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# Parity violation in metric-torsion theories of gravitation

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The general structure of metric-torsion theories of gravitation is shown to allow a parity-violating contribution to the complete action which is linear in the curvature tensor and vanishes identically in the absence of torsion. The resulting action involves apart from the Newtonian constant a coupling which governs the strength of the predicted parity-nonconserving "interactions" mediated by torsion. We consider this theory in the presence of the Proca field and show that it leads to a parity-violating term in the field equations in contrast to the Einstein-Cartan-Sciama-Kibble theory, which we use as a particularly simple example of a metric-torsion theory of gravitation.

## I. INTRODUCTION

In 1922, soon after Einstein[1] put forward his theory of gravitation based on a pseudo-Riemannian metric of space-time, Cartan[2] proposed a generalization which endowed the space-time manifold with a nonsymmetric connection. Cartan's extension of general relativity was later rediscovered by Weyl[3] and identified as a gauge theory of the Poincaré group by Sciama[4] and Kibble[5] who generalized the earlier attempt of Utiyama.[6] This theory[7] pictures space-time as a four-dimensional manifold with a symmetric tensor ($g_{\mu\nu}$) and a linear connection $\Gamma^{\alpha}_{\mu\nu}$ which is compatible with the metric (covariant derivative of $g_{\mu\nu}$ is zero) but not, in general, symmetric ($\Gamma^{\alpha}_{\mu\nu} \neq \Gamma^{\alpha}_{\nu\mu}$). In this paper we shall consider theories based on such connections and shall motivate a new Lagrangian involving a parity-violating contribution constructed from the pseudotensor density $\epsilon^{\mu\nu\alpha\rho}$ and the curvature tensor $R_{\mu\nu\alpha\rho}$. The complete action is still restricted to be linear in the curvature but leads to new parity-violating effects in the presence of torsion not present in the Einstein-Cartan-Sciama-Kibble (ECSK) theory based as it is on a Lagrangian constructed simply from the curvature scalar. The analog of the additional term our action involves has been considered before[8] for the pure Einstein theory but is known there to vanish identically.

The Lagrangian density we propose for this theory can be written as

$$\mathcal{L}_G = \mathcal{L}_{\text{ECSK}} + \mathcal{L}_A , \tag{1}$$

where $\mathcal{L}_{\text{ECSK}}$ is the usual expression for the ECSK theory (involving the Newtonian coupling constant) and $\mathcal{L}_A$ [$\sim\epsilon^{\mu\nu\alpha\rho}R_{\mu\nu\alpha\rho}$ (Ref. 9)] is the additional contribution that we motivate in the next section.

The standard procedure for accommodating torsion into Einstein's theory is to work only with $\mathcal{L}_{\text{ECSK}}$ and does not require the introduction of any new coupling constants. For $\mathcal{L}_G$, however, an additional coupling is seen to be necessary and it governs the strength of the parity-violating interactions "mediated" by torsion.

The Lagrangian we propose for our theory still involves torsion in an algebraic form since it does not contain any terms involving derivatives of the contortion (once some total divergences have been removed)—as indeed must be the case for all theories based on Lagrangians linear in the curvature tensor. One consequence of this is that for a theory based on $\mathcal{L}_G$ torsion again vanishes by virtue of the field equations in the absence of matter—as is the case for the ECSK theory. However, if we accept the view that torsion is the geometric analog of spin just as curvature represents mass and if we accept that the graviton is a spin-2 particle, then we may reasonably require that some form of dynamic torsion be present even in the absence of matter. This "vacuum torsion" would, in some sense, represent the spin effects due to gravitation. We consider possible ways of achieving this while still restricting ourselves to Lagrangians that are linear in the curvature tensor. This leads us to examine a very restricted but dynamic (in a sense which will become apparent later) type of torsion which has been motivated also from totally different points of view in other works.[10]

