c} \frac{R_i C_i}{s-\sigma_i}, \quad R_i = T(\sigma_i) - B(\sigma_i). \tag{3.42}$$

Correspondingly, for $s > s_1$ the real part of the right-hand side of (3.22) acquires the new term

$$-\sum_{i=1}^{n_c} \frac{\mathrm{Re}\,B(s) C_i}{s-\sigma_i} \ . \tag{3.43}$$

The entire change in the integral equation is then to add to the right-hand side a term equal to the difference of (3.42) and (3.43), as shown in (3.40) and (3.41).

The derivation involved certain divisions by F_{11} which must be justified by slightly stronger assumptions on F than those made heretofore. The ratio $(\mathrm{Im} F_{12})^2/F_{11}$ appears in the 22 element of the matrix of (3.19) and also in (3.36). It must then be assumed that

$$(\mathrm{Im}F_{12})^2/F_{11} = O(1), \quad F_{11} \to 0 \, . \tag{3.44}$$

At the inelastic threshold $s = s_I$ where $F \to 0$, this is a reasonable assumption.

In order that the integral equation have decent properties the inverse of the coefficient of n on the left-hand side must exist. Otherwise, the equation is an "integral equation of the third kind" which requires special treatment.[14] Accordingly, it will be assumed that

$$\det \mathrm{Re}\,\eta(s) \neq 0, \quad s \geq s_2 \tag{3.45}$$

$$\eta_1(s) \neq 0, \quad s_I \leq s \leq s_2 \, . \tag{3.46}$$

The requirements (3.45) and (3.46) are in the same spirit as the previous assumption that H is positive, which implies that $\det\eta(s) > 0$, $s \geq s_2$. Positivity of H and (3.45) and (3.46) are guaranteed if F is sufficiently small.

IV. CONSTRUCTION OF AMPLITUDE T FROM A SOLUTION OF THE INTEGRAL EQUATION

The integral equation stated in (3.40) and (3.41) constitutes a necessary condition on a matrix $n(s)$ associated with any real-analytic, symmetric 2×2 amplitude $T(s)$ having the representation (1.4), where B_L is analytic in a domain $\hat{\Omega}$ as described in Sec. II, $H = SS^\dagger$ is positive, and condition (3.44) holds. Here $\mathrm{Im} D_+ = -\rho n$, and D is related to T through (3.3). Let us now take a different point of view, and suppose that T is not in hand, but that B_L and F are given. Now F is not known to be a possible absorption matrix for an analytic unitary T with left-hand-cut term B_L; it is merely some Hermitian matrix function with support in the region $s > s_I > s_1$ [and compatible with condition (2.12), let us say]. Can the integral equations (3.40) and (3.41) be used to construct a satisfactory amplitude T?

A candidate for T is obtained by solving (3.22) for T:

$$T(s) = B(s) + \left[\Lambda(s) + \frac{1}{\pi}\int_{s_1}^{\infty} \frac{\mathrm{Re}\,B(s')\rho(s')n(s')ds'}{s'-s}\right] D^{-1}(s) \, . \tag{4.1}$$

Here n is a solution of (3.40) and (3.41), D is given by (3.20), B by (3.21), and Λ by (3.22) [the latter with the term (3.42) added if there are CDD poles]. The functions T_{+21} and $\mathrm{Im}\,T_{+22}$ that appear in B and Λ are expressed in terms of n by (3.32) and (3.34). A tilde will be temporarily affixed to the functions of (3.32) and (3.34) in the following discussion ($\tilde{T}_{+21}$, $\mathrm{Im}\,\tilde{T}_{+22}$) to emphasize that they have not yet been proved equal to the corresponding functions T_{+21}, $\mathrm{Im}\,T_{+22}$ computed from (4.1).

The input parameters are assumed to have the following properties:

(a) $B_L(s) = B_L^*(s^*)$ is analytic in a region $\hat{\Omega}$ as described in Sec. II. In particular, its discontinuity over the real axis, $\Delta B_L = \Delta_L T$, has the form (2.4) for $s \geq s_1$.

