# Asymptotic behavior of the proton electromagnetic form factor with pion exchanges in the ladder approximation

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(Received 26 March 1980)

The asymptotic behavior of the proton electromagnetic form factor due to virtual pion exchanges is studied by restricting to a class of ladder diagrams. We set up an off-mass-shell integral equation for the modified proton-proton-photon vertex due to virtual pion exchanges by summing up an infinite number of ladder diagrams. This integral equation is converted into a set of coupled differential equations involving Lorentz-invariant amplitudes. From the solutions of the differential equations it is found that the usual form factor is modified by a factor $\exp\{[-1/2 \pm (1/2)(1-g^2/\pi^2)^{1/2}]\ln t\}$ at large momentum transfers, where $g$ is the pion-nucleon coupling constant.

## I. INTRODUCTION

The asymptotic behavior of the electromagnetic form factor of the proton with radiative corrections has been investigated by a number of authors[1-3] in the local relativistic quantum field theory. The dominant contributions in the high-energy limit for the Feynman diagrams are obtained in all these calculations. Jackiw[1] has derived in connection with the dynamics at infinite momentum a set of diagrammatic rules which are similar to those discovered by Weinberg,[4] and has obtained an integral equation for the vertex function in spinor electrodynamics. He has approximated the kernel of the integral equation in a fashion which is accurate at high energies, and which is easily tractable. He has considered ladder as well as crossed-ladder diagrams. Recently, Mueller[5] has calculated the asymptotic behavior of the form factor in quantum electrodynamics in an Abelian gauge theory with nonzero photon mass. His result is essentially the same as that found by Sudakov.[2] The asymptotic behavior of the elastic nucleon form factor in the context of the renormalization group has been discussed in a number of recent papers.[6] The dominant high-energy behavior of the Feynman diagrams has also been investigated by a number of authors[7-9] in connection with the Regge behavior of the amplitudes.[10] The eikonal approximation of quantum electrodynamics in the high-energy limit has also been used in summing up the diagrams to all orders. Appelquist and Primack[3] have calculated the modified vertex function due to virtual pion exchanges in perturbation theory. In the asymptotic region where the momenta involved are much larger than any of the masses and where the processes are not dominated by resonances, one might hope that a field-theoretic approach would have some value even for strong interactions. All these results, however, indicate that summing Feynman graphs does not provide an explanation of the observed rapid decrease of the form factor ($\sim 1/t^2$) at large momentum transfers.[11]

In the present paper we have tried to solve the problem in a completely different manner. We set up an off-mass-shell integral equation for the modified vertex function due to virtual pion exchanges by summing up an infinite number of ladder diagrams. This integral equation is converted into a set of coupled differential equations involving Lorentz-invariant amplitudes. The asymptotic form factors are then obtained from the solutions of the differential equations. We have retained here the exact kernel of the integral equation. All the previous calculations consist of approximating the kernel of the integral equation by its dominant asymptotic behavior. The coupled differential equations obtained by us are, however, solved here in an approximate fashion by the iteration method.[12] We also consider the off-mass-shell effect of the integral equation. Our results are simi-lear to those of Jackiw[1] and Appelquist and Primack.[3] The method adopted by us is analogous to those of Johnson, Baker, and Willey,[13] Haag and Maris,[14] Bose and Biswas,[15] and others[16] in the nonperturbative approach to quantum electrodynamics. The method has also been applied to find the high-energy behavior of lowest-order weak amplitudes with electromagnetic radiative corrections.[17]

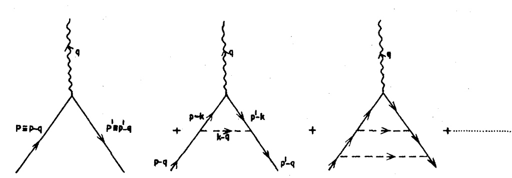

FIG. 1. Pion-exchange diagrams for the vertex function in the ladder approximation.

