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Multimonopole solutions in the Prasad-Sommerfield limit

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A variational search for multimonopole solutions of the Yang-Mills-Higgs equations in the Prasad-Sommerfield limit is performed. An ansatz where two monopoles are superimposed at the origin is shown to lead to a minimal energy differing by less than one percent from the Bogomolny bound, with the discrepancy attributable to the truncation error. Thus strong numerical evidence is obtained for the existence of two-monopole solutions, the symmetry properties of which are discussed.

I. INTRODUCTION

The existence of classical, static multimonopole solutions to the SU(2) Yang-Mills-Higgs equations in the limit of vanishing Higgs potential[1] has been conjectured since the discovery that there are no long-range Coulombic forces between equally charged monopoles.[2] Furthermore, formal proofs have been given that the interaction energy must decrease faster than any power of the distance,[3] that the energy of any field configuration with n units of magnetic charge is bounded below by precisely n times the mass of a unit monopole,[4] and that any solution to the equations for the saturation of the bound (and *a fortiori* also a solution to the Yang-Mills-Higgs equations) would admit a number of infinitesimal deformations compatible with the existence of noninteracting multimonopole field configurations.[5] However, thus far no analytical proof or numerical evidence has been given that these solutions do indeed exist and nothing is known about their actual symmetry properties.

In this article we present a study of multimonopole field configurations done by analytical and numerical methods. We consider the limiting case in which all the monopoles are superimposed in a single location and show the following:

(i) It is possible to write an ansatz for the fields which is regular and analytic everywhere for any integer value of the magnetic charge, which is symmetric under axial rotations and parity reflections, and which satisfies all the appropriate boundary conditions.

(ii) A variational search for a charge-two solution to the equations of motion within this ansatz produces an interaction energy which is less than 1% of the total energy of the system and which can be consistently attributed to the truncation error. Thus, we obtain strong numerical evidence that configurations of two noninteracting monopoles do indeed exist in the limit of vanishing Higgs potential and are able to explore some of their properties.

The paper is organized as follows: In Sec. II we present the axially symmetric ansatz and derive the equations of motion and boundary conditions leading to the n-monopole solution. In Secs. III and IV we illustrate the variational procedure and the numerical results. In Sec. V we present final considerations.

II. THE AXIALLY SYMMETRIC ANSATZ

The SU(2) Yang-Mills-Higgs Lagrangian density in the Prasad-Sommerfield limit is[1]

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \tfrac{1}{2}(D_\mu\Phi)^a(D^\mu\Phi)^a\,, \tag{2.1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc}A^b_\mu A^c_\nu\,, \tag{2.2}$$

$$(D_\mu\Phi)^a = \partial_\mu\Phi^a + e\epsilon^{abc}A^b_\mu\Phi^c \tag{2.3}$$

and the asymptotic boundary condition

$$\Phi^a\Phi^a \to v^2 \tag{2.4}$$

is imposed. (Throughout this paper, summations over repeated indices are implied.)

After rescaling

$$\Phi'^a = \frac{\Phi^a}{v}\,,\quad A'^a_\mu = \frac{A^a_\mu}{v}\,,\quad x'_\mu = evx_\mu\,, \tag{2.5}$$

the dependence of $\mathcal{L}$ on any parameter reduces to an overall factor e^2v^4 and the total energy $\mathcal{E}$ of the system becomes $4\pi v/e$ times a pure number. In particular, for static configurations with no electric fields ($F^a_{i0}=0$)

$$\mathcal{E} = \frac{4\pi v}{e}\int\frac{d^3x}{4\pi}\left[\tfrac{1}{2}B^a_iB^a_i + \tfrac{1}{2}(D_i\Phi)^a(D_i\Phi)^a\right] \equiv \frac{4\pi v\epsilon}{e}\,, \tag{2.6}$$

where

$$B_i^a = \tfrac{1}{2}\epsilon_{ijk} F^a_{jk}, \tag{2.7}$$

and we have dropped the prime from the symbols of the rescaled fields. Equation (2.6) may be rearranged into the form

$$\epsilon = \frac{1}{8\pi}\int d^3x [B_i^a \mp (D_i\Phi)^a]^2 \pm \frac{1}{4\pi}\int d^3x\, B_i^a (D_i\Phi)^a. \tag{2.8}$$

The second term in the right-hand side,

$$\pm\frac{1}{4\pi}\int d^3x\, B_i^a (D_i\Phi)^a = \pm\frac{1}{4\pi}\int d^3x [\partial_i (B_i^a\Phi^a) - \Phi^a (D_i B_i)^a] = \pm\frac{1}{4\pi}\int d^3x\, \partial_i (B_i^a \Phi^a), \tag{2.9}$$

is the integral of a divergence and equals $\pm n$, where n is the total magnetic charge of the system. The resulting inequality

$$\epsilon \geq |n| \tag{2.10}$$

is known as "Bogomolny's bound."[4] Saturation of the bound corresponds to a solution of the first-order equations

$$B_i^a = \pm (D_i\Phi)^a, \tag{2.11}$$

a situation which is well known both in the study of multi-instantons and in the study of multivortex solutions to the Ginzburg-Landau equations.

