PHYSICAL REVIEW D VOLUME 22, NUMBER 8 15 OCTOBER 1980

Composite systems viewed as relativistic quantal rotators: Vectorial and spinorial models

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(Received 15 April 1980)

The constrained relativistic Hamiltonian dynamics of Dirac is applied to two models for composite objects having a nontrivial internal space. One model has a four-vector as internal manifold, the other a real four-spinor. The corresponding quantum mechanics is developed, and the two models each possess a Regge trajectory (mass related to spin) having the minimal $(2S + 1)$ degeneracy, unlike the Regge-Hanson spherical top model with $(2S + 1)^2$ degeneracy. The spinorial model is most promising and allows (in the quantal case) electromagnetic interactions *via minimal coupling*, leading, for example, to intrinsic spin magnetic moments. An extension to "relativistic SU(6)" is shown to exist.

I. INTRODUCTION AND SUMMARY

There is considerable experimental evidence indicating that Regge trajectories (one-parameter families of particles with increasing spin, and with mass related to spin) may be interesting as objects of study by themselves.[1] Since mass and spin are the two invariants of relativistic (Poincaré) symmetry, such a trajectory relation constitutes a constraint on the dynamics; accordingly, any such study of Regge trajectories, *per se*, in quantum mechanics is necessarily a study of constrained relativistic Hamiltonian mechanics of composite objects having a nontrivial internal space.

Similar motivations led Regge and Hanson to write their classic paper on the relativistic spherical top.[2] Their paper mainly treated the classical theory of such a system; further developments were given in Hojman's thesis.[3] There are several difficulties with the Regge-Hanson model. First, the $(2S+1)^2$ degeneracy (where S is the spin)—which is characteristic of the symmetric top—is not found experimentally in hadronic Regge trajectories. Second, there is a technical difficulty in this model (which uses an antisymmetric second-rank tensor as the internal manifold) in that the constraint used is not linear in the momenta. Third, as we discuss briefly in Sec. V, the antisymmetric tensor characteristic of the model is constrained to be perpendicular to the four-momentum. As a consequence, minimal coupling to the electromagnetic field cannot generate a magnetic moment, in contrast, for example, to the situation in the Dirac equation. Thus, magnetic moments have to be introduced explicitly and in a somewhat complicated manner. Fourth, the Dirac brackets for the position coordinates of the top, to which Regge and Hanson were led by their classical singular Lagrangian,[4] are nonzero. Thus, in a straightforward quantization (replacing Dirac brackets by commutators) one finds noncommuting position observables. This leads to factor-ordering problems when one tries to quantize the theory in interaction with an external electromagnetic field.

In this paper we discuss in detail two models for Regge trajectories which have only the $(2S+1)$-fold degeneracy characteristic of spin S. To distinguish these models from the symmetric top model we call them "rigid-rod models." The first model uses a four-vector as the internal manifold. The second uses a real four-spinor for the internal space. Both models are related to the group SO(3,2), but whereas the representations of the Lorentz subgroup [SO(3,1)] are highly reducible for the first model, the representation of this subgroup remains irreducible for the second model.

The first model—which we call the "vectorial model"—is discussed in its classical form in Secs. II and III. Besides having a $(2S+1)$ degeneracy the vectorial model is algebraically simpler than that of Ref. 2, and its analysis can be carried out completely. One finds, however, second-class constraints which lead to nonvanishing Dirac brackets for the position coordinates x^μ of the particle. Just as for the spherical top, this leads to difficulties when one tries to quantize the vectorial model in interaction with an electromagnetic field. The free, noninteracting model can be quantized and this is done in Sec. V A.

The second model—the "spinorial model"—is very promising. The classical description of this model is contained in Sec. IV. The model has only one primary constraint and therefore the Dirac brackets are the same as the original Poisson brackets. This has the very desirable consequence that the position variables commute. Thus, *there is no problem in carrying out the quantization, even in interaction with the electromagnetic field*; this is discussed in Secs. V B and V C.

Several things are worth noting about the spinorial model. First, since the quantized version contains each value of spin, integer as well as half integer, each once, there are relations to supersymmetry.[5] Second, *magnetic moments are obtained via minimal electromagnetic interaction* closely analogous to what happens for the Dirac equation. Third, the model can easily be extended to produce the spectrum of the string model. Fourth, and perhaps most important, there exists an extension which gives a relativistic quantum-mechanical SU(6) model with minimal electromagnetic interactions, with the implication of nonvanishing anomalous magnetic moments for the hadrons lying on the associated trajectory.

Attempts to construct composite models are, of course, hardly new, and we do not claim any particular competence to present a full review of this field. One may, however, distinguish two very different approaches in the literature: in one, the notion of pointlike constituents (two in the simplest case) and the necessary independent space-time degrees of freedom are assumed from the outset; by contrast, the other approach, which is more algebraic in spirit, uses the language of infinite-component wave equations and the internal space may, or may not, correspond to physical mass points. The models of the present paper are of the latter type, with the added feature that we have a classical Lagrangian starting point. For a more comprehensive discussion of the subject let us cite the papers of Takabayasi, including especially his recent review.[6] We would also call attention to the works of Dominici, Gomis, and Longhi,[7] which are, in part, too recent to appear in the review of Ref. 6.

II. THE VECTORIAL MODEL: LAGRANGIAN AND POISSON BRACKETS

Let us assume our object to have a world line $x^\mu(s)$, where s is an arbitrary parameter labeling the events along the world line. Let there be associated to every s also a unit spacelike vector $a^\mu(s)$, $a^\mu(s)a_\mu(s)=1$, $g_{00}=-1$. Thus, our object has a seven-dimensional configuration space, with generalized coordinates made up of x^μ and a^μ: the former transforms in the usual way under inhomogeneous Lorentz transformations, the latter is a translation-invariant four-vector. Let differentiation with respect to s be denoted by a dot, i.e., $\dot{x}^\mu(s)=(d/ds)x^\mu(s)$; clearly the fixed length of a implies $a^\mu(s)\dot{a}_\mu(s)=a\cdot\dot{a}=0$.

We assume the Lagrangian to be invariant for inhomogeneous Lorentz transformations and to contain first derivatives with respect to s only. This implies that it must be a function of the invariants $\dot{x}^2$, $\dot{x}\cdot a$, $\dot{x}\cdot\dot{a}$, and $\dot{a}^2$ ($a^2=1$, $a\cdot\dot{a}=0$). Let $\mathcal{L}$ be such a function; then P_μ, the momentum conjugate to x^μ, is defined by

$$P=\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\ . \tag{2.1}$$

To restrict $\mathcal{L}$ further, we impose two additional requirements. The first is that (2.1) should lead to $P\cdot a=0$ as a primary constraint. Since $P\cdot a$ depends on momenta only linearly, this requirement can be easily met: $\mathcal{L}$ must be a function of only three variables,

$$u_1=(\dot{x}\cdot a)^2-\dot{x}^2,\quad u_2=\dot{x}\cdot\dot{a},\quad u_3=\dot{a}^2\,. \tag{2.2}$$

The second requirement is that the action $\int ds\,\mathcal{L}$ must be invariant for changing the arbitrary parameter s to another one, $\sigma(s)$. This invariance is denoted in the literature by the nondescriptive (and rather ungainly) term "reparametrization invariance." A better label, we feel, is "chronometric invariance," and we shall use this term in the following.

The requirement of chronometric invariance implies that $\mathcal{L}$ must be homogeneous of degree $\frac{1}{2}$ in the u's. Introducing

$$\xi=\frac{u_1}{u_2}\ ,\quad \eta=\frac{u_1}{u_3}\ , \tag{2.3}$$

the most general Lagrangian can be written as

$$\mathcal{L}=\sqrt{u_1}\,f(\xi,\eta)\,, \tag{2.4}$$

with f an arbitrary function of two variables.

The requirement of chronometric invariance is cumbersome even for a single-point particle without structure. In that case the chronometric-invariant action $\int ds[-\dot{x}(s)^2]^{1/2}$ is often replaced by the noninvariant $\int d\tau\,\dot{x}(\tau)^2$, where τ is the proper time.[8] The latter is also more convenient than the former for path-integral quantization.[9] Here, however, as in Ref. 2, chronometric invariance is what produces a mass-spin relation as a primary constraint. The Lagrangian (2.4) implies therefore at least two constraints, and is called singular. For such Lagrangians the introduction of a Hamiltonian formulation (a first step towards quantization) is somewhat complicated. Methods to handle such "constrained Hamiltonian systems" were developed by Dirac, Bergmann, Anderson, and others.[4] The application of these methods to our model will be the subject of the next sections.

In order to set up a phase space for our Lagrangian theory, we must define a suitable canonical conjugate to a^μ, keeping in mind the restriction $a^2=1$. One way is to (at least locally) eliminate a^0 (say) in favor of the three independent space components a^j of a, introduce three independent momenta conjugate to these, and then define the conventional canonical Poisson brackets (PB). An equivalent but more elegant and globally well-defined procedure is to introduce instead an antisymmetric tensor $S_{\mu\nu}$ by

$$S_{\mu\nu}=a_\mu \frac{\partial\mathcal{L}}{\partial\dot{a}^\nu} - a_\nu \frac{\partial\mathcal{L}}{\partial\dot{a}^\mu} \tag{2.5}$$

and regard this as the momentum conjugate to a^μ. That there are only three algebraically independent quantities here is assured by the identities

$$a_\lambda S_{\mu\nu}+a_\mu S_{\nu\lambda}+a_\nu S_{\lambda\mu}=0\,. \tag{2.6}$$

The nonvanishing fundamental Poisson brackets for our system are then

$$\begin{aligned}&\{x^\mu, P_\nu\}=\delta^\mu_\nu, \quad \{S_{\mu\nu}, a_\lambda\}=g_{\mu\lambda}a_\nu - g_{\nu\lambda}a_\mu\,,\\ &\{S_{\mu\nu}, S_{\rho\sigma}\}=g_{\mu\rho}S_{\nu\sigma}-g_{\nu\rho}S_{\mu\sigma}+g_{\mu\sigma}S_{\rho\nu}-g_{\nu\sigma}S_{\rho\mu}\,.\end{aligned} \tag{2.7}$$

The basic variables of our Lagrangian theory are then x^μ, a^μ and their conjugate momenta P_μ, $S_{\mu\nu}$. The algebraically independent quantities together form a 14-dimensional phase space Γ. The Legendre mapping from the Lagrangian variables into this phase space for a singular Lagrangian is such that the range of the mapping on Γ is a subset of Γ, i.e., we have primary constraints.