## II. THE NEW LAGRANGIAN

Let us begin by outlining the usual considerations[8] which lead, for the pure Einstein case, to the unique (up to a cosmical term) Lagrangian density[11]

$$\mathcal{L}_E \sim \sqrt{-g}\tilde{R}\,. \tag{2}$$

The proof of this begins by noting that the Riemann-Christoffel tensor

$$\tilde{R}_{\mu\nu\lambda}{}^{\kappa} = \tilde{\Gamma}^{\kappa}_{\nu\lambda,\mu} - \tilde{\Gamma}^{\kappa}_{\mu\lambda,\nu} + \tilde{\Gamma}^{\kappa}_{\mu\sigma}\tilde{\Gamma}^{\sigma}_{\nu\lambda} - \tilde{\Gamma}^{\kappa}_{\nu\sigma}\tilde{\Gamma}^{\sigma}_{\mu\lambda} \tag{3}$$

is the only tensor that can be constructed from the metric tensor and its first and second derivatives and which is linear in the second derivatives. This tensor is, therefore, the simplest object at our disposal when we come to write down an action for gravity. We must now begin to contract indices and construct all possible scalars linear in the curvature from $\tilde{R}_{\mu\nu\lambda\sigma}$ $(=g_{\sigma\kappa}\tilde{R}_{\mu\nu\lambda}{}^{\kappa})$. The most general Lagrangian would then just be a sum of these scalars with appropriate couplings.

It turns out[8] that for Einstein's theory only two such scalars can be constructed. However, one of them [$\sim\eta^{\mu\nu\lambda\sigma}\tilde{R}_{\mu\nu\lambda\sigma}$ (Ref. 12)] vanishes identically, and thus only the scalar $\tilde{R}$ $(=g^{\mu\nu}\tilde{R}^{\lambda}{}_{\mu\nu\lambda})$ remains.

The generalization to the case wherein torsion is present begins with the curvature tensor formed out of the nonsymmetric connection $\Gamma^{\alpha}_{\mu\nu}$ (the antisymmetric part of which is the torsion). The expression for this is as in (3) with the tildes removed. It is immediately clear from this definition that the curvature tensor is antisymmetric in its first two indices. In the general case (i.e., without any assumptions of metricity, etc.) this is the only[13] symmetry property of $R_{\mu\nu\alpha}{}^{\beta}$. If we demand metricity, we gain, in addition, antisymmetry in the last two indices.

These two antisymmetry properties are sufficient to ensure that the Ricci tensor $(R_{\nu\lambda}\equiv R_{\mu\nu\lambda}{}^{\mu})$ and the Ricci scalar $(R\equiv R_{\nu}{}^{\nu})$ are the only essential contractions of $R_{\mu\nu\lambda}{}^{\kappa}$.

Now we come to the important question of whether we can form a nonzero scalar using the pseudotensor density $\epsilon^{\mu\nu\lambda\sigma}$.

Recall that the scalar so constructed in the Einstein case vanished identically by virtue of the cyclicity property of $\tilde{R}_{\mu\nu\alpha\beta}$. When torsion is present, no such relation holds and so the scalar

$$\eta^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} \tag{4}$$

is a perfectly good quantity which can contribute to the total action of a metric-torsion theory. Indeed, the general structure of such theories allows this term and, therefore, allows parity-violating "interactions."

The new Lagrangian density may therefore be written as

$$\mathcal{L}_G = \frac{1}{16\pi G_N}\sqrt{-g}\,R + \frac{1}{16\pi G_P}\epsilon^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}\,, \tag{5}$$

where $G_N$ is the Newtonian coupling constant and $G_P$ is the analogous quantity which governs the strength of the parity-nonconserving interactions present in $\mathcal{L}_G$.[14]

In the next section we simplify the form of this expression, and compare and contrast this action with the one used in the ECSK theory. There, we will find that when we have removed some total divergences the Lagrangian contains, apart from the simple Einstein expression, terms quadratic in $K$. Since $K$ is a tensor, one might consider the most general quadratic expression in $K$ as forming the Lagrangian for contortion. This is also discussed in some detail.