We wish to investigate the existence of axially symmetric solutions to Eqs. (2.11) with magnetic charge n. For the purpose, we introduce a set of orthonormal vectors

$$\begin{aligned} \vec{u}_1^{(n)} &= (\cos n\phi, \sin n\phi, 0) \equiv (u_1^{a(n)}), \\ \vec{u}_2^{(n)} &= (0, 0, 1) \equiv (u_2^{a(n)}), \\ \vec{u}_3^{(n)} &= (\sin n\phi, -\cos n\phi, 0) \equiv (u_3^{a(n)}), \end{aligned} \tag{2.12}$$

where $\phi = \arctan(y/x)$ is the polar angle. Defining $r = (x^2+y^2)^{1/2}$, one easily finds

$$\begin{aligned} \partial_i u_\alpha^{a(n)} &= \frac{n}{r} u_3^{i(1)} \epsilon_{\alpha 2\gamma} u_\gamma^{a(n)}, \\ \epsilon^{abc} u_\alpha^{b(n)} u_\beta^{c(n)} &= \epsilon_{\alpha\beta\gamma} u_\gamma^{a(n)}. \end{aligned} \tag{2.13}$$

We expand then the fields as follows:

$$\begin{aligned} A_i^a &= u_\alpha^{i(1)} u_\beta^{a(n)} W^\beta_\alpha, \\ \Phi^a &= u_\alpha^{a(n)} \phi^\alpha. \end{aligned} \tag{2.14}$$

This still implies no restriction on the field configuration: the 12 functions A_i^a, Φ^a have been simply reexpressed in terms of the 12 expansion coefficients W^β_α, ϕ^α. But we now demand invariance under rotations around the z axis and parity reflections. One always has to be careful in the definition of symmetry properties in the presence of gauge degrees of freedom. We impose axial rotational invariance requiring explicit symmetry under transformations generated by $J_z + nI_z$, where $\vec{J}$ is the total angular momentum of the system, $\vec{I}$ the generator of global isospin rotations. This, of course, guarantees invariance of all physical quantities under rotations around the z axis.

It is straightforward to verify that the symmetry requirements imply $W_1^1 = W_2^1 = W_1^2 = W_2^2 = W_3^3 = \phi_3 = 0$ and that the six residual expansion coefficients W_i^3 W_3^i, ϕ_i $(i=1,2)$ must depend only on r and z. The boundary condition $\Phi^a\Phi^a \to 1$ as $|\vec{x}| \to \infty$ reduces to $\phi_1^2 + \phi_2^2 \to 1$ as $(r^2+z^2)^{1/2} \to \infty$. If this condition is met and the fields are regular throughout space, the topological winding number of the Higgs field is n, which also equals the total magnetic charge of the system.

It is convenient to define

$$\begin{aligned} &x^1 = r, \quad x^2 = z, \\ &W^3_\alpha = V_\alpha, \quad \alpha = 1,2 \\ &W_3^1 = -\frac{\eta_1}{r}, \\ &W_3^2 = \frac{n-\eta_2}{r} \end{aligned} \tag{2.15}$$

and to adopt the convention that, whenever the functions x^α, ϕ^α, V_α, or η_α are involved, indices run only from 1 to 2. Then one finds

$$\begin{aligned} B_i^a &= \frac{1}{r}\epsilon^{\mu\nu}(\partial_\mu \eta_\alpha - \epsilon_{\alpha\beta} V_\mu \eta_\beta) u_\nu^{i(1)} u_\alpha^{a(n)} + \epsilon^{\mu\nu}(\partial_\mu V_\nu) u_3^{i(1)} u_3^{a(n)}, \\ (D_i\Phi)^a &= (\partial_\mu \phi_\alpha - \epsilon_{\alpha\beta} V_\mu \phi_\beta) u_\mu^{i(1)} u_\alpha^{a(n)} + \frac{1}{r}(\epsilon_{\alpha\beta}\phi_\alpha \eta_\beta) u_3^{i(1)} u_3^{a(n)}, \end{aligned} \tag{2.16}$$

and Bogomolny's equations reduce to[6]

$$\begin{aligned} \frac{1}{r}\epsilon^{\mu\nu}(\partial_\mu \eta_\alpha - \epsilon_{\alpha\beta} V_\mu \eta_\beta) &= (\partial_\nu \phi_\alpha - \epsilon_{\alpha\beta} V_\nu \phi_\beta), \\ \epsilon^{\mu\nu}(\partial_\mu V_\nu) &= \frac{1}{r}\epsilon_{\alpha\beta}\phi_\alpha \eta_\beta. \end{aligned} \tag{2.17}$$

This is a set of five equations for six unknown functions but there is a residual Abelian gauge invariance

$$\begin{aligned} \phi_\alpha \to \phi'_\alpha &= \phi_\alpha \cos\chi + \epsilon_{\alpha\beta}\phi_\beta \sin\chi, \\ \eta_\alpha \to \eta'_\alpha &= \eta_\alpha \cos\chi + \epsilon_{\alpha\beta}\eta_\beta \sin\chi, \\ V_\mu \to V'_\mu &= V_\mu + \partial_\mu \chi. \end{aligned} \tag{2.18}$$

