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Origin of Higgs fields

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We show that Higgs fields can be interpreted as the extra pieces of gauge fields arising from viewing them as connections in a fiber bundle. This geometrical interpretation restricts the number of independent Higgs components.

Ever since the Weinberg-Salam model was introduced there has been a mystery about the origin of the Higgs fields.[1-3] We shall show that if one adopts the view that conventional gauge fields are pieces of connections in a fiber bundle,[4,5] the Higgs fields have a natural interpretation. Since a fiber bundle is a higher-dimensional manifold, gauge fields on these manifolds have extra components. Those extra components can be interpreted as Higgs fields. With this interpretation the number of independent components for a Higgs field is restricted to be the number of gauge group generators, N, plus one. The extra component goes away in chiral[6] models.

We shall describe the situation for a simple example. The gauge group is SU(2). In this case one imagines that the physical world is essentially a seven-dimensional manifold, P, which we shall call the bundle[4,5] [in fact $P \approx \mathrm{R}^4 \times \mathrm{SU}(2)$]. However, this seven-dimensional manifold has an action by the group SU(2) (called right multiplication, see Appendix). This action sends points around orbits generated by SU(2), which are isomorphic to the group manifold for SU(2). These orbit spaces are called the fibers of the fiber bundle. See Fig. 1. If one takes the equivalence classes formed by identifying all points in a single orbit, one finds the four-dimensional manifold R^4. Thus we write $P/\mathrm{SU}(2) \approx \mathrm{R}^4$. The map sending points $p \in P$ into their equivalence class representative $x \in \mathrm{R}^4$ is called the canonical projection π, $\pi: P \to \mathrm{R}^4$.[4,5]

If one were to write down a gauge field in this seven-dimensional manifold, P, one would naturally give it seven times three components, naively, that is, in terms of a global coordinate patch for P with x^μ the R^4 coordinates and φ^k the SU(2) coordinates. We use $\Omega = \exp(\varphi^k \Lambda_k)$, and we will be as cavalier as usual about coordinate singularities. We can write

$$B^k \Lambda_k = B^k_\mu \Lambda_k dx^\mu + B^k_a \Lambda_k d\varphi^a \tag{1}$$

with 4×3 components in the first term and 3×3 components in the second and

$$\Lambda_k = \tfrac{1}{2} i \sigma_k \,. \tag{2}$$

One can show that because of the orbit action for SU(2) a considerable simplification must occur. Indeed, as we explain in the Appendix,

$$B^k(x,\varphi)\Lambda_k = \Omega^{-1}(\varphi) A_\mu(x) dx^\mu \,\Omega(\varphi) + \Omega^{-1}(\varphi) d\Omega(\varphi) \tag{3}$$

with

$$d\Omega(\varphi) = \partial_a \Omega(\varphi) d\varphi^a \,. \tag{4}$$

Here A_μ *depends* only on x. We can write

$$\Omega(\varphi) = e^{(i/2)(\varphi^k \sigma_\mu)} \,. \tag{5}$$

A section of P is a choice of slicing R^4 through P, so that one point in each orbit is picked (smoothly varying by neighbors). See Fig. 1. It amounts to setting φ^a equal to a (smooth) function of x. We shall write the fields on the section,

$$\Omega(x) = e^{(i/2)(\varphi^k(x)\sigma_k)} \,, \tag{6}$$

$$B^k_\mu(x)\Lambda_k dx^\mu = \Omega^{-1}(x) A_\mu(x) dx^\mu \,\Omega(x) + \Omega^{-1}(x) d\Omega(x) \,, \tag{7}$$

and now $d\Omega(x) = \partial_\mu \Omega(x) dx^\mu$.