 

## II. THE INTEGRAL EQUATION FOR THE MODIFIED VERTEX FUNCTION AND ITS SOLUTION

The unrenormalized proton-proton-photon vertex function due to virtual pion exchanges in the ladder approximation is obtained by summing up the Feynman diagrams (see Fig. 1). It is found that the modified vertex function satisfies the following off-mass-shell integral equation:

$$\Gamma_\mu(p,p',q)=\gamma_\mu-\frac{ig^2}{(2\pi)^4}\int\frac{d^4k}{(k-q)^2-\mu^2}\,\gamma_5\,\frac{\not{p}'-\not{k}+m}{(p'-k)^2-m^2}\,\Gamma_\mu(p,p',k)\,\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\,\gamma_5\,, \tag{1}$$

where $P\equiv p-q$ and $P'\equiv p'-q$ are the external proton momenta, $P-P'=p-p'=q=$ momentum transfer, $m$ is the mass of the proton, $\mu$ is the mass of the pion, and $g$ is the pion-proton coupling constant.

Since the momenta involved here are much larger than the masses of the particles, we neglect the mass terms. In order to convert the integral equation (1) into an equivalent set of coupled differential equations we use the identity

$$\Box_q^{\,2}\frac{1}{(k-q)^2}=4\pi^2 i\delta^{(4)}(k-q)\,, \tag{2}$$

where

$$\Box_q^{\,2}\equiv\frac{\partial^2}{\partial q_\mu\partial q^\mu}\,.$$

This method of converting an integral equation into differential equations was originally developed by Green[18] in connection with the Bethe-Salpeter equation and was later on used by many authors in different problems.[15,17]

Applying the operator $\Box_q^{\,2}$ on (1) and using (2) we obtain the differential equation

$$\Box_q^{\,2}\Gamma_\mu(p,p',q)=c\gamma_5(\not{p}'-\not{q})\Gamma_\mu(p,p',q)(\not{p}-\not{q})\gamma_5\,, \tag{3}$$

where

$$c=\frac{g^2}{4\pi^2}\,\frac{1}{(p'-q)^2(p-q)^2}\,. \tag{4}$$

Next we express all the momentum factors of (3) and (4) in terms of the center-of-mass momentum $E$ ($\equiv P+P'$) and the momentum transfer $q$ ($\equiv P-P'$). We define $s$ and $t$ as

$$s=E^2=(P+P')^2\,,$$
$$t=-q^2=-(P-P')^2\,.$$

For the off-shell vertex function $\Gamma_\mu(E,q)$ we have the following Lorentz-invariant structure:

$$\begin{aligned}\Gamma_\mu(E,q)=&f_1\gamma_\mu+(if_2/\sqrt{s})\sigma_{\mu\nu}q^\nu+(f_3/\sqrt{s})E_\mu+(f_4/\sqrt{s})\not{E}\gamma_\mu+(f_5/s)\not{q}E_\mu+(f_6/s)\not{E}E_\mu+(f_7/s)\not{q}\not{E}\gamma_\mu\\&+(f_8/s^{3/2})\not{q}\not{E}E_\mu+(f_9/\sqrt{s})q_\mu+(f_{10}/s)\not{E}q_\mu+(f_{11}/s)\not{q}q_\mu+(f_{12}/s^{3/2})\not{q}\not{E}q_\mu\,,\end{aligned} \tag{5}$$

where the amplitudes $f_i=f_i(E^2,q^2,E\cdot q)$ are functions of $E^2$, $q^2$, and $E\cdot q$ and are dimensionless.