## III. OTHER LAGRANGIANS

Consider the Lagrangian densities

$$\mathcal{L}_{\rm ECSK} \sim \sqrt{-g}R \tag{6}$$

and

$$\mathcal{L}_A \sim \epsilon^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}\,. \tag{7}$$

$\Gamma^{\lambda}_{\mu\nu}$ are the components of the nonsymmetric connection and can, upon imposition of metricity, be written as

$$\Gamma^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu} - K_{\mu\nu}{}^{\lambda}\,, \tag{8}$$

where

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \tfrac{1}{2}g^{\lambda\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})\,, \tag{9}$$

$$K_{\mu\nu}{}^{\lambda} = -S_{\mu\nu}{}^{\lambda} + S_{\nu}{}^{\lambda}{}_{\mu} - S^{\lambda}{}_{\mu\nu} = -K_{\mu}{}^{\lambda}{}_{\nu}\,, \tag{10}$$

and

$$S_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}_{[\mu\nu]} = \tfrac{1}{2}(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}) \tag{11}$$

is the torsion. Note that the position of the indices is important and we work with the usual[15] convention that in all covariant derivatives the first of the lower indices on the $\Gamma$'s is the differentiating index.

In order to obtain the field equations we must choose a suitable set of independent fields for variational purposes. Because of metricity

$$g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^{\tau}_{\lambda\mu}g_{\tau\nu} - \Gamma^{\tau}_{\lambda\nu}g_{\mu\tau} = 0 \tag{12}$$

we have, apart from the ten $g_{\mu\nu}$'s, another 24 independent components in $\Gamma^{\lambda}_{\mu\nu}$. We shall take these to be the 24 components of the contortion tensor $K_{\mu\nu}{}^{\lambda}$.

Before plunging ourselves into variation of the Lagrangians written above, it is advisable to obtain their simplest form using the symmetry properties of $g_{\mu\nu}$, $K_{\mu\nu}{}^{\lambda}$, etc. Thus we obtain

$$\mathcal{L}_{\rm ECSK} \sim \sqrt{-g}[\tilde{R} + g^{\mu\lambda}(K_{\nu\sigma}{}^{\nu}K_{\mu\lambda}{}^{\sigma} - K_{\mu\sigma}{}^{\nu}K_{\nu\lambda}{}^{\sigma})] + (\text{total divergence})\,, \tag{13}$$

which may be written as

$$\mathcal{L}_{\text{ECSK}} \sim \mathcal{L}_B + \mathcal{L}_C + (\text{total divergence}) \quad (14)$$

and

$$\mathcal{L}_A \sim \epsilon^{\mu\nu\lambda\beta} g_{\beta\kappa} K_{\mu\sigma}{}^{\kappa} K_{\nu\lambda}{}^{\sigma} + (\text{total divergence}). \quad (15)$$

We see that the contortion terms enter both Lagrangians quadratically and that no derivatives of $K$ appear anywhere once total divergences are removed. This is a general consequence of restriction to theories linear in $R$ and is, therefore, unchanged even with the addition of $\mathcal{L}_A$. Stated differently, if we use a linear combination of $\mathcal{L}_A$ and $\mathcal{L}_{\text{ECSK}}$ as the Lagrangian density of our system, we will not obtain propagating torsion.

The interesting thing to note, however, is that the effective contribution of contortion to $\mathcal{L}_{\text{ECSK}}$ is a particular linear combination of two of the three possible scalars quadratic in $K$ (contracting with $g_{\mu\nu}$), the third being $K_{\nu\sigma}{}^{\lambda} K^{\nu\sigma}{}_{\lambda}$. One may, at this point, argue that an equally valid approach to determine an action for the torsion would be to consider all possible linear combinations of quadratics in $K$ and simply add these to $\mathcal{L}_B$. Such an approach would, however, necessitate introduction of at least three other arbitrary parameters.

