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Nonlocal currents as Noether currents

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The first two nonlocal currents in the general two-dimensional chiral models are derived as Noether currents. The associated infinitesimal field transformations are shown to obey a group integrability condition. A subset of the structure constants of the symmetry group responsible for these conserved currents is calculated.

I. INTRODUCTION

Two-dimensional chiral models have many features in common with four-dimensional Yang-Mills theory. We expect that in addition to local gauge invariance and the 15-parameter conformal symmetry the non-Abelian gauge theory has a so-far-unknown symmetry which, when found, will yield information about the nonperturbative sector. In the chiral models, the signal of this symmetry is the existence of an infinite set of nonlocal conserved charges.

In the classical nonlinear σ model there exist, in addition to the familiar space-time conserved currents, two infinite sets of currents. The first, given by Pohlmeyer,[1] can be expressed as products of local fields. They are moments of the energy-momentum tensor. A symmetry group associated with this set is the infinite-parameter conformal group in two dimensions.[2] The second set, found by Luscher and Pohlmeyer,[3] is nonlocal.

The symmetry group responsible for the nonlocal charges has not been identified. In this paper we exhibit nonlocal infinitesimal transformations which (1) are symmetries of the equations of motion, (2) shift the Lagrangian density by a total divergence without the use of the equations of motion, and thus (3) give rise to the nonlocal currents as Noether currents.

For these infinitesimal transformations we check the integrability condition[4] to see if they generate finite values which form a group. We find that the law of composition for the set of infinitesimal transformations associated with the first nonlocal current gives a transformation associated with the second current.

This pattern is consistent with the notion that there is a Lie group responsible for the nonlocal conservation laws and that all the currents must be used in order to close the algebra. This implies an infinite number of generators which can be used to construct the infinite-parameter group of finite field transformations.

The infinitesimal transformations provide a realization of these generators and thus an unambiguous method to calculate the structure constants. We explicitly calculate a finite subset of these, relating the generators corresponding to the first two currents.

We stress that we have found expressions for the infinitesimal field transformations which shift the Lagrangian density by a total divergence for arbitrary field configurations, not just for field solutions. We have thus demonstrated that the symmetry responsible for the nonlocal charges is a symmetry of the entire space of fields not just a symmetry of the set of solutions. This is necessary for the symmetry to give rise to Noether currents in the standard fashion.

Derivation of these "hidden" symmetry currents as Noether currents permits identification of the symmetry associated with the analogous path-dependent quantities in the functional formulation of Yang-Mills theory. The crucial characteristic of the nonlocal symmetry is that it has the form of global isospin where the global parameters have been replaced by a particular function of space-time. The extra sought for symmetry in Yang-Mills theory is invariance under a special path-dependent gauge transformation.[5]

In Sec. II we display the explicit form of the infinitesimal transformations for the first two nontrivial nonlocal currents in the general chiral models. We show that their Noether currents are equivalent to the standard expressions[6] for the nonlocal currents when the fields are solutions. We describe the method we used to find these infinitesimal transformations and how to use it to calculate the relevant quantities for the higher currents.

In Sec. III, the isospinlike property of these transformations is discussed in the particular example of the O(3) nonlinear σ model. To identify the symmetry group associated with the nonlocal transformations we check the group integrability condition and exhibit a subset of the structure constants. Calculations are performed in Euclidean space, but can be extended without difficulty to Minkowski space.

II. INFINITESIMAL TRANSFORMATIONS

The Lagrangian density for the general class of chiral models is

$$\mathcal{L}(x)=\tfrac{1}{16}\operatorname{tr}\partial_\mu g(x)\partial_\mu g^{-1}(x)\,. \tag{2.1}$$

