Vacuum behavior in quantum chromodynamics. II

M. Claudson and A. Yildiz
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

P. H. Cox
Research Laboratory of Mechanics, University of New Hampshire, Durham, New Hampshire 03824
(Received 21 November 1979)

Vacuum polarization contributions due to fermions or scalars produced by gluon background-field fluctuations indicate vacuum instability in quantum chromodynamics; the instability is quantitatively related to pair-creation processes. The results extend previous work on gluonic contributions, and generalize the similar quantum-electrodynamic results of Euler and Heisenberg and Schwinger.

Virtual fermion pair creation by quantum fluctuations of a constant electric field in quantum electrodynamics is known to lead to instability of the vacuum. For a pure magnetic field, however, the vacuum is stable.[1] Combined magnetic and electric background fields in general also make the vacuum unstable. Indeed, pair creation (vacuum polarization) suggests a bridge between a fermion and antifermion pair, so that the Coulomb potential will acquire a correction due to vacuum polarization.

Recent experimental discoveries of heavy mesons and spectroscopy of such families show us that an effectively linear potential will be taking over at large separation of the fermion-antifermion (quark-antiquark) pair.[2] The answer to this type of potential lies outside the quantum-chromodynamics picture (so far), but a number of questions naturally appear within quantum chromodynamics. The background techniques that we employ reveal some relevant and useful information pertinent to the vacuum structure of chromodynamics, which eventually may be a crucial step for the understanding of confinement and hadron spectroscopy.[3]

In a previous paper[4] we obtained the effective-action contribution due to gluon creation in a background gluon field, at the one-loop level, and analyzed its effect on the behavior of the vacuum. In the present paper we extend these results to include fermionic and scalar contributions.

The calculation in these cases, as for gluons, closely parallels the classic work[5] in quantum electrodynamics. We assume a background gauge field B_μ^a, with field strengths

$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g f^{abc} B^b_\mu B^c_\nu , \quad (1)$$

where f^{abc} are the (totally antisymmetric) structure constants of the gauge group. [The symmetry group is left arbitrary, although we expect to identify it as SU(3) for quantum chromodynamics.] The field strengths are assumed to obey a constant-field condition

$$D^{ab}_\lambda F^b_{\mu\nu} = 0 , \quad (2)$$

where the covariant derivative operator D^{ab}_λ occurs here as

$$D^{ab} = \delta^{ab}\partial_\lambda + g f^{acb} B^c_\lambda . \quad (3)$$

Adding fermion or scalar fields to the theory introduces other representations of the gauge group. We will use the notation T_a for matrices for either representation, obeying $[T_a, T_b] = i f_{abc} T_c$, since no serious ambiguity will arise. The covariant derivative acting on these fields is then

$$(D_\lambda)^{ij} = \delta^{ij}\partial_\lambda - ig(T_a)^{ij} B^a_\lambda . \quad (4)$$

Introducing a matrix notation for the gauge-field strengths, with

$$B_\lambda = T_a B^a_\lambda, \quad F_{\mu\nu} = T_a F^a_{\mu\nu} , \quad (5)$$

we obtain the standard commutator relation

$$[D_\mu, D_\nu] = -ig F_{\mu\nu} , \quad (6)$$

while the constant-field condition, Eq. (2), implies

$$[F_{\kappa\lambda}, F_{\mu\nu}] = 0 . \quad (7)$$

The quantum effective-action contribution is obtained from the basic formula,[6] in the fermion case,

$$\delta W_f = -i \operatorname*{Tr}_{XQD} G\,\delta G^{-1} = i \operatorname*{Tr}_{XQD} \delta \ln G , \quad (8)$$

or in the scalar case,

$$\delta W_s = +\tfrac{1}{2} i \operatorname*{Tr}_{XQ} \Delta\,\delta\Delta^{-1} = -\tfrac{1}{2} i \operatorname*{Tr}_{XQ} \delta \ln \Delta , \quad (9)$$

where G or Δ is the respective propagator for the particles generated by a field fluctuation δA, and the trace in either case is over coordinate-space labels (X) as well as internal-symmetry indices (Q). In the fermion case there is also a trace over Dirac indices, plus an extra minus sign from statistics and a doubling since fermion fields are complex; the scalar formula, Eq. (9), assumes real fields.