The curvature of the bundle is essentially the Yang-Mills field tensor

$$\begin{aligned} G &= dB + B\wedge B \\ &= \Omega^{-1}(dA + A\wedge A)\Omega \\ &\equiv \Omega^{-1} F \Omega \,. \end{aligned} \tag{8}$$

In components

$$\begin{aligned} G^k_{\mu\nu}\Lambda_k &= \Omega^{-1}\Lambda_i \Omega F^i_{\mu\nu} \,, \\ F^i_{\mu\nu} &= \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \epsilon^i_{jk} A^j_\mu A^k_\nu \,. \end{aligned} \tag{9}$$

We take g, the coupling constant, equal to one. If not, divide F^2 by g^2. The kinetic part of the action, $\mathfrak{A}_1$, is given as the Hilbert square of the curvature

$$\begin{aligned} \mathfrak{A}_1 &= \int \mathrm{tr}(*G^\dagger \wedge G) \\ &= -\int \mathrm{tr}(G^k_{\mu\nu}\Lambda^\dagger_k G^l_{\alpha\beta}\Lambda_l) g^{\mu\alpha} g^{\nu\beta} d^4x \\ &= -\tfrac{1}{4}\int F^k_{\mu\nu} F^l_{\alpha\beta}\delta_{kl} g^{\mu\alpha} g^{\nu\beta} d^4x \,. \end{aligned} \tag{10}$$

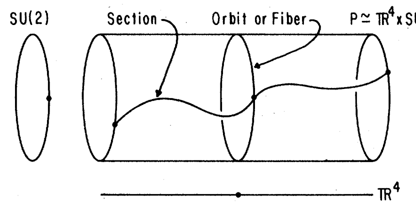

FIG. 1. Portrait of a fiber bundle as a Young cylinder.

The definition of the Hodge dual (generalized Maxwell dual), $*$, is found in Ref. 4.

We add to $\mathcal{a}_1$ an action $\mathcal{a}_2$ which gives a kinetic term for the choice of section, $\Omega(x)$. The section field $\Omega(x)$ will be the Higgs field,

$$\mathcal{a}_2 = \int m^2 \tfrac{1}{2} \operatorname{tr}(*B^\dagger \wedge B) = \int m^2 \tfrac{1}{2} \operatorname{tr}(B^\dagger_\mu B_\nu) g^{\mu\nu} d^4x \,. \tag{11}$$

m must have dimensions of mass, but is otherwise undetermined.

Introduce

$$\phi(x) = m\Omega(x) u \tag{12}$$

with

$$u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$\tfrac{1}{2} m^2 \operatorname{tr}(B^\dagger_\mu B_\nu) g^{\mu\nu} = (\partial_\mu \phi + A_\mu \phi)^\dagger (\partial_\nu \phi + A_\nu \phi) g^{\mu\nu}. \tag{13}$$

The proof is a calculation using

$$\tfrac{1}{2} \operatorname{tr}(B^\dagger \cdot B) = u^\dagger B^\dagger \cdot B u$$

or

$$\tfrac{1}{2} \operatorname{tr}(\Lambda^\dagger_i \Lambda_j) = \tfrac{1}{4} \delta_{ij} = u^\dagger \Lambda^\dagger_i \Lambda_j u \,. \tag{14}$$

The fields ϕ are "chiral" since they satisfy[6]

$$\phi^\dagger \phi = m^2 u^\dagger \Omega^\dagger \Omega u = m^2 \,. \tag{15}$$

Thus we can write the functional W as

$$W = \int [dB] \exp\left[i \int \operatorname{tr}(*G^\dagger \wedge G + \tfrac{1}{2} m^2 * B^\dagger \wedge B) \right] , \tag{16}$$

where

$$[dB] = [dA][\Omega^{-1} d\Omega][\Omega^{-1} d\Omega][\Omega^{-1} d\Omega] \,. \tag{17}$$

Use the relationship between Ω and ϕ to write

$$W = \int [d\phi^\dagger][d\phi][\delta(\bar{\phi}\phi - m^2)] \times \exp\left[i \int (-\tfrac{1}{4} F^k_{\mu\nu} F^{\mu\nu}_k + |\partial\phi + A\phi|^2) d^4x \right], \tag{18}$$

$$F^{\mu\nu}_k \equiv F^l_{\alpha\beta} \delta_{lk} g^{\alpha\mu} g^{\beta\nu} \,.$$