It is possible to express Eqs. (2.17) in gauge-invariant form. Defining

$$\begin{aligned} \phi_\alpha &= h\hat{\phi}_\alpha, \\ \eta_\alpha &= \psi\hat{\phi}_\alpha + E\epsilon_{\alpha\beta}\hat{\phi}_\beta, \\ V_\mu &= A_\mu - \epsilon_{\alpha\beta}\hat{\phi}_\alpha \partial_\mu \hat{\phi}_\beta, \end{aligned} \tag{2.19}$$

we obtain the gauge-invariant set of equations

$$\frac{1}{r}\epsilon_{\mu\nu}(\partial_\mu\psi+A_\mu E)=\partial_\nu h,$$
$$\frac{1}{r}\epsilon_{\mu\nu}(\partial_\mu E-A_\mu\psi)=-A_\nu h, \tag{2.20}$$
$$\epsilon_{\mu\nu}\partial_\mu A_\nu=-\frac{1}{r}hE.$$

h is the norm of the Higgs field and asympotically obeys a massless scalar particle equation; the potential ψ enjoys an analogous property, while the scalar field E and the vector field A_ν are exponentially damped. Thus the asymptotic equations of motion (i.e., the equations satisfied outside the "core" region where the monopoles are located) are simply

$$\frac{1}{r}\epsilon_{\mu\nu}\partial_\mu\psi=\partial_\nu h. \tag{2.21}$$

The most general asymptotic behavior with proper boundary conditions at infinity is then found to be

$$h(\rho,t)=1-\frac{n}{\rho}+\sum_{l=1}^{\infty}\frac{a_l P_l(t)}{\rho^{l+1}},$$
$$\psi(\rho,t)=-nt+\sum_{l=1}^{\infty}\frac{a_l}{l}\,\frac{(t^2-1)}{l}\,\frac{dP_l(t)}{dt}, \tag{2.22}$$

where $\rho=(r^2+z^2)^{1/2}$ is the three-dimensional radius, $t=\cos\vartheta=z/\rho$, and $P_l(t)$ are the Legendre polynomials. Unfortunately, this gauge-invariant formulation [equivalent, by the way, to a unitary gauge choice $\phi_\alpha=(0,1)$] makes use of fields [such as $h=(\phi_1{}^2+\phi_2{}^2)^{1/2}$] which do not have good analyticity properties at the origin or along the z axis and we are forced back to Eqs. (2.20) to discuss the regularity conditions to be imposed on the fields.

The advantage of the reparametrization provided by Eqs. (2.14) and (2.15) is that the new fields have simple transformation properties under the residual Abelain gauge symmetry; singular factors, however, had to be extracted. The requirement that the original fields should be regular imposes then boundary conditions on the functions $\phi_\alpha, \eta_\alpha, V_\mu$. Additional constraints come from the symmetry of the equations under parity reflections $z\to -z$ and $r\to -r$. Assuming that the actual solutions are also symmetric under these transformations, one concludes that the fields should transform as follows: V_2, ϕ_1, η_2 should be even, V_1, ϕ_2, η_1 odd under $z\to -z$. As to parity under $r\to -r$, if the total magnetic charge is odd, V_1, ϕ_2, η_2 should be even, V_2, ϕ_1, η_1 should be odd, whereas V_2, ϕ_1, ϕ_2, η should be even, V_1 odd if the magnetic charge is even.

These various constraints are summarized by the following expansions:

$$\begin{aligned}
&\phi_1=r^n F_1(r^2,z^2),\\
&\phi_2=zF_2(r^2,z^2),\\
&\eta_1=zr^n\beta(z^2)+zr^{n+2}E_1(r^2,z^2),\\
&\eta_2=n+r^2E_2(r^2,z^2),\\
&V_1=zr^{n-1}\beta(z^2)+zr^{n+1}v_1(r^2,z^2),\\
&V_2=r^n v_2(r^2,z^2),
\end{aligned} \tag{2.23}$$

where n is the total magnetic charge and F_1, F_2, β, E_1, E_2, v_1, v_2 are regular functions of their arguments. The equations are completely consistent with the behavior expressed by Eqs. (2.23) and perturbative solutions may be constructed order by order in r^2 and z^2. The perturbative solutions must be supplemented with a gauge condition. For instance, the Coulomb-gauge constraint $\partial_i A_i^a=0$ takes the form

$$\frac{1}{r}\partial_\mu(rV^\mu)=\frac{n}{r^2}\eta_1 \tag{2.24}$$

and turns out to be compatible with the regularity conditions.

An in-depth analysis of the equations in the Coulomb gauge shows that the perturbative solution still depends on an arbitrary choice for the values of three functions on the z axis, i.e., $\beta(z^2)$ may be arbitrarily chosen and only the following constraints hold:

$$\begin{aligned}
&n[v_2(0,z^2)-F_1(0,z^2)]+z^2\beta(z^2)F_2(0,z^2)=\partial_z z\beta(z^2),\\
&2E_2(0,z^2)=\partial_z zF_2(0,z^2).
\end{aligned} \tag{2.25}$$

The possibility of matching the perturbative solution with the boundary conditions at infinity seems to be related to this freedom.[7]

Finally, it should be noticed that Eqs. (2.23) imply a zero of order n at the origin for the three components of the Higgs field Φ^a. This zero follows from the assumed symmetry, angular behavior, and regularity of the field configuration. Thus, the ansatz describes a situation where n magnetic monopoles are superimposed at the origin. Additional zeros along the z axis cannot be excluded, but they would correspond to pairs of magnetic monopoles and antimonopoles and are unlikely to be found in a solution which minimizes the energy. The multiple zero at the origin, however, cannot be resolved into more zeros along the z axis without introducing root type of singularities. The symmetry properties we have assumed, in particular the axial rotational symmetry, allow only for configurations of superimposed monopoles.[8] The zero may be factored, but only in a direction different from the z axis and therefore at the expense of axial invariance.