Use of the standard fermionic propagator function, expressed in momentum space, $G=(\gamma\Pi+m)^{-1}$, where Π is the momentum-space equivalent of the covariant derivative, Eq. (4), gives the action change as

$$\delta W_f = -i \operatorname*{Tr}_{XQD} (\gamma\Pi+m)^{-1}\delta(\gamma\Pi+m)\,. \tag{10}$$

Using properties of traces and Dirac matrices, plus the identity

$$\frac{1}{x} = i\int_0^\infty ds\, e^{-isx} \tag{11}$$

(where the integral may be taken to infinity along any contour for which convergence is maintained), we obtain the net action contribution in the form

$$\Delta W_f = \tfrac{1}{2} \operatorname*{Tr}_{XQD} \int_0^\infty \frac{ds}{s} e^{-is(m^2+\Pi^2-g\sigma F)}\,, \tag{12}$$

where $\sigma F = \frac{1}{2}\sigma_{\mu\nu}F^{\mu\nu}$.

The constant-field condition, Eq. (2), enables us to separate Π and σ exponential factors, and to usefully apply proper-time techniques[7] to evaluate the coordinate-space trace. The Dirac trace can similarly be evaluated, giving a result

$$\Delta W_f = \tfrac{1}{2}\int d^4x' \frac{1}{(4\pi)^2} \times \operatorname*{Tr}_Q \int_0^\infty \frac{ds}{s^3} e^{-ism^2} \exp\left(-\tfrac{1}{2}\operatorname*{Tr}_L \frac{\sinh gF_\alpha{}^\beta s}{gF_\alpha{}^\beta s}\right) \times \left(\operatorname*{Tr}_L \cosh gF_\alpha{}^\beta s \cos g\tilde{F}_\alpha{}^\beta s\right), \tag{13}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}$ and Tr_L indicates a trace over Lorentz indices, which have been given a matrix interpretation.

Since all Lorentz components of F commute [Eq. (7)], they can be simultaneously diagonalized in color space. We can then diagonalize F as a Lorentz matrix without having difficulty due to non-Abelian color. In general, $gF_\mu{}^\nu$ will have two real and two imaginary eigenvalues in each color component; we can reassemble these into diagonal color matrices $\pm\alpha$, $\pm i\beta$ (α,β nonnegative). Due to properties of $\epsilon_{\kappa\lambda\mu\nu}$ in four-dimensional Minkowski space, expressions involving $\tilde{F}$ can also be evaluated using α and β; thus, an eigenvalue evaluation of the Lorentz traces in ΔW, Eq. (13), gives (explicitly, $(\alpha,\beta)=(g/\sqrt{2})\{[(E^2-B^2)^2+4(E\cdot B)^2]^{1/2}\pm(E^2-B^2)\}^{1/2}$)

$$\Delta W_f = \int d^4x' \frac{1}{(4\pi)^2} \operatorname*{Tr}_Q \int_0^\infty \frac{ds}{s^3} e^{-ism^2} \frac{\alpha s}{\sinh\alpha s} \times \frac{\beta s}{\sin\beta s}(2\cosh\alpha s\cos\beta s)\,, \tag{14}$$

before consideration of renormalization.

Similarly (in fact, more easily), the scalar contribution to the effective action can be obtained, using $\Delta=(\Pi^2+m^2)^{-1}$ in Eq. (9) to give

$$\Delta W_s = -\tfrac{1}{2}i\int_0^\infty \frac{ds}{s} \operatorname*{Tr}_{XQ} e^{-is(m^2+\Pi^2)} \tag{15}$$

$$= -\tfrac{1}{2}\int d^4x' \frac{1}{(4\pi)^2} \operatorname*{Tr}_Q \int_0^\infty \frac{ds}{s^3} e^{-ism^2} \times \exp\left(-\tfrac{1}{2}\operatorname*{Tr}_L \frac{\sinh gF_\alpha{}^\beta s}{gF_\alpha{}^\beta s}\right) \tag{16}$$

$$= -\tfrac{1}{2}\int d^4x' \frac{1}{(4\pi)^2} \operatorname*{Tr}_Q \int_0^\infty \frac{ds}{s^3} e^{-ism^2} \frac{\alpha s}{\sinh\alpha s}\,\frac{\beta s}{\sin\beta s}\,. \tag{17}$$