There is a constant Jacobian in the transformation ($2m^{-2}$) which we omit. $[\delta(\phi^\dagger(x)\phi(x) - m^2)]$ denotes the functional δ function. There are two standard ways of rewriting this constraint:

(1) $$[\delta(\phi^\dagger \phi - m^2)] = \lim_{\lambda \to \infty} \exp\left[\frac{-i}{2} \lambda \int d^4x (\phi^\dagger \phi - m^2)^2 \right], \tag{19}$$

(2) $$\int [da] \exp\left(\frac{-i\lambda}{2} \int a^2(x) d^4x \right) [\delta((\phi^\dagger \phi - m^2) - a(x))] = \exp\left[\frac{-i\lambda}{2} \int (\phi^\dagger \phi - m^2)^2 d^4x \right]. \tag{20}$$

The first way is analogous to finding the integral form for the gauge constraint $[\delta(\partial \cdot A)]$. The second way amounts to broadening the gauge constraint to $[\delta(\partial \cdot A - a)]$; then integrating the functional over the class of a's. In the second case one need not take $\lambda \to \infty$; instead it is an arbitrary constant. This second procedure owes its relevance to the fact that the coefficient m^2 appearing in $\mathcal{a}_2$ [Eq. (11)] need not have been a constant; instead we could let it be an arbitrary function of position $\mu^2(x)$. Case 2 above amounts to letting $\mu^2(x) = m^2 + a(x)$ and then integrating over all $a(x)$.[7]

Thus our final functional is

$$W = \int [dA][d\phi^\dagger][d\phi] \exp\left\{ i \int d^4x \left[-\tfrac{1}{4} F^k_{\mu\alpha} F^{\mu\nu}_k + |\partial\phi + A\phi|^2 - \frac{\lambda}{2} (|\phi|^2 - m^2)^2 \right] \right\}. \tag{21}$$

This is the usual Higgs action.

When one sums over all fields, one includes fields $A' = \Lambda A \Lambda^{-1} + \Lambda d\Lambda^{-1}$ and sections $\Omega'(x) = \Lambda(x)\Omega(x)$. For this particular pairing of fields and sections the action is, of course, invariant. It is this set of transformations which we call gauge invariance. We mention this seemingly obvious point since in the literature[3] one can find the statement that the $W^\pm$ and Z bosons are the gauge transforms of the massless field. This is not true. If we start from $|\partial\phi + A\phi|^2$ and write $\phi = \Omega \rho u$ and $W = \Omega^{-1} A \Omega + \Omega^{-1} d\Omega$. W is not gauge covariant; it is gauge *invariant*. If ϕ goes to $\Lambda\phi$, A goes to $\Lambda A \Lambda^{-1} + \Lambda d\Lambda^{-1}$. Indeed the whole theory can be expressed in terms of gauge-invariant fields. Gauge-invariant fields are not gauge transforms of gauge-covariant fields.

There is no 2×2 representation in which

$$\tfrac{1}{2}\,\mathrm{tr}\,(B^\dagger\cdot B)=u^\dagger B^\dagger\cdot Bu \tag{22}$$

when B takes values in $\mathrm{SU}(2)\times \mathrm{U}(1)$.

There is a four-dimensional representation in which it is true. But since the two-dimensional right-hand side is what we are interested in, we simply include that as the contribution to the action,

$$\begin{aligned}\mathfrak{A}_2&=\int u^\dagger(*B^\dagger\wedge B)um^2\\&=\int u^\dagger B_\mu^\dagger B_\nu u m^2 g^{\mu\nu}d^4x\,.\end{aligned}\tag{23}$$