The matrix field $g(x)$ is an element of some group G. The equations of motion are $\partial_\mu A_\mu=0$, where $A_\mu\equiv g^{-1}\partial_\mu g$. The infinitesimal transformations associated with the first two nonlocal currents are given by

$$\Delta^{(n)}g=-g\lambda^{(n)}\,, \tag{2.2}$$

$$\lambda^{(1)}=4[\chi^{(1)},T]\,, \tag{2.3a}$$

$$\lambda^{(2)}=4\{[\chi^{(2)},T]+\tfrac{1}{2}[\chi^{(1)},[\chi^{(1)},T]]\}\,, \tag{2.3b}$$

$$\chi^{(1)}(y,t)=\tfrac{1}{2}\int_{-\infty}^{\infty}dx\,\epsilon(y-x)A_0(x,t)\,, \tag{2.4a}$$

$$\chi^{(2)}(y,t)=-\tfrac{1}{2}\int_{-\infty}^{\infty}dx\,\epsilon(y-x)A_1(x,t)+\tfrac{1}{4}\int_{-\infty}^{\infty}dx\,\epsilon(y-x)[A_0(x,t),\chi^{(1)}(x,t)]\,. \tag{2.4b}$$

We define $T=T^e\rho^e$ for ρ^e constant and T^e the matrix generators for the group G.

The transformation (2.2) shifts the Lagrangian density by

$$\Delta^{(n)}\mathcal{L}=\tfrac{1}{8}\operatorname{tr}A_\mu\partial_\mu\lambda^{(n)}. \tag{2.5}$$

Equation (2.5) can be expressed as a total divergence without using the equations of motion:

$$\Delta^{(1)}\mathcal{L}=\partial_\mu\tfrac{1}{2}\operatorname{tr}\{(\tfrac{1}{2}\epsilon_{\mu\nu}[\partial_\nu\chi^{(1)},\chi^{(1)}]+\epsilon_{\mu\nu}A_\nu)T\}\,, \tag{2.6a}$$

$$\Delta^{(2)}\mathcal{L}=\partial_\mu\tfrac{1}{2}\operatorname{tr}\{(\epsilon_{\mu\nu}[\partial_\nu\chi^{(1)},\chi^{(2)}]+\epsilon_{\mu\nu}\tfrac{1}{8}[[\partial_\nu\chi^{(1)},\chi^{(1)}],\chi^{(1)}]+\epsilon_{\mu\nu}[A_\nu,\chi^{(1)}])T\}\,. \tag{2.6b}$$

Using the equations of motion, one finds the change in $\mathcal{L}$ for $\Delta g=-g\lambda^{(n)}$ is

$$\Delta^{(n)}\mathcal{L}=\partial_\mu\tfrac{1}{8}\operatorname{tr}A_\mu\lambda^{(n)}. \tag{2.7}$$

Equate (2.6) and (2.7) to derive the conserved quantities

$$\rho^e\partial_\mu(\tfrac{1}{2}\operatorname{tr}\mathcal{J}_\mu^{(n)}T^e)=0\,, \tag{2.8}$$

where

$$\mathcal{J}_\mu^{(1)}=[A_\mu,\chi^{(1)}]-\epsilon_{\mu\nu}A_\nu-\tfrac{1}{2}\epsilon_{\mu\nu}[\partial_\nu\chi^{(1)},\chi^{(1)}]\,, \tag{2.9a}$$

$$\mathcal{J}_\mu^{(2)}=[A_\mu,\chi^{(2)}]+\tfrac{1}{2}[[A_\mu,\chi^{(1)}],\chi^{(1)}]-\epsilon_{\mu\nu}[\partial_\nu\chi^{(1)},\chi^{(2)}]-\tfrac{1}{8}\epsilon_{\mu\nu}[[\partial_\nu\chi^{(1)},\chi^{(1)}],\chi^{(1)}]-\epsilon_{\mu\nu}[A_\nu,\chi^{(1)}]\,. \tag{2.9b}$$

Since A_μ is an element of the Lie algebra of G, $A_\mu(x)=f^e(x)T^e$, the matrix currents $\mathcal{J}_\mu^{(n)}$ are also conserved:

$$\partial_\mu\mathcal{J}_\mu^{(n)}=0\,. \tag{2.10}$$

If we now use the equations of motion, then

$$\partial_\mu\chi^{(1)}=-\epsilon_{\mu\nu}A_\nu\,,\qquad \partial_\mu\chi^{(2)}=-A_\mu-\tfrac{1}{2}\epsilon_{\mu\nu}[A_\nu,\chi^{(1)}]\,. \tag{2.11}$$