For $m\neq 0$, these integrals [Eqs. (14), (17)] are quite convergent at ∞ for any contour in the lower-half s plane that avoids the poles at $s=(n\pi/\alpha)i$. The physical choice is a fourth-quadrant contour, since correct causality boundary conditions on the Green's functions are expressed by the standard $m^2-i\epsilon$ prescription on propagator denominators, which in turn leads to exclusion of $\mathrm{Re}\,s<0$.

Taking the integration contour along the imaginary axis (displaced by ϵ), via the substitution $s\to -is$, we write these integrals as

$$\Delta W_f = -\int d^4x' \frac{2}{(4\pi)^2} \operatorname*{Tr}_Q \int_{0+i\epsilon}^{\infty+i\epsilon} \frac{ds}{s^3} e^{-m^2 s} \frac{\alpha s}{\sin\alpha s}\,\frac{\beta s}{\sinh\beta s} \times \cos\alpha s\cosh\beta s\,, \tag{18}$$

$$\Delta W_s = \int d^4x' \frac{1}{2}\,\frac{1}{(4\pi)^2} \operatorname*{Tr}_Q \int_{0+i\epsilon}^{\infty+i\epsilon} \frac{ds}{s^3} e^{-m^2 s} \frac{\alpha s}{\sin\alpha s}\,\frac{\beta s}{\sinh\beta s}\,. \tag{19}$$

Equations (18) and (19) have exactly the forms obtained in the QED calculation,[8] apart from some changes in notation. Despite their extensive history,[9] a number of properties of these expressions deserve attention, especially concerning their limit as $m\to 0$.

The ultraviolet ($s\to 0$) behavior of these integrals will as usual require renormalization; we first regularize by inserting a cutoff δ in place of 0 as the lower limit of the s integral. Use of partial-fraction expansions for the cotangent and cosecant then enables us to present ΔW as

$$\Delta W_f = \int d^4x' \frac{1}{(4\pi)^2} \operatorname*{Tr}_Q \left\{ \left[-\frac{1}{\delta^2} + \frac{2m^2}{\delta} + m^4(\gamma + \ln m^2\delta - \tfrac{1}{2})\right] - \tfrac{2}{3}(\alpha^2-\beta^2)(\gamma+\ln m^2\delta) + \alpha\beta F_1 + i\alpha\beta F_2 \right\}, \tag{20}$$

$$\Delta W_s = \int d^4x' \frac{1}{(4\pi)^2} \mathop{\mathrm{Tr}}_{Q} \left\{ \left[\frac{1}{4}\frac{1}{\delta^2} - \frac{1}{2}\frac{m^2}{\delta} - \frac{m^4}{4}(\gamma + \ln m^2\delta - \tfrac{1}{2}) \right] - \tfrac{1}{12}(\alpha^2-\beta^2)(\gamma+\ln m^2\delta) + \alpha\beta F_3 + i\alpha\beta F_4 \right\}, \quad (21)$$

where $\gamma = 0.577^{+}$ is Euler's constant and the dimensionless functions F_1, F_2, F_3, and F_4 are given by

$$F_1(\alpha,\beta,m^2) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left[\left(\coth\frac{\alpha k\pi}{\beta}\right)\int_0^\infty \frac{x\,dx}{x^2+1}e^{-(m^2\pi/\beta)kx} - \left(\coth\frac{\beta k\pi}{\alpha}\right)P\int_0^\infty \frac{x\,dx}{x^2-1}e^{-(m^2\pi/\alpha)kx}\right], \quad (22)$$

$$F_2(\alpha,\beta,m^2) = 2\sum_{k=1}^{\infty}\frac{1}{k}\left(\coth\frac{\beta k\pi}{\alpha}\right)e^{-(m^2/\alpha)k\pi}, \quad (23)$$