An even more convenient choice of a variable conjugate to a^μ is the following: we define a vector b_μ by

$$b_\mu=S_{\mu\nu}a^\nu \tag{2.8}$$

and use it in place of the tensor $S_{\mu\nu}$. One sees from (2.5) that in terms of the Lagrangian one has

$$b_\mu=a_\mu a^\nu \frac{\partial\mathcal{L}}{\partial\dot{a}^\nu} - \frac{\partial\mathcal{L}}{\partial\dot{a}^\mu}\,, \tag{2.9}$$

and from (2.6) one can recover $S_{\mu\nu}$ in this way:

$$S_{\mu\nu}=b_\mu a_\nu - b_\nu a_\mu\,. \tag{2.10}$$

Just like $S_{\mu\nu}$, b_μ also has only three algebraically independent components since $a\cdot b=0$. One now has the freedom to either view $S_{\mu\nu}$, a_μ as the basic phase space variables and b_μ as a derived variable, or to view a_μ and b_μ as the basic quantities and $S_{\mu\nu}$ as a derived object. With the latter view, the fundamental nonvanishing Poisson brackets for Γ are

$$\begin{aligned}&\{x^\mu, P_\nu\}=\delta^\mu_\nu, \quad \{a^\mu, b_\nu\}=a^\mu a_\nu - \delta^\mu_\nu\,,\\ &\{b_\mu, b_\nu\}=a_\mu b_\nu - a_\nu b_\mu\,.\end{aligned} \tag{2.11}$$

We shall use the formulation in terms of a and b, since vectors are easier to handle in the formation of invariants, etc. One may also note the relation

$$b^2=\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}\,. \tag{2.12}$$

It is interesting to observe that the set of Poisson brackets among $S_{\mu\nu}$ and b_μ, namely, the last line of (2.7) together with

$$\{S_{\mu\nu}, b_\lambda\}=g_{\mu\lambda}b_\nu - g_{\nu\lambda}b_\mu\,, \quad \{b_\mu, b_\nu\}=-S_{\mu\nu}\,, \tag{2.13}$$

corresponds to the Lie algebra of the de Sitter group SO(3, 2). Thus this group appears naturally in the kinematic structure of our model with a vector internal variables. This is an instance of a well-known construction[10]: $S_{\mu\nu}$ and a_μ yield a spacelike realization of the Poincaré group, so with b_μ defined by (2.8) we get $S_{\mu\nu}$ and b_μ realizing the group SO(3,2).

The connection between invariances of the action and the conservation laws is similar for singular and for nonsingular Lagrangians.[11] The invariance of the Lagrangian (2.4) for inhomogeneous Lorentz transformations implies the usual ten conservation laws. The conserved generators for space-time displacements are

$$P_\mu=\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}=2[(\dot{x}\cdot a)a_\mu - \dot{x}_\mu]\frac{\partial\mathcal{L}}{\partial u_1}+\dot{a}_\mu\frac{\partial\mathcal{L}}{\partial u_2}\,, \tag{2.14}$$

which imply by construction $P_\mu a^\mu=0$. The conserved generators of homogeneous Lorentz transformations are

$$M_{\mu\nu}=x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu}\,, \tag{2.15}$$

where

$$\begin{aligned}S_{\mu\nu}&=a_\mu\frac{\partial\mathcal{L}}{\partial\dot{a}^\nu} - a_\nu\frac{\partial\mathcal{L}}{\partial\dot{a}^\mu}\\ &=(a_\mu\dot{x}_\nu - a_\nu\dot{x}_\mu)\frac{\partial\mathcal{L}}{\partial u_2}+2(a_\mu\dot{a}_\nu - a_\nu\dot{a}_\mu)\frac{\partial\mathcal{L}}{\partial u_3}\,.\end{aligned} \tag{2.16}$$

For later use, we note here the definition of the Pauli-Lubanski vector W_μ in the present model:

$$\begin{aligned}&W_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}=\epsilon_{\mu\nu\rho\sigma}P^\nu b^\rho a^\sigma\,,\\ &W^2=(P\cdot b)^2 - P^2 b^2\,.\end{aligned} \tag{2.17}$$

III. CONSTRAINED HAMILTONIAN FORMALISM FOR THE VECTOR INTERNAL-VARIABLE MODEL

In this section we apply the methods of Ref. 4 to the Lagrangian (2.4). First, instead of considering the most general Lagrangian (2.4), which depends on u_1, u_2, and u_3, we consider more limited models where the Lagrangian only depends on a pair of these variables. Thus in Secs. III A–III C, we consider these limited models before discussing the general model in Sec. III D. In fact, all these models share the feature also found in Ref. 2, that the Dirac brackets for the position variables do not vanish. This leads to problems when one tries to write down a quantum theory of our model in the case where there is interaction with an electromagnetic field, as we shall discuss in Sec. V. Surprisingly, perhaps, the simple two-variable models are interesting as classical theories; we now proceed with them.

A. Model independent of u_2

Let the Lagrangian depend only on u_1 and u_3; with (2.3) we write

$$\mathcal{L}=\sqrt{u_1}\,f(\eta)\,, \tag{3.1}$$

where f is a function of one variable. Denoting the derivative of f with respect to its argument by f', the vectors P and b conjugate to x and a are

$$P_\mu=\frac{1}{\sqrt{u_1}}\,(f+2\eta f')[(\dot{x}\cdot a)a_\mu-\dot{x}_\mu]\,,\qquad b_\mu=\frac{2\eta^2 f'}{\sqrt{u_1}}\,\dot{a}_\mu\,. \tag{3.2}$$

Let us write $\Phi_2=P\cdot a$, we have already secured $\Phi_2\approx 0$ as a primary constraint. We expect the chronometric invariance of $\mathcal{L}$ to lead to another independent Lorentz-invariant primary constraint Φ_1. The only available Lorentz scalars on which Φ_1 can depend are

$$P\cdot b=-2\eta^2 f'(f+2\eta f')u_2/u_1\,, \tag{3.3}$$

$$P^2=-(f+2\eta f')^2\,, \tag{3.4}$$

$$b^2=4\eta^3 f'^2\,. \tag{3.5}$$

The explicit appearance of u_2/u_1, i.e., of ξ^{-1}, in (3.3) implies that in general $P\cdot b$ will not occur in Φ_1; this constraint must arise by eliminating η between P^2 and b^2. We define the *generic case* by the conditions $f'(f+2\eta f')\neq 0$, $P^2\neq$ constant, and $b^2\neq$ constant. Then Φ_1 must be of the form

$$\Phi_1=P^2+\alpha(b^2)\approx 0\,, \tag{3.6}$$

with α a nontrivial function of its argument. This is a Regge relation between mass and spin, as will soon be clear; it is a consequence of the chronometric invariance of $\mathcal{L}$.

The starting Hamiltonian is then, by the general theory,[4]

$$\mathcal{H}=v_1\Phi_1+v_2\Phi_2\,, \tag{3.7}$$

where the v's are arbitrary. (There is no term independent of the Φ's because $\mathcal{L}$ is homogeneous of degree one in the velocities.) Hence the generalized Hamiltonian equations of motion are, to begin with

$$\dot{x}^\mu=2v_1P^\mu+v_2a^\mu\,,\quad \dot{a}^\mu=-2v_1\alpha'(b^2)b^\mu\,,\qquad \dot{P}_\mu=0\,,\quad \dot{b}_\mu=2v_1\alpha'(b^2)b^2a_\mu+v_2P_\mu\,. \tag{3.8}$$

As a consequence we find

$$\dot{\Phi}_1\approx 2v_2\alpha'(b^2)P\cdot b\,,\quad \dot{\Phi}_2\approx -2v_1\alpha'(b^2)P\cdot b\,. \tag{3.9}$$

For consistency these expressions must vanish. To avoid both v_1 and v_2 vanishing, resulting in no motion at all, and to avoid constraining b^2 by demanding $\alpha'(b^2)=0$ and thus making P^2 constant, we are led to impose the secondary constraint[4]

$$\chi=P\cdot b\approx 0\,. \tag{3.10}$$

Then, a fresh consistency condition arises, as we must have

$$\dot{\chi}\approx v_2P^2\approx 0\,. \tag{3.11}$$

This determines that $v_2\approx 0$, v_1 remains arbitrary, and the constraint analysis is complete. Φ_1 comes through as a primary first-class constraint,[4] Φ_2 and χ form a second-class pair, and the Hamiltonian is an arbitrary multiple of Φ_1 alone:

$$\mathcal{H}=v_1\Phi_1\,. \tag{3.12}$$

Now we can see that in this model, due to the secondary constraint (3.10), the Pauli-Lubanski vector W_μ has a square equal to $-P^2b^2$ [see (2.17)], so that the constraint (3.6) is, as it stands, a relation between mass and intrinsic spin.