Note that in (2.11), $\chi^{(n)}$ is only defined when $\partial_\mu A_\mu=0$. In (2.4), $\chi^{(n)}$ is defined for all A_μ. Since previous derivations[6,7] of these symmetry currents made use of equations similar to (2.11), it is clear that those derivations could only discuss the symmetry on the solution set. In the derivation presented here, this restriction is eliminated. The charges $Q^{(n)}=\int_{-\infty}^{\infty}dy\,\mathcal{J}_0^{(n)}(y,t)$ formed from (2.9) are the nonlocal charges in standard form. The currents $\mathcal{J}_\mu^{(n)}$ however are different from the expressions in the literature[6,7] in that they are defined for all field configurations. For solutions, the currents $\mathcal{J}_\mu^{(n)}$ reduce to the standard form.

The infinitesimal transformations (2.2) are valid for the general class of chiral models. We guessed this form from the following considerations of the restricted case of the O(N) nonlinear σ model. In their theory, $g_{ab}(x)\equiv\delta_{ab}-2\phi_a(x)\phi_b(x)$, with $\phi^2=1$, i.e., $g^{-1}=g$. Using the Dirac brackets of the charges with the field to generate the infinitesimal transformations, we found

$$\{Q^{(n)},g\}_{\text{Dirac}}\equiv-\delta^{(n)}g=-[\lambda^{(n)},g]\,, \tag{2.12}$$

where we use the charges

$$Q^{(1)}=\int_{-\infty}^{\infty}dy\{-A_1(y,t)+\tfrac{1}{2}[A_0(y,t),\chi^{(1)}(y,t)]\}\,, \tag{2.13a}$$

$$Q^{(2)}=\int_{-\infty}^{\infty}dy\{-[A_1(y,t),\chi^{(1)}(y,t)]-\tfrac{1}{3}[\chi^{(1)}(y,t),[A_0(y,t),\chi^{(1)}(y,t)]]\} \tag{2.13b}$$

and $\lambda^{(n)}$ is given in (2.3). We then relaxed the constraint $g^{-1}=g$. The condition that $\delta^{(n)}g$ in (2.12) be an invariance of the equations of motion (i.e., if g is a solution so is $g+\delta^{(n)}g$) is

$$\Box\lambda^{(n)}+[g\partial_\mu g^{-1},\partial_\mu\lambda^{(n)}]=0 \tag{2.14a}$$

and

$$\Box\lambda^{(n)}+[A_\mu,\partial_\mu\lambda^{(n)}]=0\,. \tag{2.14b}$$

Equation (2.14a) is the condition when $\delta^{(n)}g=\lambda^{(n)}g$ and (2.14b) comes from the part $\delta^{(n)}g=-g\lambda^{(n)}$. Equations (2.14) are equivalent for $g=g^{-1}$. For the general models, we chose the transformation $\delta^{(n)}g=\Delta^{(n)}g=-g\lambda^{(n)}$ with $\lambda^{(n)}$ given in (2.3) and found that it is a symmetry of the solutions and of the Lagrangian density.

The infinitesimal transformations for the higher currents ($n>2$) can be found in this manner. To show that $\Delta^{(n)}g=-g\lambda^{(n)}$ shifts the Lagrangian density by a total divergence is a complicated calculation which differs significantly for each n. Owing to the tedium involved we have not calculated additional transformations, and we have not been able to simply generalize our results for arbitrary n.

III. GROUP PROPERTIES

The nonlocal transformation discussed in Sec. II is similar to the known global invariance of (2.1) except that the global parameters are particular functions of space-time. For example, for the O(3) nonlinear σ model, the finite global isospin transformation is $\varphi_a'(y,t)=\exp(\lambda_b\epsilon_{abc})\varphi_c(y,t)$ with λ_b constant. This has the infinitesimal form

$$\delta\varphi_a(y,t)=\lambda_b\epsilon_{abc}\varphi_c(y,t)\,. \tag{3.1}$$

Clearly for λ_b not constant, (3.1) is in general not a symmetry. From (2.3), however, a transformation in terms of the $\varphi_a(y,t)$ field coordinates is also given by (3.1) with $\lambda_b=\frac{1}{2}\epsilon_{abc}\lambda_{ac}^{(1)}(y,t)$:

$$\Delta^{(1)}\varphi_a(y,t)=\lambda_{ac}^{(1)}(y,t)\varphi_c(y,t)\,. \tag{3.2}$$

That is to say, for the specific space-time function $\lambda_{ac}^{(1)}(y,t)$ the infinitesimal global isospin becomes local and nonlinear in the fields.