III. THE VARIATIONAL COMPUTATION

The basic idea of a variational computation is straightforward. Equations (2.14) and (2.15) provide an ansatz for the fields. The ensuing expressions for B_i^a and $(D_i\Phi)^a$ are inserted into the formula defining the energy, Eq. (2.6) or Eq. (2.8), and one looks for the minimal value that the energy takes as the arbitrary functions in Eq. (2.14) are varied. In particular, if this minimum equals the bound $|n|$ then Eqs. (2.11) are solved. In this and the following section we shall restrict our attention to systems with total magnetic charge $n=2$.

In practice, the computation must proceed through a numerical analysis and one encounters the obvious limitations arising from the fact that only a finite-dimensional subset of the space of all available functions may be explored. Thus we face the problem of further restricting the ansatz by an expansion of the functions ϕ_α, η_α, and V_μ, which will make them dependent on a finite number of variational parameters. In the formulation of this expansion we shall be guided by the requirements that the large-ρ asymptotic behavior expressed by Eqs. (2.19) and (2.22) and the proper boundary conditions, Eqs. (2.23), be maintained. We must also fix a gauge

A radial gauge where

$$\begin{aligned}\phi_1 &= \frac{r}{\rho}h(r,z)\,,\\ \phi_2 &= \frac{z}{\rho}h(r,z)\end{aligned} \tag{3.1}$$

leads to a very simple asymptotic configuration of fields [we recall the definitions $\rho=(r^2+z^2)^{1/2}=|\vec{x}|$, $t=z/\rho=\cos\vartheta$]. However, such a gauge is not compatible with a regular behavior of the fields along the z axis. We have therefore modified the gauge prescription, defining

$$\begin{aligned}\phi_1 &= \frac{rR(r,z)}{\rho}h(r,z)\,,\\ \phi_2 &= \frac{zh(r,z)}{\rho}\,,\end{aligned} \tag{3.2}$$

where the function R is given by

$$R=\frac{r}{(r^2+\lambda^2/\cosh 2\rho)^{1/2}}\,. \tag{3.3}$$

R approaches unity exponentially as ρ becomes large, but vanishes like r as one approaches the z axis. It has a regular expansion for small r and z, is odd in r and even in z. λ is a scale parameter which may also be varied; in the limit $\lambda=0$, R becomes identically one. Equation (3.2) guarantees that the chosen gauge approaches the radial gauge exponentially as one leaves the core of the monopole, but at the same time preserves the small-r behavior $\phi_1\approx r^2$. The function R is also used to ensure a proper boundary behavior of the other fields, which are defined as follows:

$$\begin{aligned}\eta_1 &= zrRH_1(r,t)\,,\\ \eta_2 &= 2-r^2H_2(r,t)\,,\\ V_1 &= zR(2-R^2)\mathcal{V}_1(r,t)\,,\\ V_2 &= rR\mathcal{V}_2(r,t)\,.\end{aligned} \tag{3.4}$$

The new functions h, H_1, H_2, $\mathcal{V}_1$, and $\mathcal{V}_2$ are now expressed as a sum of asymptotic terms, which reproduce the multipole expansion of Eqs. (2.19) and (2.22), plus core corrections, which decay off exponentially as $e^{-\rho}$:

$$\begin{aligned}\frac{h}{\rho} &= F_{\mathrm{as}}(r^2,z^2)+F_{\mathrm{core}}(r^2,z^2)\,,\\ H_i &= H_{i,\mathrm{as}}(r^2,z^2)+H_{i,\mathrm{core}}(r^2,z^2)\,,\\ \mathcal{V}_i &= \mathcal{V}_{i,\mathrm{as}}(r^2,z^2)+\mathcal{V}_{i,\mathrm{core}}(r^2,z^2)\,,\quad i=1,2.\end{aligned} \tag{3.5}$$

We introduce auxiliary functions

$$\begin{aligned}g_n(\rho) &= \frac{\rho^{2n}}{(2n)!}(\cosh\rho)^{-1}\,,\\ f_n(\rho) &= 1-\sum_{i=0}^{n}g_n(\rho)\,.\end{aligned} \tag{3.6}$$

The functions $g_n(\rho)$ are used in the expansion of the core part of the fields; the inverse of the hyperbolic cosine is used as a cutoff rather than an exponential to preserve the correct parity properties. The functions $f_n(\rho)$, which approach 1 exponentially for large ρ but behave as ρ^{2n+2} near the origin, are used to modify the inverse powers of ρ appearing in the expansion of the asymptotic part, so that this conforms to the regular behavior expected near the origin. We then expand

$$\begin{aligned}F_{\mathrm{as}} &= -\frac{\tanh\rho}{\rho}+\frac{2f_0(\rho)}{\rho^2}-\sum_{l=2}^{2N}a_l\frac{f_l(\rho)}{\rho^{l+2}}P_l(t)\,,\\ H_{1,\mathrm{as}} &= \frac{2f_0(\rho)}{\rho^2}-\sum_{l=2}^{2N}a_l\frac{f_l(\rho)}{lt\rho^{l+2}}\frac{dP_l(t)}{dt}\,,\\ H_{2,\mathrm{as}} &= \frac{2f_0(\rho)}{\rho^2}-\sum_{l=2}^{2N}a_l\frac{f_l(\rho)t}{l\rho^{l+2}}\frac{dP_l(t)}{dt}\,,\\ \mathcal{V}_{1,\mathrm{as}} &= \frac{f_0(\rho)}{\rho^2}\,,\\ \mathcal{V}_{2,\mathrm{as}} &= -\frac{f_0(\rho)}{\rho^2}\,,\end{aligned} \tag{3.7}$$