With the Hamiltonian (3.12) we can give the solution to the equations of motion (3.8). This solution will contain one arbitrary function of s, due to the arbitrariness in v_1. First, choose $x^\mu(0)$, $P_\mu(0)$, $a_\mu(0)$, and $b_\mu(0)$ so that all kinematical conditions and constraints are obeyed at $s=0$,

$$a^2-1=a\cdot b=P\cdot a=P^2+\alpha(b^2)=P\cdot b=0\,. \tag{3.13}$$

Then the solution is

$$\begin{aligned} &P_\mu(s)=P_\mu(0),\quad x^\mu(s)=x^\mu(0)+\frac{P^\mu\phi(s)}{\alpha'(b^2)\sqrt{b^2}}\,,\\ &a^\mu(s)=a^\mu(0)\cos\phi(s)-\frac{b^\mu(0)}{\sqrt{b^2}}\sin\phi(s)\,,\\ &b^\mu(s)=b^\mu(0)\cos\phi(s)+a^\mu(0)\sqrt{b^2}\sin\phi(s)\,,\\ &\dot{\phi}(s)=2\alpha'(b^2)\sqrt{b^2}v_1,\quad \phi(0)=0\,. \end{aligned} \tag{3.14}$$

(Note that b^2 is a constant of the motion.) The space-time "orbit" is completely known, since just one unknown function $\phi(s)$ appears. Note that neither "end," $x+a$ or $x-a$, moves at the velocity of light; this is because unlike the string this model is not purely geometric.

By eliminating the second-class pair Φ_2, χ we can set up a system of Dirac brackets (DB) to replace Poisson brackets (2.11). (These may be called preliminary Dirac brackets in the sense that no gauge constraint conjugate to Φ_1, such as $\chi' = x^0 - s \approx 0$, is imposed; we leave the model in a manifestly four-dimensional form with v_1 completely free.) As $\{\Phi_2, x\} \approx -P^2$, one obtains the DB (denoted with an asterisk)

$$\{f,g\}^* \approx \{f,g\} + \frac{1}{P^2}\left(\{f, P\cdot b\}\{P\cdot a, g\} - \{f, P\cdot a\}\{P\cdot b, g\}\right). \quad (3.15)$$

We find for the basis variables

$$\{P_\mu, P_\nu\}^* = \{P_\mu, a_\nu\}^* = \{a_\mu, a_\nu\}^* = 0\,, \quad (3.16)$$

$$\{x^\mu, P_\nu\}^* = \delta^\mu_\nu, \quad \{x^\mu, a_\nu\}^* = -a^\mu \frac{P_\nu}{P^2}, \quad \{b^\mu, P_\nu\}^* = 0\,,$$

$$\{b_\mu, a_\nu\}^* = g_{\mu\nu} - a_\mu a_\nu - P_\mu P_\nu / P^2\,, \quad (3.17)$$

$$\{x^\mu, x^\nu\}^* = (a^\mu b^\nu - a^\nu b^\mu)/P^2, \quad \{x^\mu, b_\nu\}^* = -b^\mu P_\nu / P^2\,,$$

$$\{b_\mu, b_\nu\}^* = a_\mu b_\nu - a_\nu b_\mu\,. \quad (3.18)$$

The constraints $\Phi_2 = 0$ and $\chi = 0$ can be treated as identities, only one constraint Φ_1 remains, and for any f the equation of motion is

$$\frac{df}{ds} \approx \frac{\partial f}{\partial s} + v_1\{f, \Phi_1\}^*\,. \quad (3.19)$$

Note particularly that $\{x_\mu, x_\nu\}^* \neq 0$. The nonvanishing of these brackets is the cause of difficulties which occur when one tries to quantize this model in interaction with the electromagnetic field. Let us therefore briefly indicate the algebraic reasons for these brackets not vanishing. First, the Dirac brackets involving P_μ are fixed by the requirement that space-time translations commute and by the transformation rules of x, a, and b. Second, $P\cdot a = 0$ is preserved by the Dirac bracket, i.e.,

$$\{x_\mu, P\cdot a\}^* \approx a_\mu + \{x_\mu, a_\nu\}^* P^\nu \approx 0\,, \quad (3.20)$$

which shows why $\{x_\mu, a_\nu\}^*$ has the value given in (3.17). The fact that $\{x_\mu, x_\nu\}^*$ *cannot* vanish then follows from the Jacobi identity for x_λ, x_μ, and a_ν.

Next let us briefly look at the nongeneric cases of the present model with $\mathfrak{L}$ of the form (3.1). Actually these are all physically uninteresting, but for completeness we mention their features. There are three exceptional cases: (i) $f'(f + 2\eta f') \neq 0$, $P^2 = \text{const}$; (ii) $f'(f + 2\eta f') \neq 0$, $b^2 = \text{const}$; (iii) $f'(f + 2\eta f') = 0$. One finds that cases (i) and (ii) coincide. and correspond to f of the form

$$f(\eta) = C_0 + C_1/\sqrt{\eta}, \quad C_0 C_1 \neq 0\,. \quad (3.21)$$

This implies that P^2 is constant and therefore is not what we were looking for; we want an interesting mass-spin relation with neither mass nor spin constrained to a single value. Case (iii) either implies $f = \text{const}$ [$C_1 = 0$ in (3.21)] which again makes P^2 constant, or $f + 2\eta f' = 0$ [$C_0 = 0$ in (3.21)] which gives $P^2 = 0$. Thus every nongeneric case has f of the form (3.21) and is of no interest.

B. Model independent of u_3

Now we choose the Lagrangian to depend on u_1 and u_2 alone and to have the form

$$\mathfrak{L} = \sqrt{u_1}\, f(\xi)\,. \quad (3.22)$$

This give the conjugate variables P_μ and b_μ the values

$$P_\mu = \frac{1}{\sqrt{u_1}}(f + 2\xi f')[(\dot{x}\cdot a)a_\mu - \dot{x}_\mu] - \frac{1}{\sqrt{u_1}}\xi^2 f' \dot{a}_\mu\,,$$
$$b_\mu = -\frac{1}{\sqrt{u_1}}\xi^2 f'[(\dot{x}\cdot a)a_\mu - \dot{x}_\mu]\,. \quad (3.23)$$

The significant Lorentz scalars for formation of a constraint Φ_1 are

$$b^2 = -\xi^4 f'^2\,,$$
$$P\cdot b = \xi^2 f'(f + \xi f')\,, \quad (3.24)$$
$$P^2 = -f(f + 2\xi f') + \xi^4 f'^2 \frac{u_3}{u_1}\,.$$

The explicit appearance of u_3/u_1, i.e., η^{-1}, in P^2 means that now in general one must eliminate ξ between b^2 and $P\cdot b$ to find Φ_1. This generic case is characterized by $f' \neq 0$, $b^2 \neq \text{const}$, $P\cdot b \neq \text{const}$. But let us first dispose quickly of the exceptional cases. These are (i) $f' \neq 0$, $b^2 = \text{const}$; (ii) $f' \neq 0$, $P\cdot b = \text{const}$; (iii) $f' = 0$. Cases (i) and (ii) again coincide, and they correspond to f of the form

$$f(\xi) = C_0 + C_1/\xi, \quad C_1 \neq 0\,. \quad (3.25)$$

This leads, after more algebra, to no mass-spin relation at all, and can be discarded. Case (iii) implies $f = \text{const}$ [$C_1 = 0$ in (3.25)] and right away gives P^2 constant.

We now proceed with the generic case. This has two primary constraints,

$$\Phi_1 = P\cdot b - \alpha(b^2) = 0, \quad \Phi_2 = P\cdot a = 0\,, \quad (3.26)$$

with α a nontrivial function of its argument; so the starting Hamiltonian is

$$\mathfrak{H} = v_1\Phi_1 + v_2\Phi_2 \quad (3.27)$$

with v's arbitrary. This gives the equation of motion

$$\dot{x}^\mu \approx v_1 b^\mu + v_2 a^\mu, \quad \dot{a}^\mu \approx v_1[-P^\mu + 2\alpha'(b^2)b^\mu]\,,$$
$$\dot{P}_\mu \approx 0, \quad \dot{b}_\mu \approx v_1[P\cdot b - 2\alpha'(b^2)b^2]a_\mu + v_2 P_\mu\,. \quad (3.28)$$

Consistency demands that

$$\dot{\Phi}_1 \approx [P^2 - 2\alpha'(b^2)P\cdot b]v_2\,,$$
$$\dot{\Phi}_2 \approx -[P^2 - 2\alpha'(b^2)P\cdot b]v_1 \quad (3.29)$$

must both vanish. As $v_1 = v_2 = 0$ implies no motion, we must impose as a secondary constraint

$$\chi = P^2 - 2\alpha'(b^2)P\cdot b \approx 0 ,$$

or, with (3.26) equally well,

$$\chi = P^2 - 2\alpha(b^2)\alpha'(b^2) \approx 0 . \tag{3.30}$$

For $\dot{\chi}$ one now finds

$$\dot{\chi} \approx -4\alpha(b^2)[\alpha(b^2)\alpha''(b^2) + \alpha'(b^2)^2]v_2 .$$

If we do not set $v_2 = 0$, its coefficient here must vanish, but that leads to a constant P^2. To avoid this, we must choose $v_2 = 0$; then Φ_1 survives as primary first class, Φ_2 and χ form a second-class pair, and v_1 remains arbitrary. With the resulting equations of motion, b^2 and $P\cdot b$ are both conserved. The space-time trajectory of the system can again be explicitly exhibited. As P^μ is constant, it proves expedient to resolve b into its components parallel and perpendicular to P:

$$b_\parallel = P\frac{P\cdot b}{P^2}, \quad b_\perp = b - b_\parallel . \tag{3.31}$$