It is this qualitative feature which is relevant to functional Yang-Mills theory. In that case, the functional Lagrangian is symmetric under (1) arbitrary local gauge transformations [the analog of global isospin (3.1)], and under (2) a specific path-dependent gauge transformation [the analog of the "hidden" symmetry (3.2)].

We now return to the general chiral models. To see that the infinitesimal transformations associated with the nonlocal conserved quantities meet the necessary conditions to generate a group, we check the integrability condition as follows. Assume that $\delta_\rho(g)$ is the infinitesimal form of a finite transformation law $T_\rho(g)$. Then $T_\rho(g)=g+\delta_\rho(g)+O(\rho^2)$. Since T_ρ is a group element we can define its inverse as $T_{\rho^{-1}}$ and it must be that

$$T_{\sigma^{-1}}(T_{\rho^{-1}}(T_\sigma(T_\rho(g))))=\mathcal{T}(g)\,, \tag{3.3}$$

where $\mathcal{T}$ is also an element of the group. The infinitesimal form of (3.3) is the integrability condition

$$\delta_\sigma(g+\delta_\rho(g))-\delta_\sigma(g)-\delta_\rho(g+\delta_\sigma(g))-\delta_\rho(g)=d_{\sigma\rho}(g)\,, \tag{3.4}$$

where $d_{\sigma\rho}$ must be an infinitesimal transformation in the group. For $\delta_\rho(g)=-g\lambda^{(1)}$ from (2.2) we explicitly find in (3.4)

$$d_{\sigma\rho}(g)=C_{bca}\rho_b\sigma_c\tfrac{1}{4}\lambda_a^{(\tilde{2})}\,,$$

$$\lambda_a^{(\tilde{2})}(y,t)=4\left\{\tfrac{1}{2}\int_{-\infty}^{\infty}dx\,\epsilon(y-x)[\partial_0\chi^{(1)}(x,t),T^a]+\tfrac{1}{4}\int_{-\infty}^{\infty}dx\,\epsilon(y-x)[[A_0(x,t),\chi^{(1)}(x,t)],T^a]+\tfrac{1}{2}[\chi^{(1)}(y,t),[\chi^{(1)}(y,t),T^a]]\right\}. \tag{3.5}$$

Here C_{bca} are the structure constants of the Lie algebra of the group of fields G: $[T^b,T^c]=C_{bca}T^a$. Equation (3.5) is a different infinitesimal transformation from $\delta_\rho(g)=-g\lambda_\rho^{(2)}$. It is, however, a symmetry of both the equations of motion and the Lagrangian density and gives rise to (2.13b), the second nonlocal charge, as its Noether charge.[8]

Thus we see that the infinitesimal transformations from the first nonlocal charge (2.13a) do not generate a group by themselves but mix with those of the higher currents. We speculate that it is the infinite set of infinitesimal transformations which close the algebra.

From these considerations it is now easy to calculate the structure constants between the first two sets of generators. The generators $M_e^{(n)}$ can be constructed from the nonlocal infinitesimal transformations in standard fashion.[9] For $\Delta^{(n)}g\equiv\rho^e\Delta_e^{(n)}(g)$,

$$M_e^{(n)}=-\int d^2y\,\Delta_e^{(n)}(g(y))\frac{\delta}{\delta g(y)}\,. \tag{3.6}$$