$$F_3(\alpha,\beta,m^2) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(\frac{1}{\sinh(\alpha k\pi/\beta)}\int_0^\infty \frac{x\,dx}{x^2+1}e^{-(m^2\pi/\beta)kx} - \frac{1}{\sinh(\beta k\pi/\alpha)}P\int_0^\infty \frac{x\,dx}{x^2-1}e^{-(m^2\pi/\alpha)kx}\right), \quad (24)$$

$$F_4(\alpha,\beta,m^2) = \tfrac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\frac{1}{\sinh(\beta k\pi/\alpha)}e^{-(m^2/\alpha)k\pi}, \quad (25)$$

where P indicates that the principal value of the integral is to be taken at the pole.

The first term of each of these contributions ΔW can be absorbed in a redefinition of the zero point of energy, provided that δ is taken to be real. Comparing the original action expression for the gauge fields,

$$W_0 = \int d^4x' \left(-\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a\right) = \int d^4x' \mathop{\mathrm{Tr}}_{Q}\frac{\alpha^2-\beta^2}{2g^2C_2} \quad (26)$$

[C_2 is the Casimir operator for the representation T, $C_2\delta_{ab} = \mathrm{Tr}\,T_aT_b$; this Eq. (26) differs from that used in our previous paper because T_a rather than the structure constants were used to define matrices $F_{\mu\nu}$], we observe that the second term of each ΔW, proportional to $\alpha^2-\beta^2$, can be absorbed in a redefinition of g^2:

$$\frac{1}{g_R^2} = \frac{1}{g^2} - \frac{2}{(4\pi)^2}\left[C_{2f}\tfrac{2}{3}(\gamma+\ln m^2\delta) + C_{2s}\tfrac{1}{12}(\gamma+\ln m^2\delta)\right], \quad (27)$$

where subscripts distinguish the fermion and scalar Casimir operators, and we recognize the Gell-Mann–Low β-function coefficients $-\frac{4}{3}C_{2f}$ and $-\frac{1}{6}C_{2s}$ in the coefficients of the logarithms. The remaining terms in ΔW, Eqs. (20) and (21), then correspond to physical effects, modifications of vacuum structure. In particular, the imaginary part of W gives a decay rate for the vacuum state. From the explicit forms of $\mathrm{Im}\,W = \alpha\beta F_2$, $\alpha\beta F_4$, given in Eqs. (23) and (25), we see that for all values of field strengths except $\alpha = 0$, the vacuum state has a finite width for decays related to either interaction, i.e., any configuration except pure-magnetic-type will decay by pair production of both fermions and scalars. The width, per unit space-time volume, is given by the imaginary part of the integrands in ΔW [Eqs. (20) and (21)]; $|\langle 0|0\rangle|^2 = |\exp(iW)|^2 = \exp(-2\,\mathrm{Im}\,W) = \exp(-2VT\,\mathrm{Im}\mathcal{L})$, where VT is the volume of space-time and $\mathcal{L}$ is the Lagrangian whose integral is the action W. Here we obtain

$$2\,\mathrm{Im}\mathcal{L}_f = \frac{1}{8\pi^2}\mathop{\mathrm{Tr}}_{Q}\alpha\beta F_2 = \frac{1}{4\pi^2}\mathop{\mathrm{Tr}}_{Q}\sum_{k=1}^{\infty}\frac{\alpha\beta}{k}\left(\coth\frac{\beta k\pi}{\alpha}\right)e^{-(m^2/\alpha)k\pi}, \quad (28)$$

$$2\,\mathrm{Im}\mathcal{L}_s = \frac{1}{8\pi^2}\mathop{\mathrm{Tr}}_{Q}\alpha\beta F_4 = \frac{1}{16\pi^2}\mathop{\mathrm{Tr}}_{Q}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\frac{\alpha\beta}{\sinh(\beta k\pi/\alpha)}e^{-(m^2/\alpha)k\pi}. \quad (29)$$

These formulas also apply in the case of Abelian symmetry, where we recover the known pair-creation probability in an electric field ($\beta = 0$, $\alpha = e|\vec{E}|$).[5] In that Abelian case, these contributions give the entire decay rate, and we determine that a magnetic-field-only configuration in the vacuum is stable. In the non-Abelian case, however, we must include the gluonic action contribution calculated in our first paper[4] and find that only the zero-fields configuration can be stable; consequences of this instability are discussed in that paper.