Then, if a choice of $x^\mu(0)$, $a^\mu(0)$, $b^\mu(0)$, and P^μ is made so as to obey all the constraints and kinematic conditions at $s = 0$,

$$a^2 - 1 \approx a\cdot b = P\cdot a = P\cdot b - \alpha(b^2) = P^2 - 2\alpha(b^2)\alpha'(b^2) = 0 , \tag{3.32}$$

the solution to (3.28) (with $v_2 = 0$) is

$$P_\mu(s) = P_\mu(0),$$

$$x^\mu(s) = x^\mu(0) + \frac{P^\mu}{4[\alpha'(b^2)]^2}\phi(s) + \frac{a^\mu(s) - a^\mu(0)}{2\alpha'(b^2)} , \tag{3.33}$$

$$a^\mu(s) = a^\mu(0)\cos\phi(s) + \frac{b_\perp^\mu(0)}{(b_\perp^2)^{1/2}}\sin\phi(s) ,$$

$$b_\perp^\mu(s) = b_\perp^\mu(0)\cos\phi(s) - a^\mu(0)(b_\perp^2)^{1/2}\sin\phi(s) ,$$

$$b_\parallel^\mu(s) = b_\parallel^\mu(0) , \tag{3.34}$$

$$\dot{\phi}(s) = 2(b_\perp^2)^{1/2}\alpha'(b^2)v_1, \quad \phi(0) = 0 .$$

(Note that $b_\perp^2$ is a constant of motion.) When compared with the solution (3.14) in model A, the interesting point is that now the motion of x^μ has a rotating component that was previously absent.

In the present model the trajectory relation (3.30) is not, as it stands, a relation expressing the squared mass as a function of the intrinsic spin. This is because now, combining (2.17) and (3.32), we only express W^2 as some function of b^2:

$$W^2 \approx \alpha(b^2)^2 - 2b^2\alpha(b^2)\alpha'(b^2) . \tag{3.35}$$

One must eliminate b^2 between (3.30) and (3.35), get a relation between P^2 and W^2, and identify $-W^2/P^2$ with the square of the intrinsic spin, to get the conventional trajectory statement.

Of the Dirac brackets obtained on eliminating Φ_2 and χ we will only mention two points: (i) the Dirac brackets now depend on the function $\alpha(b^2)$ so that unlike model A, there is no longer a clean separation of kinematic and dynamic constraints in the four-dimensional sense; (ii) the bracket $\{x_\mu, x_\nu\}^*$ does not and cannot vanish, for the same reasons which were discussed in Sec. IIIA.

C. Model independent of u_1

When we assume the Lagrangian to be dependent on u_2 and u_3 only, we find, by arguments similar to those in the preceding subsections, that the mass spectrum is of the form P^2 constant in all cases, generic as well as exceptional.

D. General model

To return to the most general case, with Lagrangian (2.4): $\mathcal{L} = \sqrt{u_1}f(\xi,\eta)$. In this case the algebra is more complicated, we shall not pursue it in all details as again it leads to Dirac brackets for the position variables which are nonzero. One calculates again P_μ, b_μ and the three quantities P^2, b^2, and $P\cdot b$. The two parameters ξ and η can, in general, be eliminated between these three quantities, producing a primary constraint of the form

$$0 = \Phi_1 = P^2 + \alpha(P\cdot b, b^2) ,$$

which joins the other primary constraint

$$0 \approx \Phi_2 = P\cdot a .$$

This gives again a Hamiltonian with arbitrary coefficients v_1, v_2: $\mathcal{H} = v_1\Phi_1 + v_2\Phi_2$. Consideration of $d\Phi_1/ds \approx v_2\{\Phi_1, \Phi_2\}$, and $d\Phi_2/ds \approx -v_1\{\Phi_1, \Phi_2\}$ leads to the secondary constraint

$$0 \approx \chi = \{\Phi_1 \Phi_2\} ,$$

as one does not want to eliminate all motion with the choice $v_1 = v_2 = 0$. The demand that $0 \approx d\chi/ds$, leads to $v_2 = 0$. Therefore, $0 = \Phi_1$ is first class and the pair $0 \approx \Phi_2$, $0 \approx \chi$ is second class. The constraint Φ_1 is first class in all these models as it corresponds to chronometric invariance in the parameters.

IV. SPINORIAL MODEL

The model developed in Secs. II and III describes a relativistic object which, after quantization and in its rest frame, exhibits a spin spectrum with each integer spin value appearing just once. This was achieved by asking for $P\cdot a \approx 0$ as a primary constraint, in addition to the consequence of chronometric invariance. This fact led directly to a nontrivial constraint analysis, the emergence of a secondary constraint, the existence of a second-class pair, and values for Dirac brackets different from Poisson brackets. As a consequence, the x^μ are noncommuting operators in the quantum theory, resulting in difficulties with interactions. The spinor model now to be developed is simpler in its constraint algebra: the x^μ remain commuting quantities and interaction with the electromagnetic field exists.

Consider two conjugate pairs of variables q_1, p_1 q_2, p_2, to be denoted on occasion as q_1, q_3, q_2, q_4. We can form a column vector

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{4.1}$$

and the Poisson brackets may be written as

$$\{Q_a, Q_b\} = \gamma^0_{ab}, \quad \gamma^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad a, b = 1,2,3,4. \tag{4.2}$$

The ten independent quadratic forms in the q's allow one to write a set of generators of SO(3,2) as follows:

$$\begin{aligned} &S_{12} = \tfrac{1}{2}(q_2 q_3 - q_1 q_4), \quad S_{23} = \tfrac{1}{2}(q_1 q_2 + q_3 q_4), \\ &S_{31} = \tfrac{1}{4}(q_1^2 + q_3^2 - q_2^2 - q_4^2), \quad S_{01} = \tfrac{1}{4}(q_1^2 - q_3^2 - q_2^2 + q_4^2), \\ &S_{02} = \tfrac{1}{2}(q_3 q_4 - q_1 q_2), \quad S_{03} = \tfrac{1}{2}(q_1 q_3 + q_4 q_2), \\ &S_{05} = \tfrac{1}{4}(q_1^2 + q_3^2 + q_2^2 + q_4^2), \quad S_{15} = \tfrac{1}{2}(q_2 q_4 - q_1 q_3), \\ &S_{25} = \tfrac{1}{2}(q_1 q_4 + q_2 q_3), \quad S_{35} = \tfrac{1}{4}(q_1^2 - q_3^2 + q_2^2 - q_4^2). \end{aligned} \tag{4.3}$$

These have the SO(3,2) Poisson brackets

$$\{S_{AB}, S_{CD}\} = g_{AC} S_{BD} - g_{BC} S_{AD} + g_{AD} S_{CB} - g_{BD} S_{CA}, \quad ABCD = 0,1,2,3,5. \tag{4.4}$$

Restricting to the subgroup SO(3,1) for $A, B = \mu$, $\nu = 0,1,2,3$, $S_{\mu 5} = V_\mu$ transforms as a four-vector, and $S_{\mu\nu}$ as an antisymmetric tensor. The column Q of (4.1) is a real four-spinor under SO(3,2), and under the subgroup SO(3,1) in particular we have

$$\begin{aligned} &\{V_\mu, Q\} = -\tfrac{1}{2}\gamma Q, \\ &\{S_{\mu\nu}, Q\} = -\tfrac{1}{4}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) Q. \end{aligned} \tag{4.5}$$

Note that all the γ_μ are real and obey

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu}, \quad g_{00} = -1. \tag{4.6}$$

The generators of this SO(3,2) realization obey several characteristic identities:

$$S_{CA} S^C_B = 0, \text{ i.e., } V_\mu V^\mu = 0, \; S_{\lambda\mu} S^\lambda_\nu = V_\mu V_\nu \; (S_{\mu\nu} S^{\mu\nu} = 0), \tag{4.7}$$

$$\epsilon^{ABCDE} S_{BC} S_{DE} = 0, \text{ i.e., } \epsilon^{\mu\nu\rho\sigma} S_{\mu\nu} S_{\rho\sigma} = 0, \; \epsilon^{\lambda\mu\nu\rho} S_{\mu\nu} V_\rho = 0.$$

The representation (4.3) is the classical analog[12] of what Dirac calls the remarkable unitary representation of SO(3,2).[13] The latter representation is associated with the Majorana equation.[14] Note that the invariants in (4.7) are quite different from what one obtains in the quantum version, where $S_{\mu\nu} S^{\mu\nu} = \frac{3}{2}$ and $V_\mu V^\mu = \frac{1}{2}$ (in units of $\hbar^2$).[15]

We wish to construct a Lagrangian out of the internal variable Q, a space-time position four-vector x^μ, and their derivatives $\dot{Q}, \dot{x}^\mu$, with respect to an evolution parameter s. It must have these properties: (i) it must lead to q_1 and q_3 being a canonical pair, and q_2, q_4 another; (ii) it must be Lorentz and chronometric invariant. Keeping in mind the identities (4.7), one finds that the most general such Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\dot{Q}^T \gamma^0 Q + \sqrt{-\dot{x}^2} f(\xi), \quad \xi = \frac{\dot{x} \cdot V}{\sqrt{-\dot{x}^2}}, \tag{4.8}$$

with f an arbitrary function of one variable. The first term in $\mathcal{L}$ is fully SO(3,2) invariant, whereas the second one is only invariant for SO(3,1). (Q is invariant under space-time translations.) Since the dependence on the "velocities" $\dot{Q}$ is no more than linear, the application of canonical methods with respect to this variable leads to constraints. But as is easy to check, a prior application of the Dirac formalism leads to the anticipated result that q_1, q_3 and q_2, q_4 form canonical pairs. Hereafter, we may assume this "in principle necessary" analysis has been completed, and concentrate on the constraint due to the functional form of f. We need now to set up a canonical conjugate to x^μ alone:

$$P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = f' V_\mu + \frac{1}{\sqrt{-\dot{x}^2}} (\xi f' - f) \dot{x}_\mu. \tag{4.9}$$