and

$$
\begin{aligned}
F_{\text{core}} &= \sum_{n=0}^{N} \sum_{l=2n}^{2N} c_{1,n,l}\, g_n(\rho) P_l(t)\,, \\
H_{1,\text{core}} &= \sum_{n=0}^{N} \sum_{l=2n}^{2N} c_{2,n,l} \frac{g_n(\rho)(t^2-1)}{lt} \frac{dP_l(t)}{dt}\,, \\
H_{2,\text{core}} &= \sum_{n=0}^{N} \sum_{l=2n}^{2N} c_{3,n,l} \frac{g_n(\rho)t}{l} \frac{dP_l(t)}{dt}\,, \\
\mathcal{V}_{1,\text{core}} &= \sum_{n=0}^{N} \sum_{l=2n}^{2N} c_{4,n,l} \frac{g_n(\rho)(t^2-1)}{lt} \frac{dP_l(t)}{dt}\,, \\
\mathcal{V}_{2,\text{core}} &= \sum_{n=0}^{N} \sum_{l=2n}^{2N} c_{5,n,l} \frac{g_n(\rho)t}{l} \frac{dP_l(t)}{dt}\,.
\end{aligned}
\tag{3.8}
$$

l ranges only through even values in all sums. The variables a_l and $c_{i,n,l}$ are variational parameters, but Eqs. (2.23) demand $c_{4,n,0}=\frac{1}{2}c_{1,n,0}$. λ, in the definition of R[Eq. (3.3)], must also be considered a variational parameter. The integer N determines the cutoff in the expansions. For given N, the number of variational parameters is $\frac{1}{2}[5(N+1)(N+2)]$.

Given the expansions, it is only a matter of straightforward (although tedious) algebra to find the integrand to be inserted into the formula for the energy. We have used Eq. (2.8) to evaluate ϵ, rather than the original expression Eq. (2.6). The second term in the left-hand side of Eq. (2.8) equals 2 (for $n=2$); the first integral was computed numerically by Gaussian quadrature. A 9-point interpolation formula was used to cover the range of $\cos\vartheta$ (between 0 and 1); this guarantees the exact integration of polynomials in $\cos\vartheta$ of degree ≤ 17; the function R, however, does introduce a nonpolynomial dependence. The subsequent integral over ρ was evaluated by dividing the range $0\leq\rho\leq 10$ into six parts and using a 10-point interpolation formula in each. The advantage of Eq. (2.8) over Eq. (2.6) is that the integrand in the first term of the right-hand side becomes exponentially damped when the fields have the correct asymptotic expansion, and integration over a finite range is sufficient for great numerical precision. Occasionally we have compared results obtained by numerical integrations of Eqs. (2.6) and (2.8) to check the accuracy. This and other tests have convinced us that the error in the integration never exceeds a few parts times 10^{-4}.

The search for a minimum of ϵ was performed by a standard minimization routine, which follows a method introduced by Fletcher and Powell.[9] The basic idea is to evaluate the gradient of the energy in the space of the variational parameters and to move in a direction opposite to it. Metric corrections, which aim at taking into account the curvature of the energy surface, are, however, performed to improve convergence. In all our actual numerical computations the variational procedure appeared to converge reasonably fast, without instabilities. The results are illustrated in Sec. IV.

IV. NUMERICAL RESULTS

We have performed variational searches for a minimum energy with $N=1$, 2, and 3 (and therefore 15, 30, 50 independent variational parameters, respectively) and have found the following results:

$$
\begin{aligned}
N&=1, \quad \epsilon_{\min}=2.0353\,, \\
N&=2, \quad \epsilon_{\min}=2.0253\,, \\
N&=3, \quad \epsilon_{\min}=2.0182\,.
\end{aligned}
\tag{4.1}
$$

These numbers give strong evidence that the absolute minimum of ϵ is 2, i.e., that Bogomolny's bound may be saturated. Of course, proof of this statement is beyond the reach of a numerical computation, but the minimal values are indeed very close to 2 and, what is more relevant, they decrease as N increases in a way compatible with the difference $\epsilon_{\min}-2$ being entirely due to the truncation error.

The values of the variational parameters at the minimum for $N=3$ are reproduced in Table I.

We have performed checks to assess the importance of various elements in the computation. A minimization with $N=3$, but only the terms with $l=0$ in the expansion, produced $\epsilon_{\min}=2.1078$, which indicates the relevance of the angular dependence. A computation without the constraint $c_{4,0,l}=\frac{1}{2}c_{1,0,l}$ gave $\epsilon_{\min}=2.0247$ (versus $\epsilon_{\min}=2.0253$) for $N=2$. This shows that the boundary behavior $\eta_1\approx rV_1$ as $r\to 0$ [see Eq. (2.33)], which one expects of the exact solution, is closely satisfied also by the minimum in the truncated space of functions.