Here u is as before [Eq. (12)]. The field B_μ is now given by

$$\begin{aligned}&B=\Omega^{-1}A_\mu dx^\mu\,\Omega+\Omega^{-1}d\Omega\,,\\&\Omega=e^{(i/2)(\varphi^\alpha\sigma_\alpha)}\,,\end{aligned}\tag{24}$$

with

$$\sigma_\alpha=(I,\vec{\sigma})_\alpha\,.$$

The final action, after replacing m^2 by $\mu^2(x)=m^2+a(x)$ and integrating[7] is

$$W=\int[dA][d\phi^\dagger][d\phi]\exp\left\{i\int d^4x\left[-\tfrac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha+|\partial\phi+A\phi|^2-\frac{\lambda}{2}(|\phi|^2-m^2)^2\right]\right\}.\tag{25}$$

In this case it is interesting to note that the fourth degree of freedom can be obtained from Ω, though in an unnatural way. We take

$$\begin{aligned}\mu^2(x)&=m^2+a(x)\\&=m^2(1+|\ln\det\Omega|^2)\\&=m^2(1+|\mathrm{tr}\,\ln\Omega|^2)\end{aligned}\tag{26}$$

so that

$$a(x)=m^2(|\mathrm{tr}\,\ln\Omega|^2-1)=m^2[\tfrac{1}{4}(\varphi^0)^2-1]\tag{27}$$

with φ^0 the component arising in Eq. (24), and therefore

$$\mathfrak{A}_2=\int u^\dagger(*B^\dagger\wedge B)um^2[|\mathrm{tr}\,\ln\Omega|^2-1]\,.\tag{28}$$

The fourth function can be extracted this way since the three SU(2) parameters suffice to cover the constrained manifold.

It is more conventional to take the adjoint Higgs fields to be not the group element $\Omega=e^{m^{-1}\phi}$ but the algebra element $\phi=(d\Omega/dm^{-1})|_{m^{-1}=0}=m\ln\Omega$. The group covariant derivative of ϕ is defined as

$$\begin{aligned}D_\mu\phi&=\frac{dB_\mu}{dm^{-1}}\bigg|_{m^{-1}=0}=\frac{d}{dm^{-1}}[\Omega^{-1}(\partial_\mu\Omega+A_\mu\Omega)]\bigg|_{m^{-1}=0}\\&=\partial_\mu\phi+[A_\mu,\phi]\,.\end{aligned}\tag{29}$$

There is no intrinsic constraint or potential for ϕ. Assuming reflection symmetry, the general potential is[8]

$$V(\phi)=\frac{\mu^2}{2}\,\mathrm{tr}\phi^2+\frac{\lambda_1}{4}(\mathrm{tr}\phi^2)^2+\frac{\lambda_2}{4}\mathrm{tr}\phi^4\,.$$

Inclusion of fermions is straightforward.[9]

Thus we have seen that the Higgs fields can be ascribed to the extra degrees of freedom that come from viewing a gauge field as a connection form. The Higgs fields are the section fields. A restriction on the number of Higgs fields arises. There are essentially as many as the number of generators in the group. One can include one extra field by exploiting the difference in dimensionality between F^2 and B^2. We must multiply B^2 by something of dimension m^2. If that something is made into a field, we can obtain the conventional Higgs action. If not, we obtain the chiral Higgs action.[6] The chiral Higgs action certainly seems more natural in the context we have described. It is interesting to note that recent progress in its quantization has been made.[10]

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APPENDIX

The claim that the gauge field in the bundle can be expressed as

$$B(x,\varphi)=\Omega^{-1}(\varphi)A_\mu(x)dx^\mu\Omega(\varphi)+\Omega^{-1}(\varphi)d\Omega(\varphi)\tag{A1}$$

is merely an application of the definition of a connection form on a principal fiber bundle. Since the notation is probably unfamiliar we shall briefly review some definitions.

Let M be a manifold and G a Lie group. A principal fiber bundle with group G consists of a manifold P and an action of G on P such that[5] we have the following:

(B1) G acts freely on the right $(u,\Lambda)\in P\times G \to u\Lambda = R_\Lambda u \in P$.