The chronometric invariance must lead to a Lorentz-invariant primary constraint Φ which can only depend on the scalars P^2 and $P \cdot V$. These have the values

$$P^2 = (\xi f')^2 - f^2, \quad P \cdot V = \xi(\xi f' - f). \tag{4.10}$$

It is natural to define the generic case as obtaining when neither P^2 nor $P \cdot V$ reduces to a constant. Then we can eliminate ξ between these two variables to obtain a single primary constraint of the form

$$\Phi = P^2 - \alpha(P \cdot V) \approx 0, \tag{4.11}$$

with α being a nontrivial function. To recognize this as a mass-spin relation, we remark that the conserved generators for Poincaré transformations in the present model are P_μ for translations and

$$M_{\mu\nu}=x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu} \tag{4.12}$$

for homogeneous Lorentz transformations. Hence the Pauli-Lubanski vector is

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu S^{\rho\sigma}. \tag{4.13}$$

Using some of the identities (4.7) for $S_{\mu\nu}$, V_μ we find its square has the value

$$W^2 = S_{\mu\nu}P^\nu \cdot S^{\mu\rho}P_\rho = \xi^2(\xi f' - f)^2 ,$$

i.e.,

$$W^2 = (P\cdot V)^2 . \tag{4.14}$$

Thus though the constraint (4.11) is not, as it stands, a relation between mass and intrinsic spin, it can easily be so interpreted.

Before proceeding with the generic case, let us dispose of the exceptional cases. Assuming $P\cdot V$ = constant leads to P^2 constant, and vice versa. These correspond to f of the form

$$f(\xi)=C_0\xi = C_1/\xi,\quad P^2=-4C_0C_1,\quad P\cdot V = -2C_1 . \tag{4.15}$$

The vice versa means that *we do not have a classical version of the Majorana equation*, at least within the framework of the present Lagrangian approach; we can get at most a projection onto one mass level of that equation. In fact, it is also impossible to obtain any trajectory of the form $P^2 = a_0 P\cdot V + a_1(P\cdot V)^2$ without each side separately being constant, i.e., again restricting to one mass level. Thus, in particular, trajectories with mass proportional to spin are excluded in our model.

Now we resume the analysis of the generic case. Since there is only one primary constraint and $\mathcal{L}$ is homogeneous of degree one in the velocities, the Hamiltonian is an arbitrary multiple of Φ, there is no constraint analysis needed, and Φ remains first class:

$$\mathcal{H} = v\Phi . \tag{4.16}$$

The nonvanishing Poisson brackets among the basic variables are of course just (4.2) and

$$\{x^\mu, P_\nu\} = \delta^\mu_\nu . \tag{4.17}$$

We get the equations of motion for x, P, and Q by computing their Poisson brackets with $\mathcal{H}$:

$$\dot{x}^\mu = v[2P^\mu - \alpha'(P\cdot V)V^\mu],\quad \dot{P}_\mu = 0 ,$$
$$\dot{Q} = -\tfrac{1}{2}v\alpha'(P\cdot V)P^\mu\gamma_\mu Q . \tag{4.18}$$

Those for $S_{\mu\nu}$ and V_μ follow from the Q equation of motion, or directly from $\mathcal{H}$:

$$\dot{S}_{\mu\nu} = -v\alpha'(P\cdot V)(P_\mu V_\nu - P_\nu V_\mu) ,$$
$$\dot{V}_\mu = v\alpha'(P\cdot V)S_{\mu\nu}P^\nu . \tag{4.19}$$

Recognizing that P_μ and $P\cdot V$ are conserved, we can exhibit the solution to these equations explicitly in terms of one arbitrary function $\phi(s)$ corresponding to the arbitrariness in v:

$$x^\mu(s) = x^\mu(0) + \left(\frac{2}{\alpha'[P\cdot V(0)]} - \frac{P\cdot V(0)}{P^2}\frac{P^\mu}{\sqrt{-P^2}}\right)\phi(s) - \frac{1}{P^2}[S^{\mu\nu}(s) - S^{\mu\nu}(0)]P_\nu ,$$

$$Q(s) = \left(\cos\frac{\phi(s)}{2} - \frac{P^\mu\gamma_\mu}{\sqrt{-P^2}}\sin\frac{\phi(s)}{2}\right)Q(0) ,$$

$$V_\mu(s) = P_\mu\frac{P\cdot V(0)}{P^2} + \left(V_\mu(0) - P_\mu\frac{P\cdot V(0)}{P^2}\right)\cos\phi(s) + \frac{1}{\sqrt{-P^2}}S_{\mu\nu}(0)P^\nu\sin\phi(s) , \tag{4.20}$$

$$S_{\mu\nu}(s) = S_{\mu\nu}(0) - \frac{1}{\sqrt{-P^2}}[P_\mu V_\nu(0) - P_\nu V_\mu(0)]\sin\phi(s) + \frac{1}{P^2}[P_\mu S_{\nu\rho}(0) - P_\nu S_{\mu\rho}(0)]P^\rho[1-\cos\phi(s)] ,$$

$$\dot{\phi}(s) = v\alpha'[P\cdot V(0)]\sqrt{-P^2},\quad \phi(0)=0 .$$

Thus, allowing for the fact that Q, being a spinor, cannot be "represented" in space-time, the classical space-time "trajectory" for our model is completely known. The initial values P_μ and $Q(0)$ are of course restricted by the constraint (4.11), which restricts the motion to an 11-dimensional region in the full 12-dimensional phase space.

Let us indicate by general arguments that choices for the "trajectory function" $\alpha(P\cdot V)$ exist which lead to physically acceptable structures at the classical level. For the moment, write z for $P\cdot V$. A sufficient condition that would ensure the absence of spacelike solutions for P^μ in (4.11) is to take for $\alpha(z)$ a negative definite function of z. It does not matter that (4.11) contains P both on the left-hand and on the right-hand side; there can be no solutions to (4.11) with $P^2 > 0$ if $\alpha(z)$ is negative for all (real) z. Next let us imagine for simplicity that $\alpha(z)$ is an even function of z; then positive and negative timelike P^μ are present symmetrically. Let us now ask if $\alpha(z)$ can be so chosen as to make the world line $x^\mu(s)$ everywhere timelike; in other words, we require $\dot{x}^2 < 0$. From the x equation of motion and the constraint (4.11) this is

$$\dot{x}^2 = 4v^2[P^2 - P\cdot V\alpha'(P\cdot V)] = 4v^2[\alpha(z) - z\alpha'(z)] = -4v^2z^2\left(\frac{\alpha(z)}{z}\right)' < 0 , \tag{4.21}$$

i.e.,

$$\left(\frac{\alpha(z)}{z}\right)' > 0 .$$

We examine the consequences of demanding that for all real z, $\alpha(z)$ be negative and $\alpha(z)/z$ a monotonically increasing function of z. It is clear that this cannot be achieved if $\alpha(z)/z$ is continuous through $z=0$, for then this function would have to be positive when $z<0$, negative when $z>0$, and be monotonically increasing. However, if we permit a discontinuity in $\alpha(z)/z$ at $z=0$, both conditions can be met. A simple example is

$$\alpha(z) = -z/\sinh z .$$

We do not intend that this α be taken seriously but present it just to show that models with timelike P^2 and $\dot{x}^2$ certainly exist. In a practical case, it could also happen that the physical range of z may be a half-line and not the entire real line, in which case less restrictive conditions than were described above may suffice.

It is useful, for this as yet noninteracting model, to establish the connection with what we have called the Thomas form of the generators of the Poincaré group.[1,5,15] To do this we carry the analysis one step further, in analogy with Ref. 2, and demand that the parameter s equal the laboratory time x^0. This is done by imposing the gauge constraint

$$\chi = x^0 - s \approx 0 . \tag{4.22}$$

The pair of constraints Φ, χ is now second class, with

$$\{\chi, \Phi\} = 2P^0 - V^0\alpha'(P\cdot V) . \tag{4.23}$$

Demanding that $\chi \approx 0$ be preserved for all s fixes the hitherto arbitrary function v of (4.16):

$$\frac{d\chi}{ds} \approx 0 \Rightarrow \frac{\partial\chi}{\partial s} + v\{\chi, \Phi\} \approx 0 \Rightarrow v = [2P^0 - V^0\alpha'(P\cdot V)]^{-1} . \tag{4.24}$$

We may now convert the constraints $\Phi\approx 0$, $\chi\approx 0$ into identities by passing from Poisson brackets to the appropriate Dirac brackets:

$$\{f,g\}^* = \{f,g\} + [2P^0 - V^0\alpha'(P\cdot V)]^{-1}[\{f,x^0\}\{\Phi,g\} - \{f,\Phi\}\{x^0,g\}] . \tag{4.25}$$

This produces a nonsingular bracket defined on a ten-dimensional phase space, for which the independent quantities $\vec{x}$, $\vec{P}$, and Q are canonical coordinates. In fact, these variables have standard values for their Dirac brackets:

$$\{x^j, x^k\}^* = \{P^j, P^k\}^* = \{x^j, Q\}^* = \{P^j, Q\}^* = 0 ,$$
$$\{x^j, P^k\}^* = \delta_{jk}, \quad \{Q_a, Q_b\}^* = \gamma^0_{ab} . \tag{4.26}$$