(b) $B_L = B_L^T$.

(c) The absorption matrix F from which η is constructed is Hermitian, has support in the region $s > s_I > s_1$ with $F(s_I) = 0$, and satisfies (3.44).

(d) $C_i = C_i^*$, $\det C_i = 0$; the σ_i are distinct real points, $\sigma_i < s_1$.

(e) $R_i = R_i^*$, $R_i^T - R_i = r_i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, r_i = any real constant.

The property of $R_i^T - R_i$ is motivated by the definition (3.42) that the CDD parameter R_i had in the case of a proper amplitude T: since $T = T^T$ in that case, and since $B^{(1)} = B^{(1)T}$ follows from $B_L = B_L^T$ and Hermiticity of η, one has

$$R_i^T - R_i = B^{(2)}(\sigma_i) - B^{(2)}(\sigma_i)^T = \begin{bmatrix} 0 & -B_{21}^{(2)}(\sigma_i) \\ B_{21}^{(2)}(\sigma_i) & 0 \end{bmatrix}. \tag{4.2}$$

Further it will be assumed that

$$\det D(s) \neq 0 \tag{4.3}$$

in the cut plane, including points on the cut. This is the condition of "no ghost poles."

The following properties of T given by (4.1) will be established:

(i) $T(s) = T^*(s^*) = T^T(s)$.

(ii) $$T(s)=B_L(s)+\frac{1}{\pi}\int_{s_1}^{\infty}\frac{T_+(s')\rho(s')T_-(s')ds'}{s'-s}+\frac{1}{\pi}\int_{s_1}^{\infty}\frac{F'(s')ds'}{s'-s},$$

where $F'(s)$ is an Hermitian matrix with $F'(s_I)=0$.

(iii) The expressions for F' in terms of $M=D_-D_+^{-1}$ are as follows:

(a) For $s>s_2$,

$$F'=F+\tfrac{1}{4}\rho^{-1}\eta[\rho-M\rho M^\dagger]\eta^\dagger\rho^{-1}.$$

(b) If $s_I<s<s_2$,

$$F'_{11}=-\frac{1}{4\rho_1}[1-(\eta M)_{11}(\eta^*M^*)_{11}],$$

$$F'_{12}=-\frac{1}{4\rho_1}[(\eta M)_{11}(\eta^*M^*)_{12}+(\eta M)_{12}]=F'^*_{21},$$

$$F'_{22}=F_{22},$$

where η is given by (3.13).

(iv) T_+ has the representation

$$T_{+ij}=\frac{1}{2i}\rho_i^{-1}[\eta M-1]_{ij},\quad s>s_i.$$

Note that there is no such representation of T_{+22} for $s_1<s<s_2$.

(v) For small amplitudes, $F'=F$ to first order:

$$F'=F+O(\lambda^2),\quad \lambda\to 0,\quad \text{uniformly in } s \text{ for } s\geqslant s_I,$$

where λ is a strength parameter. It is assumed that

$$B_L=O(\lambda),\quad \mathrm{Re}F=O(\lambda),\quad \mathrm{Im}F=O(\lambda^2).$$

It is seen that the integral equation yields a properly analytic, symmetric, and unitary amplitude, with the left-hand-cut part having the input value. The output absorption matrix F' has the proper support ($s>s_I$) and is close to the input value F in the case of small amplitudes.

To prove the listed properties of T, a lemma must first be established; namely, $N=TD$ computed from (4.1) has properties like those ascribed to N when the equation was derived:

$$\mathrm{Im}N_{+ij}=[\tfrac{1}{2}\rho^{-1}(1-\mathrm{Re}\eta)\,\mathrm{Re}D+\tfrac{1}{2}\rho^{-1}(\mathrm{Im}\eta)\rho n]_{ij},\quad s>s_i \tag{4.4}$$

$$\mathrm{Im}N_{+2j}=\mathrm{Im}\,\tilde{T}_{+21}\,\mathrm{Re}D_{+1j}+\mathrm{Im}\,\tilde{T}_{+22}\,\mathrm{Re}D_{+2j}-\mathrm{Re}\,\tilde{T}_{+21}\rho_1 n_{1j},\quad s_1<s<s_2 \tag{4.5}$$

$$\mathrm{Re}N_{+ij}=[\tfrac{1}{2}\rho^{-1}(1+\mathrm{Re}\eta)\rho n+\tfrac{1}{2}\rho^{-1}\,\mathrm{Im}\eta\,\mathrm{Re}D]_{ij},\quad s>s_i. \tag{4.6}$$