For the radially symmetric case ($s$ wave), we obtain, by substituting (5) into (3), the following set of coupled differential equations:

$$16sq^2f_1''+32sf_1'+8f_{11}=c[s(E^2-q^2)f_1-2E^2q^2f_7]\,,$$
$$16q^2f_2''+48f_2'=c(E^2+q^2)f_2\,,$$
$$16sq^2f_3''+32sf_3'+16f_{12}=c[-2sq^2f_2+s(E^2-q^2)f_3+2sE^2f_4-2E^2q^2f_8]\,,$$
$$16sq^2f_4''+32sf_4'-8f_{12}=-cs(E^2+q^2)f_4\,,$$
$$16q^2f_5''+48f_5'=c[2sf_1+(E^2+q^2)f_5+2E^2f_7]\,,$$
$$16q^2f_6''+32f_6'=c[-2sf_1-(E^2+q^2)f_6+2q^2f_7]\,,$$
$$16q^2f_7''+48f_7'=-c[2sf_1+(E^2-q^2)f_7]\,,$$
$$16q^2f_8''+48f_8'=c[-2s(f_2+f_3+f_4)+(q^2-E^2)f_8]\,,$$
$$16sq^2f_9''+48sf_9'=c[2sE^2f_4+s(E^2-q^2)f_9-2E^2q^2f_{12}]\,,$$
$$16q^2f_{10}''+48f_{10}'=c[-2sf_1+2q^2f_7-(E^2+q^2)f_{10}]\,,$$
$$16q^2f_{11}''+64f_{11}'=c[2sf_1+2E^2f_7+(E^2+q^2)f_{11}]\,,$$
$$16q^2f_{12}''+64f_{12}'=c[-2sf_4-2sf_9+(q^2-E^2)f_{12}]\,,$$

where the prime denotes derivative with respect to $q^2$. Now introducing the dimensionless variable $z=t/s$ we obtain the following set of differential equations:

$$z(z-1)^2f_1''+2(z-1)^2f_1'+A(z+1)f_1=-2Azf_7+\tfrac{1}{2}(z-1)^2f_{11}\,, \tag{6a}$$

$$z(z-1)f_2''+3(z-1)f_2'+Af_2=0\,, \tag{6b}$$

$$z(z-1)^2f_3''+2(z-1)^2f_3'+A(z+1)f_3=(z-1)^2f_{12}-2A(zf_2+f_4+zf_8)\,, \tag{6c}$$

$$z(z-1)f_4''+2(z-1)f_4'+Af_4=-\tfrac{1}{2}(z-1)f_{12}\,, \tag{6d}$$

$$z(z-1)^2f_5''+3(z-1)^2f_5'-A(z-1)f_5=-2A(f_1+f_7)\,, \tag{6e}$$

$$z(z-1)^2f_6''+2(z-1)^2f_6'+A(z-1)f_6=2A(f_1+zf_7)\,, \tag{6f}$$

$$z(z-1)^2f_7''+3(z-1)^2f_7'-A(z+1)f_7=2Af_1\,, \tag{6g}$$

$$z(z-1)^2f_8''+3(z-1)^2f_8'-A(z+1)f_8=2A(f_2+f_3+f_4)\,, \tag{6h}$$

$$z(z-1)^2 f_9'' + 3(z-1)^2 f_9' + A(z+1) f_9 = -2A(f_4 + z f_{12}), \tag{6i}$$

$$z(z-1)^2 f_{10}'' + 3(z-1)^2 f_{10}' + A(z-1) f_{10} = 2A(f_1 + z f_7), \tag{6j}$$

$$z(z-1)^2 f_{11}'' + 4(z-1)^2 f_{11}' - A(z-1) f_{11} = -2A(f_1 + f_7), \tag{6k}$$

$$z(z-1)^2 f_{12}'' + 4(z-1)^2 f_{12}' - A(z+1) f_{12} = 2A(f_4 + f_9), \tag{6l}$$

where $A = g^2/(4\pi^2)$.