The general equation of motion prior to imposing $\chi\approx 0$ reads

$$\frac{df}{ds} \approx \frac{\partial f}{\partial s} + v\{f, \Phi\} . \tag{4.27}$$

Here v is completely arbitrary, and f of course can be any function of x^μ, P_μ, Q, and s. Once v has been fixed by (4.24) and we adopt the Dirac bracket (4.25), a general dynamical variable f must be thought of as some function of $\vec{x}$, $\vec{P}$, Q, and s; and its equation of motion will appear in the Hamiltonian form with a suitable dynamical variable playing the role of the Hamiltonian within a Dirac bracket. In fact, if we imagine the constraint $\Phi\approx 0$ to be solved for P^0,

$$\Phi = \vec{P}^2 - P^{02} - \alpha(\vec{P}\cdot\vec{V} - P^0V^0) = 0 ,$$
$$P^0 = H(V^0, \vec{V}\cdot\vec{P}, \vec{P}\cdot\vec{P}) , \tag{4.28}$$

we have the equation of motion with respect to $x^0 = t$,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}^* . \tag{4.29}$$

The generators of the Poincaré group behave analogously. That the Poisson brackets among the ten quantities P^μ, $M^{\mu\nu}$ correspond to the Poincaré algebra is kinematically obvious. We now set $\chi\approx 0$, and imagine these generators to be expressed in terms of $\vec{x}$, $\vec{P}$, Q, and t: it is then automatic that the Dirac brackets among these expressions again correspond to the Poincaré algebra. This is assured by the fact that the Poisson brackets $\{P_\mu, \Phi\}$, $\{M_{\mu\nu}, \Phi\}$ vanish. We then end up with the set of Poincaré generators (with respect to the Dirac bracket)

$$\vec{P}, H(V^0, \vec{V}\cdot\vec{P}, \vec{P}\cdot\vec{P}), \quad J_{jk} = x_jP_k - x_kP_j \pm S_{jk} ,$$
$$K_j = x_jH - tP_j + S_{0j}, \quad jk = 1,2,3 . \tag{4.30}$$

Presenting the Poincaré generators of our model in this form is exactly analogous to presenting them for the ordinary free Dirac equation in the form

$$\vec{P}, H = \vec{\alpha}\cdot\vec{P} + \beta m, \quad J_{jk} = x_jP_k - x_kP_j - i\tfrac{1}{4}\hbar[\alpha_j, \alpha_k] ,$$
$$K_j = x_jH - i\tfrac{1}{2}\hbar\alpha_j , \tag{4.31}$$

in which the Hamiltonian "has not been diagonalized." In particular, we shall see in Sec. V that, as in the case of the Dirac equation the form (4.28) for the Hamiltonian of our system is the proper one in which interaction with an external electromagnetic field is correctly represented by the minimal replacement for $\vec{P}$.

For the free case in our model, we get the Thomas form of the Poincaré generators if in place of the position three-vector $\vec{x}$ we use the Newton-Wigner position. This three-vector can be defined in terms of the generators given in

(4.30), and the space part $\vec{W}$ of the Pauli-Lubanski vector, as

$$\vec{x}_{NW} = \frac{\vec{K}}{H} + \frac{\vec{P}x\vec{W}}{MH(M+H)}, \quad M = (H^2 - \vec{P}\cdot\vec{P})^{1/2}. \tag{4.32}$$

However, whereas for the free system the Thomas form of the generators and the form (4.30) are canonically equivalent, the latter is to be preferred because of its capacity to allow for minimal electromagnetic coupling.

V. QUANTIZATION AND THE ELECTROMAGNETIC INTERACTION

This section is divided into three subsections in which we discuss, in turn, the quantization of the free vectorial model, that of the free spinorial model, and finally, minimal coupling to an external electromagnetic field followed by quantization. Under the last mentioned heading the major part of our analysis will relate to the spinor model.

A. Quantization of the free vector model

In Sec. III A the description of the generic classical model of type A was left in the following form: we had a system of manifestly Lorentz-invariant Dirac brackets among the four-vectors P^μ, a^μ, x^μ, and b^μ; and only one primary constraint in the form of a trajectory relation (3.6) was still to be imposed. These Dirac brackets, listed in (3.18), resulted in elimination of the two trajectory-independent second-class constraints $P\cdot a \approx 0$, $P\cdot b\approx 0$. There are now two possible approaches to quantization of this classical model. The first one would be to impose a gauge constraint $\chi = \chi^0 - s \approx 0$, conjugate to Φ_1 (in the sense of the Dirac brackets already defined), and using the iterative property of Dirac brackets one could now eliminate the pair Φ_1, χ and arrive at a final set of Dirac brackets. These last would be brackets among three-dimensional quantities $\vec{x}, \vec{P}, \vec{a}, \vec{b}$, and could be converted into a set of commutation relations for Hermitian operators. However, this reduction of the kinematics to three-dimensional form results in commutation relations that depend on the trajectory function α and so are hard to solve. We shall therefore adopt an alternative approach. We convert the Dirac brackets (3.18) into a system of manifestly Lorentz-invariant commutation relations and find the simplest and most natural solution to them, in a suitable Hilbert space. The constraint $\Phi_1 \approx 0$ is then implemented as a generalized Klein-Gordon-type wave equation on wave functions.

We begin then by postulating a set of commutation relations suggested by the classical Dirac brackets (3.18):

$$[P_\mu, P_\nu] = [P_\mu, a_\nu] = [a_\mu, a_\nu] = 0, \tag{5.1a}$$

$$\begin{aligned}
&[x_\mu, P_\nu] = i\hbar g_{\mu\nu}, \quad [x_\mu, a_\nu] = -i\hbar a_\mu P_\nu/P^2, \\
&[b_\mu, P_\nu] = 0, \\
&[b_\mu, a_\nu] = i\hbar(g_{\mu\nu} - a_\mu a_\nu - P_\mu P_\nu/P^2),
\end{aligned} \tag{5.1b}$$

$$\begin{aligned}
&[x_\mu, x_\nu] = i\hbar(a_\mu b_\nu - a_\nu b_\mu)/P^2, \\
&[x_\mu, b_\nu] = -i\hbar b_\mu P_\nu/P^2, \\
&[b_\mu, b_\nu] = i\hbar(a_\mu b_\nu - a_\nu b_\mu).
\end{aligned} \tag{5.1c}$$

(There should be no risk of confusion between the operators of the present discussion and the classical variables of Sec. III.) One may now check the following points: (i) all the Jacobi identities are obeyed; (ii) one may consistently demand that P, a, x, and b all be Hermitian; (iii) one may also consistently demand that a^2-1, $P\cdot a$, and $P\cdot b$ vanish identically. [However, one cannot demand $a\cdot b=0$; this would conflict with the result $b\cdot a - a\cdot b = (a\cdot b)^\dagger - a\cdot b = 2i\hbar$ that follows from Hermiticity and the $[b,a]$ commutation relation.]

We now find a solution to (5.1). Clearly, one may assume P^μ and a^μ to be simultaneously diagonal. To avoid a proliferation of symbols, we shall use the same symbols P, a for operators and their eigenvalues. In other words, our representation space shall consist of wave functions $\psi(P,a)$ where P and a are two independent real four-vectors. The conditions $P\cdot a=0$, $a^2-1=0$ are implemented by defining the norm of ψ via

$$\begin{aligned}
\|\psi\|^2 &= \int d^4P\, d^4a\, \delta(P\cdot a)\delta(a^2-1)\,|\psi(P,a)|^2 \\
&\equiv \int d\mu(P,a)\,|\psi(P,a)|^2.
\end{aligned} \tag{5.2}$$

Thus, though ψ may be imagined as being defined on an eight-dimensional space of pairs of four-vectors P, a, it is only the values of ψ on the six-dimensional manifold defined by $P\cdot a = a^2-1=0$ that are relevant. The commutation relations (5.1a) are then automatically obeyed. We turn now to the set (5.1b): they essentially tell us how the operators x_μ and b_μ, when applied to a wave function $\psi(P,a)$, alter the arguments of ψ. For an infinitesimal vector ϵ^μ let us define

$$U(\epsilon) = 1 + i\epsilon\cdot x, \quad V(\epsilon) = 1 + i\epsilon\cdot b. \tag{5.3}$$

We want these to be unitary operators and, according to (5.1b), to obey

$$\begin{aligned}
&U(\epsilon)PU^{-1}(\epsilon) = P - \hbar\epsilon, \\
&U(\epsilon)aU^{-1}(\epsilon) = a + \hbar\epsilon\cdot aP/P^2, \\
&V(\epsilon)PV^{-1}(\epsilon) = P, \\
&V(\epsilon)aV^{-1}(\epsilon) = a - \hbar\epsilon + \hbar a a\epsilon\cdot a + \hbar P\epsilon\cdot P/P^2.
\end{aligned} \tag{5.4}$$

This way of presenting the commutation relations (5.1b) tells us that $U(\epsilon)$ and $V(\epsilon)$ must act on a wave function $\psi(P,a)$ in this way:

$$(U(\epsilon)\psi)(P,a) \approx \lambda(P,a;\epsilon)\,\psi(P-\hbar\epsilon,\, a+\hbar\epsilon\cdot aP/P^2)\,,$$
$$(V(\epsilon)\psi)(P,a) \approx \mu(P,a;\epsilon) \times \psi(P, a-\hbar\epsilon+\hbar a\epsilon\cdot a+\hbar P\epsilon\cdot P/P^2)\,. \tag{5.5}$$