To derive the structure constants, compute the commutators $[M_e^{(n)},M_d^{(m)}]$:

$$\begin{aligned}[M_e^{(n)},M_d^{(m)}]&=\int d^2y\,\Delta_e^{(n)}(g(y))\frac{\delta}{\delta g(y)}\left(\int d^2x\,\Delta_d^{(m)}(g(x))\frac{\delta}{\delta g(x)}\right)-\int d^2x\,\Delta_d^{(m)}(g(x))\frac{\delta}{\delta g(x)}\left(\int d^2y\,\Delta_e^{(n)}(g(y))\frac{\delta}{\delta g(y)}\right)\\&=\int d^2x\left(\int d^2y\,\Delta_e^{(n)}(g(y))\frac{\delta}{\delta g(y)}\,\Delta_d^{(m)}(g(x))\right)\frac{\delta}{\delta g(x)}-\int d^2y\left(\int d^2x\,\Delta_d^{(m)}(g(x))\frac{\delta}{\delta g(x)}\,\Delta_e^{(n)}(g(y))\right)\frac{\delta}{\delta g(y)}\\&=\int d^2x[\Delta_d^{(m)}(g(x)+\Delta_e^{(n)}(g(x)))-\Delta_d^{(m)}(g(x))]\frac{\delta}{\delta g(x)}-\int d^2y[\Delta_e^{(n)}(g(y)+\Delta_d^{(m)}(g(y)))-\Delta_e^{(n)}(g(y))]\frac{\delta}{\delta g(y)}\,.\end{aligned} \tag{3.7}$$

Whenever the infinitesimal transformations $\Delta_e^{(n)}(g)$ obey the integrability condition (3.4), the structure constants are defined as $f_{eda}^{(n)(m)(j)}$ such that

$$[M_e^{(n)},M_d^{(m)}]=f_{eda}^{(n)(m)(j)}M_a^{(j)}\,. \tag{3.8}$$

For $\Delta^{(1)}g=-g\lambda^{(1)}$ and $\Delta^{(\tilde{2})}g=-g\lambda^{(\tilde{2})}$ we find using (3.5)

$$[M_e^{(1)}, M_d^{(1)}] = C_{eda} M_a^{(2)}. \tag{3.9}$$

Thus the structure constants $f_{eda}^{(1)(1)(\bar{2})} = C_{eda}$, the structure constants of the field group G.

Finally, integration of (3.4) gives finite transformation laws for the fields.[10] These are finite in the group parameters $\rho_e^{(n)}$. One such law is given for the O(3) nonlinear σ model (for simplicity) by the formal expression

$$\varphi_d'(y) = \exp\left[\epsilon_{abc}\rho_e \int d^2x\, \chi_{be}(x)\varphi_c(x)\frac{\delta}{\delta\varphi_a(x)}\right]\varphi_d(y). \tag{3.10}$$

For global isospin symmetry the formula analogous to (3.10) is

$$\varphi_d''(y) = \exp\left[\epsilon_{abc}\rho_b \int d^2x\, \varphi_c(x)\frac{\delta}{\delta\varphi_a(x)}\right]\varphi_d(y). \tag{3.11}$$

Owing to the linearity of (3.11), it is easily shown to be equal to the familiar useful equation

$$\varphi_d''(y) = \exp(\epsilon_{dbc}\rho_b)\varphi_c(y). \tag{3.12}$$

IV. CONCLUSION

The new results of this paper are the derivation of the first two nonlocal currents in the general two-dimensional chiral models as Noether currents and the exhibition that the infinitesimal field transformations giving rise to these Noether currents are related by a group integrability condition.

The two-dimensional chiral models have many features in common with the non-Abelian gauge theory and it is thus natural to expect an analog of the infinite set of conservation laws, or correspondingly, an associated symmetry group in the physical theory. As a step towards this goal we have chosen to examine the two-dimensional models in the familiar language of continuous symmetries. This is a new step, since although these theories exhibit infinite sets of conservation laws, they are derived using techniques like Bäcklund transformations which restrict the calculations to the space of solutions and obscure the underlying symmetry.

Furthermore, the search for this symmetry in four-dimensional gauge theories by mimicking the inverse scattering techniques has led to singularities in the functional field calculations.[11] These difficulties should be avoided by computing the hidden-symmetry currents as path-dependent Noether currents.[5]

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