Another interesting test consists of performing a minimization with $\lambda=0$ (i.e., equating to 1 the

TABLE I. Values of the variational parameters at the minimum for $N=3$ with $\lambda=0.3500$, $\epsilon_{\min}=2.0182$, $a_2=-0.0004$, $a_4=0.0017$, $a_6=0$.

n	l	$c_{1,n,l}$	$c_{2,n,l}$	$c_{3,n,l}$	$c_{4,n,l}$	$c_{5,n,l}$
0	0	−0.1217	−0.8932	−0.0489	−0.0609	0.9068
1	0	0.0522	−0.0601	0.0248	0.0261	0.0922
	2	0.0640	0.1266	−0.0695	0.2289	−0.2019
2	0	−0.0299	0.0184	−0.0071	−0.0149	−0.0331
	2	−0.0281	−0.0602	0.0214	−0.0811	0.0619
	4	0.0011	0.0001	0.0032	0.0002	0.0061
3	0	0.0091	−0.0041	0.0028	0.0045	0.0076
	2	0.0055	0.0129	−0.0088	0.0187	−0.0116
	4	0.0008	0.0003	−0.0015	0.0009	−0.0027
	6	0.0005	0.0001	−0.0001	0.0007	0.0002

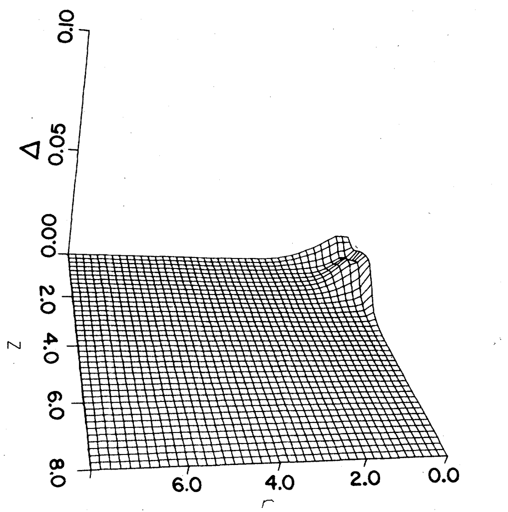

FIG. 1. The surface representing $\Delta = [B_i^a - (D_i\Phi)^a]^2$ as function of r and z, for the configuration of minimal energy and $N=3$.

function R, which we have introduced to enforce the correct boundary behavior of the fields). Although this limit produces singularities in the field configuration, the energy remains finite as long as $c_{4,0,l} = \frac{1}{2} c_{1,0,l}$. With $\lambda = 0$ and $N = 3$ we have found $\epsilon_{\min} = 2.0188$. Thus, the advantage of enforcing the correct expansion for $r \to 0$ is only marginal, or, more accurately, we should say that we have not been able to reproduce the correct boundary behavior, within the ansatz, in an appropriate way. We shall return to this point later.

More insight into the properties of the configurations of minimal energy can be obtained from a graphic display of various quantities as functions of r and z. Figure 1 illustrates the surface $\Delta = \Delta(r,z)$, where

$$\Delta = [B_i^a - (D_i\Phi)^a]^2 \tag{4.2}$$

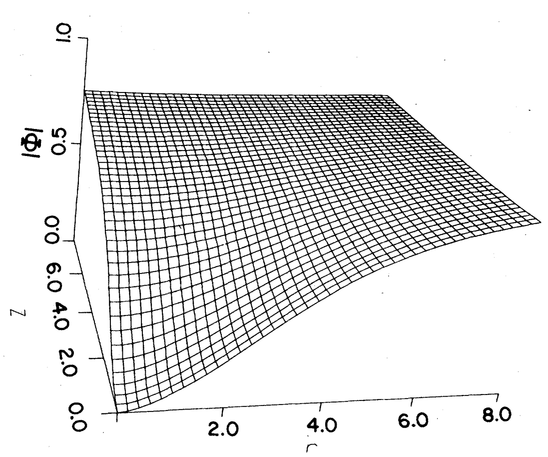

FIG. 2. The modulus of the Higgs field in the minimal configuration.

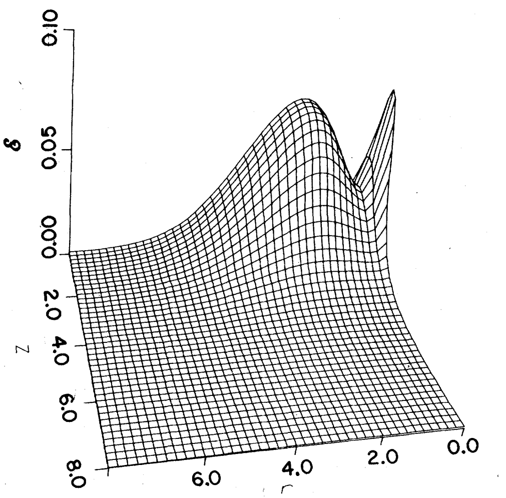

FIG. 3. The energy distribution in the minimal configuration.

is the integrand of the first term of the right-hand side of Eq. (2.8). (All figures refer to the minimal configurations with $N=3$.) Δ should be zero for an exact solution and we see that, in our approximate solution, Δ is very close to zero everywhere, apart from a small region near the z axis.