Here λ and μ are two multipliers to be determined presently. One can confirm that, to first order in ϵ, if the arguments P, a on the left-hand sides in (5.5) obey the restrictions $P\cdot a = a^2-1=0$, the arguments of ψ on the right-hand sides obey the same restrictions. To fix λ and μ we invoke the unitarity of $U(\epsilon)$ and $V(\epsilon)$. In the former case, let us write $P'=P-\hbar\epsilon$, $a'=a+\hbar\epsilon\cdot aP/P^2$. Then the measure appearing in the inner product (5.2) behaves as follows:

$$\begin{aligned} d\mu(P',a') &= d^4P'd^4a'\,\delta(P'\cdot a')\delta(a'^2-1) \\ &= d^4Pd^4a\left(1+\hbar\frac{\epsilon\cdot P}{P^2}\right)\delta(P\cdot a) \\ &\quad\times\delta\left(a^2-1+2\hbar\epsilon\cdot a\frac{P\cdot a}{P^2}\right) \\ &= \left(1+\hbar\frac{\epsilon\cdot P}{P^2}\right)d\mu(P,a)\,. \end{aligned}$$

Unitarity of $U(\epsilon)$ then gives us

$$|\lambda(P,a;\epsilon)|^2 \simeq 1+\hbar\frac{\epsilon\cdot P}{P^2}\,,$$

the simplest and most natural solution to which is

$$\lambda(P,a;\epsilon)\simeq 1+\hbar\frac{\epsilon\cdot P}{2P^2}\,. \tag{5.6}$$

In an analogous fashion we find

$$\mu(P,a;\epsilon)\simeq 1+\hbar a\cdot\epsilon\,. \tag{5.7}$$

One can then write our proposed solution to (5.1b) in the form

$$\begin{aligned} x_\mu &= -\frac{i\hbar}{2}\frac{P_\mu}{P^2}+i\hbar\left(\frac{\partial}{\partial P^\mu}-\frac{a_\mu}{P^2}P\cdot\frac{\partial}{\partial a}\right), \\ b_\mu &= -i\hbar a_\mu+i\hbar\left(\frac{\partial}{\partial a^\mu}-a_\mu a\cdot\frac{\partial}{\partial a}-\frac{P_\mu}{P^2}P\cdot\frac{\partial}{\partial a}\right). \end{aligned} \tag{5.8}$$

For ease in computation we may imagine that our wave functions $\psi(P,a)$ are defined for all independently chosen arguments P, a, and the partial derivatives $\partial/\partial P^\mu$, $\partial/\partial a^\mu$ also treat the eight variables P_μ, a_μ as being independent; the specific combinations of operators appearing in (5.8) are guaranteed to respect the restrictions $P\cdot a=a^2-1=0$. Keeping this remark in mind, one can now verify without difficulty that the last set of commution relations (5.1c) is also obeyed.

We thus have a Hermitian set of operators P, a, x, b satisfying the commutation relations (5.1) and acting on a Hilbert space of wave functions $\psi(P,a)$ with inner product (5.2). It is amusing to note that in this operator system we have $a\cdot b=-i\hbar$, whereas in the classical model $a\cdot b$ vanished. We must also note that the norm (5.2) is *not* the physical norm to be used for quantum-mechanical interpretation. That norm must be set up by a natural modification of (5.2) after imposing a "wave equation" on $\psi(P,a)$ and restricting attention to the solutions. In doing this we shall be guided by what we would have done in an analogous treatment of the Klein-Gordon equation.[9]

Equation (3.4) shows that, in a classical vectorial model of type A, the four-momentum P_μ is definitely timelike (provided, of course, the variables in the classical Lagrangian have real values). At the classical level this will imply that the function $\alpha(b^2)$ is non-negative. On quantization, b_μ becomes the Hermitian operator (5.8); and in taking over the classical constraint $\Phi_1=P^2+\alpha(b^2)\approx 0$ into quantum theory, we will assume that $\alpha(b^2)$ is a Hermitian positive definite operator. With P_μ and b_μ the operators of the present discussion, we now impose on $\psi(P,a)$, a general wave function of our representation space, the *wave equation*

$$(P^2+\alpha(b^2))\psi(P,a)=0\,. \tag{5.9}$$

Not every $\psi(P,a)$ in the representation space will, of course, obey this equation. We must isolate those that do, find a natural description of them, and set up a quantum-mechanical inner product among them alone. In (5.9), P^2 is a numeric, since in our representation P_μ is diagonal, while b^2 is a differential operator acting on the components a_μ. However, since $\alpha(b^2)$ is positive definite, we may assert that every solution $\psi(P,a)$ to (5.9) *vanishes for spacelike* P_μ. Let it be hereafter understood that we deal with such $\psi(P,a)$ alone, though P^μ may be either positive or negative timelike. So we are only concerned with functions $\psi(P,a)$ defined in the region determined by $P^2<0$, $P\cdot a=0$, $a^2-1=0$. At this point, in order to handle the operator $\alpha(b^2)$ in the wave equation, it proves expedient to switch from the variables P^μ, a^μ to a pair P^μ, a'^μ in this way. As P^μ is timelike, we can set up a Lorentz matrix $\Lambda(P)$, the one that relates P^μ to its rest frame form, in this way:

$$\begin{aligned} &\Lambda^0{}_0=\epsilon\frac{P^0}{M}\,,\quad \Lambda^j{}_0=\Lambda^0{}_j=\epsilon\frac{P_j}{M}\,, \\ &\Lambda^j{}_k=\delta_{jk}+\frac{P_jP_k}{\vec{P}\cdot\vec{P}}\left(\epsilon\frac{P^0}{M}-1\right), \\ &\epsilon=\text{sign}P^0,\quad M=\sqrt{-P^2}\,. \end{aligned} \tag{5.10}$$

We then define a'^{μ} as

$$a'^{\mu} = \Lambda^{\mu}{}_{\nu}(P)a^{\nu} . \qquad (5.11)$$

Any wave function $\psi(P, a)$ can be rewritten as some $\phi(P, a')$. The point of introducing a'^{μ} is that now the (unphysical) norm (5.2) takes the form (since $a'^2 = a^2$ and $p \cdot a = \epsilon M a'^0$)

$$\|\psi\|^2 = \int d^4p\, d^4a'\, \delta(Ma'^0)\delta(a'^2 - 1)|\phi(P, a')|^2$$

$$= \int \frac{d^4p}{2M} d\Omega(\hat{a}')|\phi(P, \hat{a})|^2 . \qquad (5.12)$$

Effectively, ϕ is a function of a (timelike) four-vector p^{μ} and a unit vector $\hat{a}'$ in three-dimensional space. The differential operator expression for b_{μ} in (5.8) can be rewritten now in terms of a' and $\partial/\partial a'$, suitable for application to a $\phi(P, \hat{a}')$, and one finds after a little algebra that, hereafter setting $\hbar = 1$,

$$b^2 = 1 + \vec{L}' \cdot \vec{L}', \quad \vec{L}' = -i\hat{a}' \times \nabla_{\hat{a}'} . \qquad (5.13)$$

(That is, $\vec{L}'$ is the ordinary orbital angular momentum operator associated with $\hat{a}'$.) It is natural now to expand $\phi(P, \hat{a}')$ in a series in terms of the spherical harmonics $Y_{lm}(\hat{a}')$:

$$\psi(P, a) \equiv \phi(P, \hat{a}') = \sum_{lm} \phi_{lm}(P) Y_{lm}(\hat{a}') . \qquad (5.14)$$

The inner product and the wave equation then become

$$\|\psi\|^2 = \int \frac{d^4p}{2M} \sum_{lm} |\phi_{lm}(P)|^2 , \qquad (5.15a)$$

$$(P^2 + \bar{\alpha}(l))\phi_{lm}(P) = 0 , \qquad (5.15b)$$

$$\bar{\alpha}(l) \equiv \alpha[1 + l(l+1)]. \qquad (5.15c)$$

The wave equation tells us $\phi_{lm}(P)$ can only be nonzero if P lies on the mass hyperboloid $M^2 = \bar{\alpha}(l)$. In turn this means that a $\psi(P, a)$ obeying (5.9) can have a nonzero value only if P lies on one of these mass hyperboloids, and then its general form is determined by the right-hand side of (5.14): it is now to be understood that the "component" $\phi_{lm}(P)$ is defined only for $P^2 = -\bar{\alpha}(l)$, or in other words this is a function of $\vec{p}$ alone (except for the possibility that $p^0 = \pm[\vec{p} \cdot \vec{p} + \bar{\alpha}(l)]^{1/2}$ can be of either sign). The physical quantum-mechanical inner product is now inferred in this way: Prior to imposition of the wave equation (5.9) or (5.15b), we write the $\|\psi\|^2$ as

$$\|\psi\|^2 = \int_0^{\infty} \frac{dM^2}{2M} \int \frac{d^3p}{(\vec{p} \cdot \vec{p} + M^2)^{1/2}} \sum_{lm} |\phi_{lm}(P)|^2 . \qquad (5.16)$$

After solving the wave equation as described above, we alter this expression for $\|\psi\|^2$ by suspending the integration over the invariant mass[9] and thus arrive at

$$\|\psi\|^2 = \sum_{lm} \frac{1}{2\sqrt{\bar{\alpha}(l)}} \int \frac{d^3P}{[\vec{P} \cdot \vec{P} + \bar{\alpha}(l)]^{1/2}} |\phi_{lm}(\vec{P})|^2 . \qquad (5.17)$$

(The sum over positive and negative signs of P^0 is, for simplicity, left unstated.) With this expression for the physical norm of our solution of the wave equation, the quantization of the free relativistic vector model (of type A) is complete. It appears as a direct sum of unitary irreducible representations $[\bar{\alpha}(l)^{1/2}, l]$ of the Poincaré group, with $l = 0, 1, \ldots$; l represents the intrinsic spin and $\bar{\alpha}(l)^{1/2}$ the rest mass. Both positive and negative energies occur symmetrically.