(B2) M is the quotient space of P induced by G. Its canonical projection $\pi : P \to P/G = M$ is differentiable.

(B3) P is locally trivial. For every x in M there exists a neighborhood U such that $\psi : \pi^{-1}(u) \to u\times G$ diffeomorphically, $\psi(u) = (\pi(u), \varphi(u))$ and $\varphi(u)\Lambda = \varphi(u\Lambda)$.

In our simple example $P = \mathrm{R}^4\times\mathrm{SU}(2)$. It has a "global" coordinate form and triviality, since we can give coordinates everywhere as $u=(x,\Omega)$ with $\Omega\in\mathrm{SU}(2)$. The property (B3) yields this fact. Furthermore (B1) amounts to $((x,\Omega),\Lambda)\to(x,\Omega\Lambda)$ and (B2) amounts to projection of (x,Ω) onto x. Since $R_\Omega(x,I)=(x,\Omega)$ we see that x can be used to label the equivalence classes under group multiplication.

Now we shall define a connection as a splitting of the tangent space $T(P)$ to the bundle P into vertical $V(P)$ and horizontal $H(P)$ sectors.[5] $T_u(P)$ *et al.* denote the tangents at $u\in P$:

(C1) $T_u(P) = V_u(P) + H_u(P)$ for $u\in P$.

(C2) $H_{u\Lambda} = (R_\Lambda)_* H_u$ (see below).

(C3) H_u varies differentiably with u.

The condition (C2) requires that the connection is invariant by right multiplication, R_Λ. $(R_\Lambda)_*$ denotes the action of right multiplication induced on the tangent bundle $T(P)$ as opposed to R_Λ which denotes the action on the bundle P.

Given a set of generators for SU(2), such as the angular momentum operators, which satisfy

$$[L_A, L_B] = f^C_{AB} L_C \tag{A2}$$

we can find a dual basis for the linear functions on SU(2). In terms of angles (e.g., Euler angles)

$$L_A = M^a_A(\varphi)\frac{\partial}{\partial\varphi^a}, \tag{A3}$$

where M^a_A is a 3×3 matrix function of φ^a.[4] Introduce $d\varphi^b$ as a dual basis to $\partial/\partial\varphi^a$ satisfying

$$\left\langle d\varphi^b \middle| \frac{\partial}{\partial\varphi^a}\right\rangle = \delta^b_a . \tag{A4}$$

Set $\Theta^A = M^A_a(\varphi)d\varphi^a$ where $M^A_a(\varphi)$ is the inverse matrix to $M^a_A(\varphi)$. Then clearly

$$\begin{aligned}\langle\Theta^B | L_A\rangle &= M^B_b M^a_A \left\langle d\varphi^b \middle| \frac{\partial}{\partial\varphi^a}\right\rangle \\ &= M^B_b M^b_A \\ &= \delta^B_A .\end{aligned} \tag{A5}$$

The "forms" Θ^B satisfy the dual of Eq. (A2). That is,

$$d\Theta^A + \tfrac{1}{2} f^A_{BC}\Theta^B \wedge \Theta^C = 0 . \tag{A6}$$

These are the Maurer-Cartan equations. d is the exterior derivative; thus

$$\begin{aligned}d\Theta^A &= dM^A_a \wedge d\varphi^a \\ &= \frac{\partial M^A_a}{\partial\varphi^b} d\varphi^b \wedge d\varphi^a \\ &\equiv M^A_{a|b} d\varphi^b \wedge d\varphi^a .\end{aligned} \tag{A7}$$

$\wedge$ denotes antisymmetrized tensor multiplication.

If we introduce Λ_A, a set of matrices representing L_A, and thus satisfying

$$[\Lambda_A, \Lambda_B] = f^C_{AB}\Lambda_C \tag{A8}$$

we can write

$$\Theta = \Theta^A\Lambda_A . \tag{A9}$$

(A6) becomes

$$d\Theta + \Theta\wedge\Theta = 0 . \tag{A10}$$

This is easily solved by $\Theta = \Omega^{-1}d\Omega$.