Figure 2 displays the modulus of the Higgs field $|\Phi| = \sqrt{\Phi^a\Phi^a}$, and Fig. 3 the energy density $\mathcal{E} = B_i^a B_i^a + (D_i\Phi)^a (D_i\Phi)^a$. The spike in this last graph is most likely artificial (i.e., would not be true of an exact solution) and probably depends on the inadequacy of our ansatz, insofar as the behavior of the fields for $r \to 0$ is concerned.

It is interesting to examine the graph of Δ in the configuration of minimal energy obtained with $\lambda = 0$. This is displayed in Fig. 4. The surfaces in Figs. 1 and 4 look rather similar, showing that only a small advantage has been obtained by im-

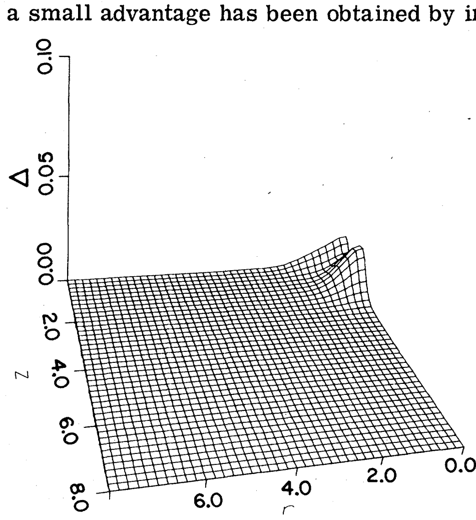

FIG. 4. The surface $\Delta = \Delta(r,z)$ for the configuration of minimal energy and $\lambda = 0$.

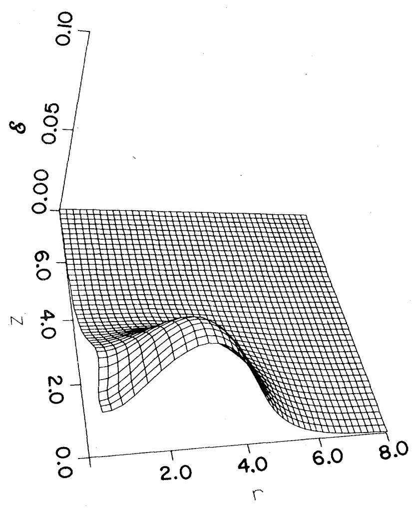

FIG. 5. The energy distribution of the minimal configuration with $\lambda=0$.

posing the correct boundary conditions through the function R. In formulating the ansatz, we have tried to interpolate between the small-r and the large-ρ asymptotic behaviors of the fields, both of which could be determined analytically. The ansatz being complete, any function with these behaviors may in principle be reproduced. But the possibility of approximating an assumed solution with few terms in the expansion depends on the specific interpolation chosen. Apparently, our ansatz becomes unsatisfactory, from this point of view, near the z axis. We believe that developing a better interpolating formula could provide an important clue for the determination of an analytic solution.

The field configuration obtained with $\lambda=0$, even if it produces a slightly higher total energy, probably gives a better representation of what the energy distribution of a true solution would be. This is displayed, seen under a different perspective, in Fig. 5. One notices that the energy distribution of the two-monopole configuration peaks away from the z axis and that it departs markedly from spherical symmetry.[10]

V. CONCLUSIONS

While we think that our arguments should give anyone a reasonable confidence on the existence of multimonopole solutions, a formal proof would obviously be welcome. One of our hopes is that our results may stimulate and encourage such a search, in analogy with the case of the multivortex problem in the Ginzburg-Landau model, where the numerical indications of existence[11] were ultimately superseded by the formal proof given by Taubes.[12,13] We must add that, as in the vortex problem, the system of equations we have studied may be replaced by a single superpotential equation, which has not proved to be very useful for the numerical analysis, but is perhaps appropriate for an analytical discussion.

The potential formulation is derived from the gauge-invariant equations, once, after elimination of A_μ, these are rewritten in the form

$$\begin{aligned}&\partial_\mu(E^2+\psi^2)-r^2\partial_\mu h^2-r\epsilon_{\mu\nu}\partial_\nu(h\psi)=0\,,\\&\frac{1}{r}hE+\epsilon_{\mu\nu}\partial_\mu\left(\frac{r\epsilon_{\nu\rho}\partial_\rho h-\partial_\nu\psi}{E}\right)=0\,.\end{aligned}\tag{5.1}$$

Defining

$$\begin{aligned}&A=h^2=\phi^\alpha\phi^\alpha\,,\\&B=E^2+\psi^2=\eta^\alpha\eta^\alpha\,,\\&C=h\psi=\eta^\alpha\phi^\alpha\,,\\&D=hE=\epsilon^{\alpha\beta}\phi^\alpha\eta^\beta\,,\end{aligned}\tag{5.2}$$

one obtains

$$\begin{aligned}&\frac{r}{2}\partial_\mu A-\frac{1}{2r}\partial_\mu B+\epsilon_{\mu\nu}\partial_\nu C=0\,,\\&\frac{1}{r}D-\frac{1}{2}\frac{\partial_\nu B}{B}\epsilon_{\mu\nu}\partial_\mu\frac{C}{D}-\partial_\mu\left(\frac{r}{2}\frac{\partial_\mu A}{D}\right)=0\,,\\&C^2+D^2=AB\,.\end{aligned}\tag{5.3}$$