B. Quantization of the free spinor model

The classical spinor model is characterized by a much simpler constraint algebra than the classical vector model. As a result the commutation relations that provide the kinematic basis of the corresponding quantum system are much simpler to state and to solve. In Sec. IV we brought the classical description to a form where, with the gauge constraint $\chi = x^0 - s \approx 0$, the Dirac brackets (4.26) are manifestly invariant only under the Euclidean group, and x^0 is a parameter rather than a dynamical variable. We convert these Dirac brackets into a set of commutation relations for Hermitian operators $\vec{x}$, $\vec{P}$, q_a:

$$[x_j, x_k] = [P_j, P_k] = [x_j, q_a] = [P_j, q_a] = 0 ,$$
$$[x_j, p_k] = i\hbar\, \delta_{jk} , \quad [q_a, q_b] = i\hbar\, \gamma^0_{ab} . \qquad (5.18)$$

These are immediately solved: with $\vec{x}$, q_1, q_2 diagonal, we have wave functions $\psi(\vec{x}, q_1, q_2)$ with the quantum-mechanical norm

$$\|\psi\|^2 = \int d^3x \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2 |\psi(\vec{x}, q_1, q_2)|^2 , \qquad (5.19)$$

and the remaining operators are

$$\vec{P} = -i\hbar \frac{\partial}{\partial \vec{x}} , \quad q_3 = -i\hbar \frac{\partial}{\partial q_1} , \quad q_4 = -i\hbar \frac{\partial}{\partial q_2} . \qquad (5.20)$$

As they stand, the expressions S_{AB} of (4.3) are now Hermitian operators and generate the two unitary representations of SO(3, 2) alluded to earlier.

To complete the quantization of the free spinor model we must discuss the relativistic aspects.

Let us assume the classical Lagrangian (4.8) leads to a negative definite "trajectory function" $\alpha(P\cdot V)$. We have a perfectly acceptable set of classical generators for the Poincaré group in the set of variables $\vec{J}$, $\vec{K}$, $\vec{P}$, H of (4.30), since via their Dirac brackets they realize the Poincaré algebra. $\vec{J}$ and $\vec{P}$ can be taken over as they stand into quantum mechanics and reinterpreted as operators. We now *assume* we can order the noncommuting factors in the classical expression $H(V^0, \vec{V}\cdot\vec{P}, \vec{P}\cdot\vec{P})$, when V^0 and $\vec{V}$ are made into Hermitian operators, such that (i) H becomes Hermitian and remains positive definite, (ii) with the rearrangement $K_j = \frac{1}{2}(x_j H + H x_j) + S_{0j}$, the nontrivial commutation relations of the Poincaré group,

$$[K_j, H] = i\hbar P_j\,, \quad [K_j, K_k] = -i\hbar\,\epsilon_{jkl}\,J_l\,, \tag{5.21}$$

are obeyed. With this we have a satisfactory relativistic quantum theory of the free spinor model. In the rest frame the system exhibits each spin value 0, $\frac{1}{2}$, 1,... *once*, with mass determined by spin via the trajectory relation.

C. Electromagnetic interactions

For the classical theories, the interaction with an external electromagnetic field with potential $A_\mu(x)$ is introduced easily enough by adding to the free Lagrangian a term, homogeneous of degree one in velocities, of the form

$$\mathcal{L}_{\rm int} = eA_\mu(x)\,\frac{dx^\mu}{ds} + g\,(-\dot{x}^2)^{1/2} S_{\mu\nu} F^{\mu\nu}(x)\ . \tag{5.22}$$

($S_{\mu\nu}$ is to be interpreted appropriately in each of the two models.) Note the gauge invariance: replacing A_μ by $A_\mu + \partial_\mu \Lambda$ changes the Lagrangian by $d\Lambda/ds$. Difficulties occur, however, when one tries to quantize the vector models after introducing the above interaction. These are caused by the occurrence, in the models of Secs. II and III, of second-class constraints. This is a feature common to all the internal vector variable models and which, moreover, they share with Ref. 2. Then the Dirac brackets are different from the Poisson brackets, and in particular one finds $\{x^\mu, x^\nu\}^* \neq 0$. This is, again, a general and inescapable feature, as argued following (3.19). At this point one runs into factor-ordering problems, as the components of the four-vector x^μ in $A_\mu(x)$ and $F_{\mu\nu}(x)$ in (5.22) do not commute with one another. For this reason, we do not pursue the vector models further.

The spinor model of Sec. IV does not share these difficulties and, as we have seen, the x^μ remain commuting quantities to the end. Let us restrict ourselves to the "minimal-coupling" case with $g=0$ in (5.22): the complete classical Lagrangian is then

$$\mathcal{L} = \tfrac{1}{2}\dot{Q}^T\gamma^0 Q + (\dot{x}^2)^{1/2} f(\xi) + eA_\mu(x)\dot{x}^\mu\ ,$$
$$\xi = \dot{x}\cdot V/(-\dot{x}^2)^{1/2}\ . \tag{5.23}$$

Now P_μ, the canonical conjugate to x^μ, is

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = f' V_\mu + (\xi f' - f)\,\frac{\dot{x}_\mu}{(\dot{x}^2)^{1/2}} + eA_\mu(x)\ . \tag{5.24}$$

Writing $\Pi_\mu = P_\mu - eA_\mu$, the primary constraint will now arise by eliminating ξ between Π^2 and $\Pi\cdot V$:

$$\Pi^2 = (\xi f')^2 - f^2\ ,$$
$$\Pi\cdot V = \xi(\xi f' - f)\ . \tag{5.25}$$

Comparing this with (4.10) and (4.11), we immediately see that the primary constraint is formed in exactly the same way as in the free case, namely,

$$\Phi = (P - eA)^2 - \alpha\,[(P - eA)\cdot V] \approx 0 \tag{5.26}$$

with the same function α as before. The canonical Poisson brackets among x^μ, P_μ, and Q are also, of course, unaffected by the presence of interaction. More to the point, we may now again impose the gauge constraint $\chi = x^0 - s \approx 0$, and then we find these consequences: (i) the Dirac bracket resulting in elimination of Φ and χ,

$$\{f,g\}^* = \{f,g\} + (2\Pi^0 - V^0\alpha'(\Pi\cdot V))^{-1}$$
$$\times[\{f,x^0\}\{\Phi,g\} - \{f,\Phi\}\{x^0,g\}]\,, \tag{5.27}$$

leads to the interaction-independent values (4.26) for the independent Dirac brackets; (ii) the quantity that now acts as the Hamiltonian is

$$H' = eA^0(\vec{x},t) + H(V^0, \vec{V}\cdot\vec{\Pi}, \vec{\Pi}\cdot\vec{\Pi})\ . \tag{5.28}$$

Thus we have obtained the result stated in Sec. IV, that the description of the free spinor model as given by us is the right one to account for the electromagnetic interaction via the minimal-coupling principle. It is also reassuring that in this model the quantum kinematics is unaffected by the interaction. The same commutation relations as in the free case, namely, (5.18), may again be postulated; and provided factor-ordering problems can be solved we have a valid Hamiltonian operator for setting up the Schrödinger equation.

The Hamiltonian (5.18) is interesting in that, due to the presence of the internal vectorial object $\vec{V}$, one can generate both orbital and intrinsic magnetic moments for all the states on the trajectory described by the free system. In fact, in principle the moments for the different spin states are interrelated since everything is determined by the single "trajectory function" α. This feature of producing magnetic moments via minimal electromagnetic interaction (similar to what happens for the Dirac equation) is missing in the top model of Regge and Hanson. This is because their Dirac Hamiltonian [their equation (3.16)] does not contain the term $P_\mu S^{\mu\nu}$, which plays a role similar to $P_\mu V^\mu$, but which in their case is equal to zero by a constraint.

Two final remarks. First, the model can be generalized so that it produces the spectrum of the string model.[16] To do this one replaces the operator V^μ, of Sec. V B based on a degenerate pair of oscillators by a sum of such operators, each having its own degenerate pair of oscillators: $V^\mu \to \sum_{N=1}^{\infty} N V^{\mu(N)}$, where the label N refers to the Nth pair of oscillators.

Second, another straightforward generalization gives a fully relativistic quantum-mechanical SU(6) model with a minimal electromagnetic interaction. This model is obtained from that of Secs. V B and V C by replacing V^μ by a degenerate triplet: $V^\mu \to \sum_{N=1}^{3} V^{\mu(N)}$. One now has six degenerate harmonic-oscillator variables and the mass is an SU(6) singlet, whereas the multiplets contain the familiar states with different spin values. For a somewhat similar case this is discussed in Ref. 17 for no interaction. The electromagnetic interaction is again by minimal substitution: $P^\mu \to P^\mu + iQA^\mu$, where now, however, Q is an operator which has eigenvalues $\frac{2}{3}$ for all quanta $N=1$, $-\frac{1}{3}$ for all quanta $N=2$ and quanta $N=3$. Thus, labeling the total number of the quanta of the three pairs of oscillators n_1, n_2, n_3, one has for the charge $Q=\frac{2}{3}n_1-\frac{1}{3}n_2-\frac{1}{3}n_3$, whereas the rest mass is given as a function of the sum $(n_1+n_2+n_3)$.

A more detailed investigation of these questions is in progress and will be reported elsewhere.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation and by the Department of Energy.

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