The massless limit of these formulas deserves special attention. For fermions, the proper-time integral for the effective-action correction [ΔW_f, Eq. (14)] is infrared divergent ($s \to \infty$) in the massless case for all contour directions unless the background field is either pure electric or pure magnetic in some Lorentz frame, i.e., either $\beta = 0$ or $\alpha = 0$, respectively. In contrast, the corresponding integral for the massless scalar contribution [ΔW_s, Eq. (17)] is convergent at $s \to \infty$.

Taking m^2 finite but small, we can separate the small-m^2 dependence of the fermionic contribution in the form

$$\begin{aligned}\alpha\beta(F_1+iF_2)=\alpha\beta\Big\{\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\Big[\Big(\coth\frac{\alpha k\pi}{\beta}-1\Big)\ln\frac{\beta}{kF_0}-\Big(\coth\frac{\beta k\pi}{\alpha}-1\Big)\ln\frac{\alpha}{kF_0}+\frac{\pi}{2}i\Big(\coth\frac{\beta k\pi}{\alpha}-1\Big)\Big]\Big\}\\ -\tfrac{2}{3}(\alpha^2-\beta^2)\Big(\ln\frac{F_0}{m^2\pi}-\gamma\Big)-\frac{4}{\pi}\int_{i\epsilon}^{x_0+i\epsilon}\frac{dx}{x^2}\Big[\beta\ln\frac{\Gamma(1+\alpha x)}{\Gamma(1-\alpha x)}+i\alpha\ln\frac{\Gamma(1+i\beta x)}{\Gamma(1-i\beta x)}\Big]\\ -\frac{4}{\pi}\int_{x_0+i\epsilon}^{\infty+i\epsilon}\frac{dx}{x^2}\Big[\beta\ln\frac{\Gamma(1+\alpha x)}{\Gamma(1-\alpha x)}+i\alpha\ln\frac{\Gamma(1+i\beta x)}{\Gamma(1-i\beta x)}-2\alpha\beta x\Big(\ln\frac{\beta}{\alpha}+\frac{\pi}{2}i\Big)\Big]\\ +2\alpha\beta\Big(\frac{2}{\pi}\ln\frac{\beta}{\alpha}+i\Big)(2-\gamma-\ln m^2\pi x_0)\,,\end{aligned} \tag{30}$$

where $\epsilon\to 0+$ defines the treatment of the poles of $\Gamma(1-\alpha x)$, and Γ is the generalized factorial function; terms which vanish as $m\to 0$ have been dropped. Here F_0 and x_0 are arbitrary dimensional parameters, introduced to divorce field-strength dependence from mass dependence, similar to the renormalization-group arbitrary dimensional parameter. The first divergent term here, $-\frac{2}{3}(\alpha^2-\beta^2)[\ln(F_0/m^2\pi)-\gamma)$, has the effect of substituting F_0 for m^2 in the coupling-constant renormalization formula, with some change in numerical additive constants, and is thus removed from the theory. The last term in this expression, however, cannot be similarly removed, and unless α or β is zero, it gives a divergent indefinite-sign contribution to the vacuum action integral and a divergent nonnegative contribution to the vacuum decay width.