As regards $\mathcal{L}_A$ one can also think of three other scalars quadratic in $K$ (contracting with $\epsilon$ and $g$) apart from the one selected by $\mathcal{L}_A$, namely, $\eta^{\mu\nu\lambda\sigma} K_{\alpha}{}^{\alpha}{}_{\mu} K_{\nu\lambda\sigma}$, $\eta^{\mu\nu\lambda\sigma} K_{\alpha\mu\nu} K_{\lambda\sigma}{}^{\alpha}$, and $\eta^{\mu\nu\lambda\sigma} K_{\alpha\mu\nu} K^{\alpha}{}_{\lambda\sigma}$. Thus, the most general such Lagrangian density for torsion would contain seven contractions all with different and arbitrary coefficients. In view of this it seems much simpler, and indeed more natural, to restrict oneself to Lagrangians obtained directly by contracting $R_{\mu\nu\lambda\sigma}$ in all possible ways to form a scalar.

## IV. NO MATTER, NO TORSION

In this section we consider the contortion field equations for both the ECSK and the new action in the absence of matter. We have already shown that $\mathcal{L}_{\text{ECSK}}$ can be decomposed as in (13).

Taking $g_{\mu\nu}$ and $K_{\alpha\beta}{}^{\sigma}$ as our independent variables, the field equations obtained by the $K$ variation are

$$K^{\lambda\mu\nu} + K^{\nu\lambda\mu} - K_{\alpha}{}^{\alpha\nu} g^{\mu\lambda} - K_{\alpha}{}^{\lambda\alpha} g^{\mu\nu} = 0 . \quad (16)$$

These are 24 equations because of the antisymmetry property of $K$. Contracting $\mu$ and $\lambda$ (or $\nu$ and $\lambda$) gives

$$K_{\nu}{}^{\nu}{}_{\beta} = 0 .$$

Using this in (16) we obtain

$$K^{\lambda\mu\nu} + K^{\nu\lambda\mu} = 0 . \quad (17)$$

By cyclically permuting (17) we get the two equations

$$K^{\mu\nu\lambda} + K^{\lambda\mu\nu} = 0 \quad (18)$$

and

$$K^{\nu\lambda\mu} + K^{\mu\nu\lambda} = 0 . \quad (19)$$

Adding (17), (18), and (19) and using (19) to simplify the sum, one can easily verify that $K^{\lambda\mu\nu} = 0$.

The same calculation can be repeated for the theory based on $\mathcal{L}_G$. The analog of Eq. (16) now reads

$$K^{\lambda\mu\nu} + K^{\nu\lambda\mu} - K_{\alpha}{}^{\alpha\nu} g^{\mu\lambda} - K_{\alpha}{}^{\lambda\alpha} g^{\mu\nu} - 2a(\eta^{\mu\nu\sigma\rho} K_{\sigma}{}^{\lambda}{}_{\rho} + \eta^{\mu\lambda\sigma\rho} K_{\sigma\rho}{}^{\nu}) = 0 . \quad (20)$$

After a certain amount of tedious algebra and index manipulation, one can again explicitly verify the result that torsion vanishes in the absence of matter. These results follow in fact from quite general considerations as outlined below.

If one has a Lagrangian which involves the contortion fields in a nondynamic manner (no second derivatives of $K$, or equivalently, terms quadratic in the derivatives of $K$), stationarity under $K$ variations will give an algebraic equation for $K$ that can, in principle, be solved for $K$. The solution of this equation must then be expressible in terms of the other quantities in the theory. In our case we have at our disposal only $g_{\mu\nu}$, $\epsilon_{\mu\nu\alpha\beta}$, $g_{\mu\nu,\alpha}$, and $g_{\mu\nu,\alpha\beta}$ out of which we must construct a three-index tensor.

It is clear, since the process of contraction always removes two indices, that no such object can be formed from $g_{\mu\nu}$, $g_{\mu\nu,\alpha\beta}$, and $\epsilon_{\mu\nu\alpha\beta}$ only. Thus, the $g_{\mu\nu,\alpha}$ must enter each term of the expression for $K_{\alpha\beta\gamma}$. But we can always choose a coordinate system where $g_{\mu\nu,\alpha} = 0$ since the partial derivative of the metric is not a tensor. Thus, $K$ will vanish in this coordinate system and (by virtue of its tensorial character), in all coordinate systems. It should be noted, of course, that we cannot use $g_{\mu\nu;\alpha}$ since this vanishes because of metricity.