It is clear that $f(s)=f^*(s^*)$ for all analytic functions f appearing in the expression for T, so that $T(s)=T^*(s^*)$. The proof of (4.4)–(4.6) is then merely a matter of computation, effectively retracing the steps which led to the integral equation. In the case of (4.6), one must use the fact that n indeed satisfies the integral equation.

Property (iv) above is just a restatement of (4.4) and (4.6): assemble $\mathrm{Re}N_+$ and $i\,\mathrm{Im}N_+$ to give (3.5), and multiply by D_+^{-1}.

The property $T=T^T$ of item (i) may be proved by generalizing a method of Bjorken and Nauenberg.[15] Show by an analyticity argument that the following function is identically zero:

$$\chi=D^T(T^T-T)D=N^TD-D^TN. \tag{4.7}$$

Since D^{-1} exists everywhere, $T=T^T$ will follow. Clearly $\chi(s)$ is analytic in the s plane with at most a cut for $s>s_1$ and possible poles from CDD poles in N and D; there is no discontinuity of χ over the half-line $s<s_1$ because $B_L=B_L^T$. There is in fact no pole in χ, since near a CDD pole position σ_i it has the form

$$\frac{1}{(s-\sigma_i)^2}C_i^T(R_i^T-R_i)C_i+O(1). \tag{4.8}$$

The residue matrix vanishes because C_i is singular and $R_i^T-R_i$ has the skew-symmetric form given in (*e*). Since χ vanishes at infinity, it must then be identically zero if $\mathrm{Im}\chi=0$, $s>s_1$. A short calculation based on (4.4) and (4.6) shows that $\mathrm{Im}\chi=0$ for $s>s_2$. A longer calculation using (4.4)–(4.6) finishes the proof of proposition (i) by showing that $\mathrm{Im}\chi=0$ for $s_1<s<s_2$. In this latter calculation one must write out all of the terms in $\mathrm{Im}\chi$ separately, use the expression (3.32) for $\mathrm{Im}\,\tilde{T}_{+21}$, and take account of the particular forms of the matrices involving η. The reader will agree that symmetry of the output T matrix was hardly obvious *a priori*, in view of the highly asymmetrical treatment of the two channels below the threshold s_2.

It is now possible to make the identification

$$T_{+12}=T_{+21}=\tilde{T}_{+21},\quad s_1<s<s_2. \tag{4.9}$$

This follows from the first row of (iv), the definition (3.13) of η, and symmetry of T.

Knowing that T is symmetric and has the representation described in (iv), one can investigate unitarity, which is to say look at the value of the matrix F' defined by

$$\mathrm{Im}\,T_+=T_+\rho\,T_-+F'+\Delta_L T,\quad s>s_1. \tag{4.10}$$

Clearly F' is Hermitian, since all other terms in (4.10) are. Expression (iii a) for F' in the region $s>s_2$ follows directly from (4.10), (iv), and the symmetry of T:

$$H'=SS^*=(1+2i\rho^{1/2}T_+\rho^{1/2})(1-2i\rho^{1/2}T_-\rho^{1/2})=1-4\rho^{1/2}F'\rho^{1/2}=SS^\dagger=(\rho^{-1/2}\eta M\rho^{1/2})(\rho^{1/2}M^\dagger\eta^\dagger\rho^{-1/2})$$
$$=(\rho^{-1/2}\eta\rho^{1/2})(\rho^{1/2}\eta^\dagger\rho^{-1/2})+\rho^{-1/2}\eta(M\rho M^\dagger-\rho)\eta^\dagger\rho^{-1/2}. \quad (4.11)$$