A small-mass scalar, on the other hand, yields a well-behaved contribution. In the scalar contribution, F_4 has a smooth limit as $m\to 0$, while F_3 has the form

$$\alpha\beta F_3=-\frac{1}{12}\Big(\alpha^2\ln\frac{\alpha}{F_0}-\beta^2\ln\frac{\beta}{F_0}\Big)+\frac{2\alpha^2\beta^2}{\pi^2}\sum_{k,n=1}^{\infty}(-1)^{n+k}\frac{\ln(\alpha k/\beta n)}{\alpha^2k^2+\beta^2n^2}-\tfrac{1}{12}(\alpha^2-\beta^2)\Big(\ln\frac{F_0}{m^2}-\gamma-\ln\frac{\pi}{2}+\frac{6}{\pi^2}\zeta'(2)\Big)\,, \tag{31}$$

where ζ' is the derivative of the Riemann zeta function; terms which vanish at $m\to 0$ have been dropped. Here the only divergent term is the one required to substitute F_0 for m^2 in the coupling-constant renormaliztion formula. The additive constant there can conveniently absorb some non-divergent terms, but in any case the result is a form with a smooth limit as $m_s\to 0$, quite different from the new singularities obtained as $m_f\to 0$.

One of us (A.Y.) would like to acknowledge conversations with J. Schwinger, C. Callan, D. Nanopoulos, and especially with D. Sutherland. This research is supported in part by the National Science Foundation under Grant No. PHY77-22864, and by the Department of Energy under Grant No. ER-78-S-02-4999.

---

[1]However, even pure magnetic fields induce unstable eigenmodes for theories containing spin-1 particles with anomalous magnetic moments; see W-y. Tsai and A. Yildiz, Phys. Rev. D 4, 3643 (1971); T. Goldman, W-y. Tsai, and A. Yildiz, *ibid.* 5, 1926 (1972). The first observation of such instabilities in quantum chromodynamics (QCD) was by N. K. Nielsen and P. Olesen, Nucl. Phys. B144, 376 (1978); Phys. Lett. 79B, 304 (1978). Other investigations of QCD instabilities include L. S. Brown and W. I. Weisberger, Nucl. Phys. B157, 285 (1979).

[2]A linear confining potential was first observed in the (1+1)-dimensional Schwinger model [J. Schwinger, Phys. Rev. 125, 397 (1962)]. Application of this potential to four-dimensional physics has been suggested by many authors, notably K. G. Wilson, Phys. Rev. D 10, 2445 (1974).

[3]For discussion of various developments in these areas see S. L. Adler, Phys. Rev. D 19, 1168 (1979); Phys. Lett. 86B, 203 (1979); Phys. Rev. D 21, 550 (1980); A. A. Belavin *et al.*, Phys. Lett. 59B, 85 (1975); C. G. Callan, Jr., R. F. Dashen, and D. J. Gross, *ibid.* 63B, 334 (1976); A. Chodos *et al.*, Phys. Rev. D 9, 3471 (1974); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976); J. E. Mandula, Phys. Rev. D 14, 3497 (1976); G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976); H. Pagels and E. Tomboulis, Nucl. Phys. B143, 485 (1978).

[4]A. Yildiz and P. H. Cox, Phys. Rev. D 21, 1095 (1980).

[5]J. Schwinger, Phys. Rev. 82, 664 (1951), reprinted in *Quantum Electrodynamics*, edited by J. Schwinger (Dover, New York, 1958), p. 209.

[6]For a derivation in the boson case see J. Schwinger, Lett. Math. Phys. 1, 43 (1975). The fermion case is

essentially identical; we note the minor differences in the text.

[7]Proper-time techniques were introduced in Schwinger's elegant paper, Ref. 5. A recent discussion appears in B. S. DeWitt, Phys. Rep. 19C, 295 (1975).

[8]This is due essentially to our assumption of covariantly constant background, which implies the simplifications due to Eq. (7). In the QED results, Ref. 5, there is naturally no color trace (a factor of two for charge trace occurs instead), while our eigenvalue matrices $\pm\alpha$, $\pm i\beta$ correspond to Schwinger's $\pm eF^{(1,2)}$ (in some order), and he combines our trigonometric and hyperbolic functions in terms of functions of $X = [\frac{1}{2}F_{\mu\nu}(F^{\mu\nu} + i\tilde{F}^{\mu\nu})]^{1/2}$. The QED results have not, however, been extended beyond this point except for the cases of either $\vec{E}=0$, or $\vec{B}=0$, or weak fields; we continue our evaluation in general.

[9]Schwinger's result, Ref. 5, is in fact an independent confirmation of the Lagrangian originally obtained by W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936).