The set of vector fields $(\partial/\partial x^\mu, L_A)$ form a basis for $T(P)$. Thus to every τ in the algebra $\mathcal{G}$ (of G) we can assign a fundamental vector field τ^* in $T(P)$ as follows:

$$\tau = \tau^A\Lambda_A \to \tau^* = \tau^A L_A . \tag{A11}$$

We note that

$$\begin{aligned}\Theta(\tau^*) &= \Theta^A\Lambda_A(\tau^B L_B) \\ &= \Lambda_A\tau^B\langle\Theta^A | L_B\rangle \\ &= \Lambda_A\tau^A \\ &= \tau .\end{aligned} \tag{A12}$$

Now we are prepared to define a connection form. Every $X\in T(P)$ can be split into $vX\in V(P)$ and $hX \in H(P)$ by the connection. Define the connection form $\omega(X)$ to be the unique $\tau\in\mathcal{G}$ such that τ^* is equal to vX.[4] Clearly $\omega(H)=0$ if and only if H is horizontal. Thus the connection form ω satisfies the following:

(F1) $\omega(\tau^*) = \tau$,

(F2) $(R_\Lambda)^*\omega = \omega_\Lambda(R_\Lambda)_* = \Lambda^{-1}\omega\Lambda \equiv \mathrm{ad}(\Lambda^{-1})\omega$,

(F3) ω is a differentiable form.

These are equivalent to (C1), (C2), and (C3). Since $\omega(H)$ vanishes when H is horizontal, (F1) $=$(C1). (F2) is merely the statement that ω respects right multiplication; thus (F2)$=$(C2).

If we begin by writing $\omega = (B^A_\mu dx^\mu + \Theta^A)\Lambda_A$ and then use $\Theta^A\Lambda_A = \Omega^{-1}d\Omega$ to satisfy (F1), we see that

$$B^A_\mu dx^\mu\Lambda_A = \Omega^{-1}A_\mu(x)dx^\mu\Omega \tag{A13}$$

in order that (F2) is satisfied for them $R_\Lambda \Omega = \Omega\Lambda$, with $\Lambda \in G$, and

$$\begin{aligned}(R_\Lambda)^*\omega &= (\Omega\Lambda)^{-1}A_\mu dx^\mu(\Omega\Lambda) + (\Omega\Lambda)^{-1}d(\Omega\Lambda) \\ &= \Lambda^{-1}(\Omega^{-1}A_\mu dx^\mu\Omega + \Omega^{-1}d\Omega)\Lambda \\ &= \Lambda^{-1}\omega\Lambda \\ &= \mathrm{ad}(\Lambda^{-1})\omega\,. \end{aligned} \tag{A14}$$

Any other dependence on Ω would violate either (F1) or (F2). We set $B \equiv \omega$.

From these facts one sees immediately that the curvature form $d\omega \circ h = d\omega + \omega \wedge \omega$ is horizontal.[5]

[1]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967); A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity* (*Nobel Symposium No. 8*), edited by N. Svartholm (Almqvist and Wiksell, Stockholm, 1968), p. 367.

[2]P. W. Higgs, Phys. Rev. Lett. 12, 132 (1964); T. W. B. Kibble, Phys. Rev. 155, 1156 (1967).

[3]E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

[4]K. I. Macrae, Phys. Rev. D 18, 3737 (1978); 18, 3761 (1978); 18, 3777 (1978); L. N. Chang, K. I. Macrae, and F. Mansouri, *ibid.* 13, 235 (1976); and see references therein for some of the historical developments of this idea.

[5]S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, New York, 1963 and 1969), Vols. I and II.

[6]M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).

[7]When m becomes dynamical, we include a $(\partial m)^2$ term.

[8]Since the action can now be constructed using only elements in the algebra, it becomes invariant under adjoint gauge transformations.

[9]Of course they are only easily included if there are no topological obstructions.

[10]A. A. Slavnov, private communication.