Now by Hermiticity of $H^{1/2}=\rho^{-1/2}\eta\rho^{1/2}$, the first term in the last line is equal to $H=1-4\rho^{1/2}F\rho^{1/2}$, and the result (iii a) follows. The expressions for F'_{ij} in (iii b) are obtained from (4.10) and the first row of (iv). Finally, one must prove the far from obvious relation $F'_{22}=F_{22}$, for $s_I<s<s_2$. The proof requires a direct calculation from (4.1), since T_{+22} does not have a representation such as (iv) for $s<s_2$. The problem is equivalent to showing that $\operatorname{Im}T_{+22}=\operatorname{Im}\tilde{T}_{+22}$, since by (4.9) and the definitions of the functions with tildes it is known that

$$\operatorname{Im}\tilde{T}_{+22}=\rho_1|\tilde{T}_{+12}|^2+F_{22}+\theta(s_L-s)\phi$$
$$=\rho_1|T_{+12}|^2+F_{22}+\theta(s_L-s)\phi. \quad (4.12)$$

With account of the expression (3.21) for B (of course, with $\operatorname{Im}T_{+22}$ replaced by $\operatorname{Im}\tilde{T}_{+22}$) and (3.27) for Λ, a calculation for $s_1<s<s_2$ gives

$$\operatorname{Im}T_{+22}=\operatorname{Im}\tilde{T}_{+22}+\{\tfrac{1}{2}\rho^{-1}(1-\operatorname{Re}\eta)\}_{22}+\sum_k(\operatorname{Re}(T_+-B_+)_{21}\operatorname{Im}D_{+1k}-[\{\tfrac{1}{2}\rho^{-1}(1-\operatorname{Re}\eta)\}\operatorname{Re}D]_{2k})\operatorname{Re}(D^{-1}_{+k2})$$
$$+\sum_k\left[\operatorname{Re}\Lambda_{+2k}+\frac{P}{\pi}\int_{s_1}^\infty\frac{[\operatorname{Re}B_+(s')\rho(s')n(s')]_{2k}ds'}{s'-s}\right]\operatorname{Im}(D^{-1}_{+k2}). \quad (4.13)$$

Now apply the trivial identities,

$$\operatorname{Re}D\operatorname{Re}D^{-1}-\operatorname{Im}D\operatorname{Im}D^{-1}=1,\quad \operatorname{Im}D\operatorname{Re}D^{-1}+\operatorname{Re}D\operatorname{Im}D^{-1}=0, \quad (4.14)$$

to eliminate $\operatorname{Re}(D^{-1}_{+k2})$ in favor of $\operatorname{Im}(D^{-1}_{+k2})$. The result is

$$\operatorname{Im}T_{+22}=\operatorname{Im}\tilde{T}_{+22}+\sum_k\Big[-\operatorname{Re}(T_+-B_+)_{21}\operatorname{Re}D_{+1k}-\{\tfrac{1}{2}\rho^{-1}(1-\operatorname{Re}\eta)\}_{21}\operatorname{Im}D_{+1k}$$
$$+\operatorname{Re}\Lambda_{+2k}+\frac{P}{\pi}\int_{s_1}^\infty\frac{[\operatorname{Re}B_+(s')\rho(s')n(s')]_{2k}ds'}{s'-s}\Big]\operatorname{Im}(D^{-1}_{+k2}). \quad (4.15)$$

If $\operatorname{Re}\Lambda_{+2k}$ in (4.15) is computed by solving (4.1) for Λ, then most of the terms cancel and one is left with

$$\operatorname{Im}T_{+22}=\operatorname{Im}\tilde{T}_{+22}+\operatorname{Re}(T_+-B_+)_{22}(\operatorname{Re}D_+\operatorname{Im}D_+^{-1})_{22}$$
$$=\operatorname{Im}\tilde{T}_{+22}-\operatorname{Re}(T_+-B_+)_{22}(\operatorname{Im}D_+\operatorname{Re}D_+^{-1})_{22}$$
$$=\operatorname{Im}\tilde{T}_{+22},\quad s_1<s<s_2, \quad (4.16)$$

since $\operatorname{Im}D_{+2j}=0$ for $s<s_2$. Thus, all of the claims of (iii) have been proved.