This system is already interesting because the functions A, B, C, D are gauge invariant and regular everywhere. But one may proceed further solving the first equation by means of the superpotential $\tau(x_1,x_2)$ (Ref. 6):

$$\begin{aligned}&A=1-\frac{1}{r}\partial_1(r\partial_1\tau)-\partial_2\partial_2\tau\,,\\&B=n^2-r^2(\partial_1\partial_1\tau+\partial_2\partial_2\tau)+r\partial_1\tau\,,\\&C=-\partial_2\tau\end{aligned}\tag{5.4}$$

and rewriting the second equation in the form

$$\begin{aligned}(D^2)^2-\frac{r}{2}\partial_2B(A\partial_1C-\tfrac{1}{2}C\partial_1A)+\frac{r}{2}\partial_1B(A\partial_2C-\tfrac{1}{2}C\partial_2A)&\\-\frac{D^2}{2}r\partial_1(r\partial_1A)-\frac{D^2}{2}r^2\partial_2\partial_2A&\\+\frac{r^2}{A}\partial_1A\partial_1D^2+\frac{r^2}{A}\partial_2A\partial_2D^2=0&\end{aligned}\tag{5.5}$$

Only $D^2=AB-C^2$ appears in this last equation. Substituting Eq. (5.4) into it, all of the original equations are reduced to a single, but unfortunately highly nonlinear, partial differential equation for the superpotential τ.

A final consideration is about the symmetry properties of multimonopole solutions. It is interesting to observe that even after the complete superposition of the two monopoles, a preferential direction is still present and, correspondingly, the number of degrees of freedom is five, two of which are rotational. This leads us to reconsider Weinberg's result[5] on the number of parameters of an n-monopole system, which is seven for $n=2$. Thus, the two superimposed monopoles must admit two nontranslational and nonrotational zero-mode fluctuations. We argue that these zero-energy modes should correspond to non-axially-symmetric deformations and that no axially symmetric solution of spatially separated monopoles exist (see also Ref. 7).

Indeed, let us remember that the locations of the monopoles are associated with the zeros of the Higgs field.[14] In our case this field has a double zero at the origin, where it behaves as

$$\begin{aligned}
&\Phi^1 \sim x^2 - y^2 ,\\
&\Phi^2 \sim 2xy ,\\
&\Phi^3 \sim z .
\end{aligned} \tag{5.6}$$

Now, while it is possible through the replacement $\Phi^1 \sim x^2 - y^2 \mp a^2$ to get two single zeros on the x or y axis, and therefore to separate the monopoles in the plane orthogonal to the z axis, there is no way to split the zero along the z axis and thus, apparently, it is impossible to preserve the symmetry once the monopoles are taken apart.

A formal argument may also be given by considering the problem of two separated monopoles with axial symmetry in terms of the previously defined gauge-invariant functions. The quantization condition for the Dirac string implies that the function ψ may assume only the values ± 1 on the z axis when the monopoles are separated and must change its sign at each monopole location. But if we insist on a physically motivated up-down symmetry we find that the gauge-invariant function $h\psi = \eta_\alpha \phi^\alpha$ must be discontinuous when $z=0$, since h is chosen to be always positive and its asymptotic behavior determines the asymptotic behavior of ψ:

$$\psi \to -\cos\theta_1 - \cos\theta_2 + \text{const}. \tag{5.7}$$

The constant in Eq. (5.7) is forced to change through the $z=0$ plane:

$$\psi = \begin{cases} -\cos\theta_1 - \cos\theta_2 - 1 , & z<0 \\ -\cos\theta_1 - \cos\theta_2 + 1 , & z<0 . \end{cases} \tag{5.8}$$

Although a jump in ψ is *per se* irrelevant (only ψ^2 is a gauge-invariant, regular function), this discontinuity reflects on the gauge-invariant quantity $h\psi$ and it is not acceptable. Thus any search for spatially separated monopoles must face the full system of Bogomolny's equations, with none of the simplifications arising from the existence of a continuous symmetry.

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[1]G. 't Hooft, Nucl. Phys. B79, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. 20, 430 (1974). [JETP Lett. 20, 194 (1974)]; M. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[2]N. S. Manton, Nucl. Phys. B126, 525 (1977).

[3]W. Nahm, CERN report, 1979 (unpublished).

[4]E. B. Bogomol'nyi, Yad. Fiz. 24, 861 (1976). [Sov. J. Nucl. Phys. 24, 449 (1976)]; S. Coleman, S. Parke, A. Neveu, and C. M. Sommerfield, Phys. Rev. D 15, 544 (1977).

[5]E. Weinberg, Phys. Rev. D 20, 936 (1979).

[6]N. S. Manton, Nucl. Phys. B135, 319 (1978); B. Houston and L. O'Raifeartaigh, Dublin Institute for Advanced Study report, 1979 (unpublished).

[7]S. Adler, private communication.

[8]B. Houston and L. O'Raifeartaigh, Dublin Institute for Advanced Study report, 1980 (unpublished).

[9]R. Fletcher and M. J. D. Powell, Comput. J. 6, 163 (1963).

[10]E. J. Weinberg and A. H. Guth, Phys. Rev. D 14, 1660 (1976).

[11]L. Jacobs and C. Rebbi, Phys. Rev. B 19, 4486 (1979).

[12]C. N. Taubes, Harvard report, 1979 (unpublished).

[13]An analytic proof of existence for multipoles at large separation has been recently given by C. N. Taubes, Harvard report, 1980 (unpublished).

[14]S. Adler, Phys. Rev. D 20, 1386 (1979).