In view of the extremely complicated nature of the nonhomogeneous coupled differential equations (6) containing the functions $f_i$, we solve them in an approximate fashion. The procedure is of course a standard one.[12] We first solve the homogeneous parts of these equations, and then substitute these solutions for the respective inhomogeneous terms in the original equations. For the homogeneous parts of these equations we note that Eqs. (6a), (6c), (6g), (6h), (6i), and (6l) are Heun's equations[17,19,20] and (6b), (6d), (6e), (6f), (6j), and (6k) are hypergeometric equations. The general solution of Heun's equation may be written down as a series of hypergeometric functions, and is rather involved. What we are interested in, however, are the asymptotic solutions of these equations and these may be readily obtained:

$$f_1^h \underset{z\to\infty}{\sim} z^{-[1\pm(1-4A)^{1/2}]/2}, \tag{7a}$$

$$f_2^h \underset{z\to\infty}{\sim} z^{-1\pm(1+A)^{1/2}}, \tag{7b}$$

$$f_9^h \underset{z\to\infty}{\sim} z^{-1\pm(1-A)^{1/2}}, \tag{7c}$$

$$f_{11}^h \underset{z\to\infty}{\sim} z^{(3/2)[1\pm(1+4A)^{1/2}]}, \tag{7d}$$

where the superscript "$h$" corresponds to the homogeneous parts of Eq. (6). The homogeneous solutions of different amplitudes satisfy the following relations in the asymptotic limit $z\to\infty$:

$$f_1^h \approx f_3^h \approx f_4^h \approx f_6^h, \tag{7e}$$

$$f_2^h \approx f_5^h \approx f_7^h \approx f_8^h, \tag{7f}$$

$$f_9^h \approx f_{10}^h, \tag{7g}$$

$$f_{11}^h \approx f_{12}^h. \tag{7h}$$

Thus, for large $z$ the inhomogeneous terms in Eq. (6) are known. We can obtain the asymptotic solutions of these equations as given in Ref. 20. We take the dominant convergent solutions of the nonhomogeneous equations. The new solutions obtained are again substituted for the inhomogeneous terms of the differential equations (6) until we obtain the same consistent solutions in the asymptotic region. The results are

$$f_1(s,t) \underset{t\to\infty}{\sim} c_1(s)\exp\{[-\tfrac{1}{2}\pm\tfrac{1}{2}(1-4A)^{1/2}]\ln t\}, \tag{8a}$$

$$f_2(s,t) \underset{t\to\infty}{\sim} c_2(s)\exp\{[-1-(1+A)^{1/2}]\ln t\}, \tag{8b}$$

$$f_5(s,t) \underset{t\to\infty}{\sim} c_5(s)\exp\{[-\tfrac{3}{2}\pm\tfrac{1}{2}(1-4A)]^{1/2}\ln t\}, \tag{8c}$$

$$f_9(s,t) \underset{t\to\infty}{\sim} c_9(s)\exp\{[-1\pm(1-A)^{1/2}]\ln t\}. \tag{8d}$$

The amplitudes $f_1(s,t)$, $f_3(s,t)$, $f_4(s,t)$, and $f_6(s,t)$ have the same exponential factor at large momentum transfers. They decrease as $1/\sqrt{t}$ in the asymptotic region and have the dominant contributions. The form factors $f_5$, $f_7$, $f_8$, $f_{11}$, and $f_{12}$ have the same behavior in the asymptotic limit, whereas $f_9$ and $f_{10}$ decrease as $\exp\{[-1 \pm (1-A)^{1/2}]\ln t\}$. Since the constant $A$ ($\equiv g^2/4\pi^2$) is greater than one, the absolute values of all the form factors except $f_2$ are independent of the pion-nucleon coupling constant in the high-$t$ limit. Thus, the asymptotic behavior in the variable $t$ of the modified vertex function is completely known. It should be mentioned here that the earlier investigations of the electromagnetic form factor with radiative corrections resulted in determining the asymptotic behavior of what is $f_1$ in this paper. We have estimated all the off-shell form factors. Our result for $f_1$ is similar to those of Jackiw[1] and Appelquist and Primack.[3] In our nonperturbative approach of converting an integral equation into an equivalent set of differential equations the photon-exchange diagrams can also be tackled in a similar manner.


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