It may now be shown that F' has the same support as F, the half-line $s>s_I$. If $s_I>s_2$, then η tends to the unit matrix as $s\to s_I+$, and in place of the computation (4.11) one has at $s=s_I$ that

$$H'=SS^*=(\rho^{-1/2}M\rho^{1/2})(\rho^{-1/2}M^*\rho^{1/2})$$
$$=1=1-4\rho^{1/2}F'\rho^{1/2}, \quad (4.17)$$

hence

$$F'(s)=0,\quad s\le s_I. \quad (4.18)$$

The same conclusion (4.18) is reached if $s_I<s_2$, as is seen by (iii b), (3.13), and the formula for M,

$$M=\begin{bmatrix}\det D_-/\det D_+ & 2i\rho_1[n_{12}\operatorname{Re}D_{11}-n_{11}\operatorname{Re}D_{12}]/\det D_+\\ 0 & 1\end{bmatrix},$$
$$s<s_2. \quad (4.19)$$

For the discussion of property (v) the magnitudes of the various functions may be measured by a norm in an appropriate Banach space.[6] Thus, $f=O(\lambda)$ means $\|f\|<\kappa\lambda$, $\lambda\to 0$, for some constant κ. Under the hypotheses of (v),

$$H^{1/2}=(1-4\rho^{1/2}F\rho^{1/2})^{1/2}$$
$$=1-2\rho^{1/2}F\rho^{1/2}+O(\lambda^2),$$
$$\eta=1-2\rho F+O(\lambda^2),$$
$$\{\tfrac{1}{2}\rho^{-1}(1-\operatorname{Re}\eta)\}=\operatorname{Re}F+O(\lambda^2), \quad (4.20)$$
$$\tfrac{1}{2}\rho^{-1}\operatorname{Im}\eta=-\operatorname{Im}F+O(\lambda^2),$$
$$\rho^{-1}\operatorname{Re}\eta\rho=1-2(\operatorname{Re}F)\rho+O(\lambda^2).$$

The solution of the integral equations (3.40) and (3.41) to lowest order is

$$n^{(1)}(s)=B_L(s)+\frac{P}{\pi}\int_{s_I}^\infty\frac{\operatorname{Re}F(s')ds'}{s'-s}, \quad (4.21)$$

$$n = n^{(1)} + O(\lambda^2)\,, \tag{4.22}$$

and

$$M = D_- D_+^{-1} = 1 + 2i\rho n + O(\lambda^2)\,. \tag{4.23}$$

Introduction of the foregoing results in the expressions (iii a and iii b) for F' leads to the conclusion (v). The hypotheses of (v) are consistent with the formulas for B_L and ReF in crossing-symmetric Regge theory, and the value (1.3) for ImF. To lowest order the expression for T_+ is

$$T_{+ij} = (n^{(1)} + i\,\mathrm{Re}F)_{ij} + O(\lambda^2), \quad s > s_i\,. \tag{4.24}$$

V. REMAINING TECHNICAL PROBLEMS

A successful application of the integral equations (3.40) and (3.41) in crossing-symmetric Regge theory will require the treatment of two technical questions, one having to do with the logarithmic end-point singularities at $s = s_2$, and the other the problem of adjusting ImF to make ReF = ReF'.

With regard to singularities, the first observation is that the integral equation entails a principal-value integration over the Cauchy singularity at $s = s'$. As was pointed out in Ref. 4, this singularity may be removed by the standard procedure of multiplying by a regularizing operator. In the generic situation, the regularized equation is expected to be equivalent to the original equation. For a numerical solution, there is the option of using either the regularized equation or a technique for direct solution without regularization as discussed by Ivanov and others.[16]

The equation also entails logarithmic singularities at $s = s_2$ due to the Cauchy integrals of a, b, and c over the interval $[s_1, s_2]$ in (3.39) and (3.40). In this case the Cauchy denominators $s' - s$ do not vanish, but the numerators are nonzero at $s' = s_2$ and produce terms proportional to $\ln(s - s_2)$. These singularities arise from using two different expressions for ImN_{+2j}, expression (3.7) for $s > s_2$, and (3.10) for $s < s_2$. When the integral equation is written as a necessary condition on n for a proper amplitude, as it was in Sec. III, the singularities will cancel, because ImN_{+2j} is continuous at $s = s_2$. On the other hand, when the equation is being used as in Sec. IV to construct amplitudes, there is no guarantee that the singularities will cancel as long as $F \neq F'$.

It is likely that a solution of (3.40) and (3.41) can be obtained, in spite of the logarithmic singularities. Moreover, if ImF can be adjusted to make $F_{22} = F'_{22}$, the singularities should disappear from n and the corresponding T. The argument in favor of this conjecture is as follows. Estimates show that a function space in which n_{1j} is bounded and n_{2j} has a logarithmic singularity at $s = s_2$ is mapped into itself by the integral operator of (3.40) and (3.41). Since $n_{2j}(s')$ is always multiplied by $\rho_2(s') = O([s' - s_2]^{1/2})$ in integrals, the singularity of n_{2j} does not propagate to appear in n_{1j}, nor does it become more potent on iteration of the integral operator. With an appropriate metric in function space, it should be possible to prove that an iterative sequence converges to a solution, locally unique in the space considered, for B_L and F sufficiently small. Reduction of the equation to a Fredholm equation, which was possible in other similar cases,[5] might also be attempted. The solution n, obtained by iteration or solution of a Fredholm equation, would have $n_{1j} = O(1)$, $n_{2j} = O(\ln^\alpha(s - s_2))$ with some $\alpha > 0$, so that $T_\pm$ constructed from (4.1) would have no singularity in its first row at $s = s_2$, but it could have one in its second row. Since $T_{12} = T_{21}$, the only amplitude that may have a singularity is T_{22}, and the singularity is at worst logarithmic. It then follows from the unitarity condition that the singularity disappears when $F'_{22} = F_{22}$. Since

$$\mathrm{Im}\,T_{+22} = \rho_1 |T_{+12}|^2 + \rho_2 |T_{+22}|^2 + F'_{22} + \theta(s_L - s)\phi\,, \tag{5.1}$$

the only way for a singularity to appear in Im$\,T_{+22}$ is through F'_{22}, thanks to the factor ρ_2 in the second term. For $s < s_2$, $F'_{22} = F_{22}$ and consequently Im$\,T_{+22}$ is bounded as s tends to s_2 from below. A calculation for $s > s_2$ based on formula (iii a) of Sec. IV shows that in general a singularity of F'_{22} is expected, so that Im$\,T_{+22}$ blows up logarithmically as s tends to s_2 from above. When $F'_{22} = F_{22}$ there is no singularity in Im$\,T_{+22}$, since F_{22} has none, and hence none in T_{22}, by analyticity as expressed in the dispersion relation (1.4).

Use of the integral equation as part of the crossing-unitarity mapping in Regge theory requires for a solution of the full crossing-symmetric scheme that $F = F'$ for each partial wave. This condition is needed to prove that partial waves generated by the N/D equation are equal to the Froissart-Gribov partial waves; that equality in turn is needed to prove that the partial waves sum up to a crossing-symmetric plane-wave amplitude. A definition of the crossing-unitarity mapping, incorporating an iterative procedure to make $F = F'$, is given in Ref. 17.

Since ImF obeys (1.3) when F corresponds to a proper amplitude, it is plausible that ImF can be adjusted to make $F = F'$ by an iteration of the form

$$\mathrm{Im}F^{(n+1)} = -\mathrm{Im}(T_+^{(n)}\rho T_-^{(n)}), \quad n = 1, 2, \ldots \tag{5.2}$$

where $T_\pm^{(n)}$ is the N/D amplitude from the nth solution of the integral equation, and Im$F^{(n)}$ the input value of ImF for the nth solution.

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