It is clear, therefore, that both the (matter-free) theories are identical to Einstein's theory of general relativity.[16] As long as torsion is algebraic, this identity between the two matter-free theories will remain.

However, as remarked in the Introduction, it is reasonable to expect torsion to be nonzero in vacuum and to represent the spin effects of gravitation.

In Sec. V we go on to consider possible ways of implementing these ideas.

## V. DYNAMIC TORSION AND FIELD EQUATIONS

We now wish to consider possible ways of incorporating dynamic torsion into the matter-free theory. One approach is to work with quadratic $R$ Lagrangians but these lead to rather cumbersome equations. Another approach consists essentially in adding to $\mathcal{L}_{\mathrm{ECSK}}$ a Lagrangian density quadratic in the covariant derivatives of $K$. However, the most general such Lagrangian density, $\mathcal{L}_K$, would contain an enormous[17] number of independent terms (see the Appendix for its explicit form) involving an equally large number of arbitrary parameters and would be quite useless unless one is able to eliminate most of these terms on some physical grounds—and this seems unlikely. So how else can one have dynamic torsion?

Recall that torsion vanished by virtue of the field equations essentially because of the nonexistence in the theory of an odd-index object using which we could construct a three-index tensor. Since torsion itself is represented by a three-index tensor, the simplest possibility for having nonzero torsion is to allow for a new one-index field in the theory in terms of which $K$ can be expressed. Coupled with the requirement that this new field be dynamical we are led to examine the following form[18] for the contortion:

$$K_{\alpha\beta\gamma}=\phi_\beta g_{\alpha\gamma}-\phi_\gamma g_{\alpha\beta}\,, \tag{21}$$

where $\phi_\alpha \equiv \phi_{,\alpha}$. Note that the form of $K$ is the same as has been motivated also in other works on completely different grounds.[10]

We may now proceed in two different ways. One is to simply substitute the motivated form of contortion into $\mathcal{L}_G$, eliminate $K$, and obtain the field equations for $g_{\mu\nu}$ and $\phi$ by variation. We prefer to avoid this approach and consider it more appropriate to treat (21) as a constraint which will be implemented by introducing an appropriate set of Lagrange multipliers into $\mathcal{L}_G$. Let us, therefore, consider the following Lagrangian density:

$$\begin{aligned}\mathcal{L}=\sqrt{-g}[\tilde{R}&+g^{\mu\lambda}(K_{\nu\sigma}{}^{\nu}K_{\mu\lambda}{}^{\sigma}-K_{\mu\sigma}{}^{\nu}K_{\nu\lambda}{}^{\sigma})]\\&+a\epsilon^{\mu\nu\lambda\beta}g_{\beta\kappa}K_{\mu\sigma}{}^{\kappa}K_{\nu\lambda}{}^{\sigma}\\&+\Lambda^{\alpha\beta}{}_{\gamma}(K_{\alpha\beta}{}^{\gamma}-\phi_\beta\delta^\gamma_\alpha+\phi_\lambda g^{\lambda\gamma}g_{\alpha\beta})\,,\end{aligned} \tag{22}$$

where the $\Lambda^{\alpha\beta}{}_{\gamma}$ are the Lagrange multipliers introduced to ensure satisfaction of (21). Variations with respect to $g_{\mu\nu}$, $K_{\alpha\beta}{}^{\gamma}$, and $\phi$ yield the following equations:

$$\begin{aligned}\delta g:\ \ &\sqrt{-g}(\tilde{R}^{ab}-\tfrac{1}{2}\tilde{R}g^{ab})+\tfrac{1}{2}\sqrt{-g}(g^{ab}g^{\mu\lambda}-g^{\mu a}g^{\lambda b}-g^{\mu b}g^{\lambda a})(K_{\mu\sigma}{}^{\nu}K_{\nu\lambda}{}^{\sigma}-K_{\nu\sigma}{}^{\nu}K_{\mu\lambda}{}^{\sigma})-\tfrac{1}{2}(\Lambda^{\alpha}{}_{\alpha}{}^{a}\phi^{b}+\Lambda^{\alpha}{}_{\alpha}{}^{b}\phi^{a})\\&\qquad+\tfrac{1}{2}(\Lambda^{ab\mu}+\Lambda^{ba\mu})\phi_\mu+a(\epsilon^{\mu\nu\lambda a}K_{\mu\sigma}{}^{b}K_{\nu\lambda}{}^{\sigma}+\epsilon^{\mu\nu\lambda b}K_{\mu\sigma}{}^{a}K_{\nu\lambda}{}^{a})=0\,,\\\delta K:\ \ &\Lambda^{abc}-\sqrt{-g}[(K^{cab}+K^{bca}-K_{\lambda}{}^{\lambda b}g^{ac}-K_{\lambda}{}^{c\lambda}g^{ab})+2a(\eta^{ab\nu\lambda}K_{\nu}{}^{c}{}_{\lambda}+\eta^{ac\nu\lambda}K_{\nu\lambda}{}^{b})]=0\,,\end{aligned}$$

and

$$\delta\phi:\ \ (\Lambda^{\alpha\beta}{}_{\alpha}-\Lambda^{\alpha}{}_{\alpha}{}^{\beta})_{,\beta}=0\,,$$

while the $\Lambda$ variation yields the desired constraint. Eliminating $K$ and $\Lambda$ from the above equations we obtain the field equations for Einstein coupled to a scalar field. Note that there is no parity-violating term remaining; this is due to the special ansatz we have taken for the contortion as can be easily seen, and implies that the vacuum theory is parity conserving.

## VI. COUPLING TO MATTER FIELDS

In this section we wish to give an example where our Lagrangian predicts parity-violating effects but where the ECSK Lagrangian does not. It is unfortunate that there are not many matter fields one can study at the Lagrangian level. In consequence, when studying matter fields on a Riemann-Cartan space-time ($U_4$), we are further restricted. For example, one cannot minimally couple gauge fields to torsion[19] in a gauge-invariant manner, so the study of gauge fields on a $U_4$ does not lead to any new physics than on a Riemann space-time ($V_4$). We cannot use the Dirac field for our present purpose as the ECSK theory already predicts a parity-violating effect for this field. So we are left with the Proca (massive-vector) field, which due to its nonzero mass does not present problems of gauge (non) invariance when minimally coupled to torsion. We take the usual Lagrangian for the Proca field:

$$\mathcal{L}_m=\sqrt{-g}(-\tfrac{1}{4}G_{\mu\nu}G^{\mu\nu}+\tfrac{1}{2}m^2A_\mu A^\mu)$$

with the field strength tensor $G_{\mu\nu}$ given by

$$\begin{aligned}G_{\mu\nu}&=\nabla_\mu A_\nu-\nabla_\nu A_\mu\\&=\partial_\mu A_\nu-\partial_\nu A_\mu-2A_\sigma S_{\mu\nu}{}^{\sigma}\\&=\tilde{\nabla}_\mu A_\nu-\tilde{\nabla}_\nu A_\mu-2A_\sigma S_{\mu\nu}{}^{\sigma}\,.\end{aligned}$$

Let us define

$$B_{\mu\nu}\equiv\tilde{\nabla}_\mu A_\nu-\tilde{\nabla}_\nu A_\mu\,,$$

then

$$G_{\mu\nu}=B_{\mu\nu}-2A_\sigma S_{\mu\nu}{}^{\sigma}$$

and $\mathcal{L}_m$ can be written as

$$\mathcal{L}_m=\sqrt{-g}\left(-\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu}+B^{\alpha\beta}S_{\alpha\beta}{}^{\sigma}A_\sigma -g^{\mu\alpha}g^{\nu\beta}S_{\alpha\beta}{}^{\sigma}S_{\mu\nu}{}^{\rho}A_\sigma A_\rho+\tfrac{1}{2}m^2A_\mu A^\mu\right). \tag{23}$$

The spin angular momentum tensor is defined by

$$\sqrt{-g}\,\tau_k{}^{ji}\equiv\frac{\delta\mathcal{L}_m}{\delta K_{ij}{}^k},$$

where $\delta/\delta K$ denotes the variational derivative.

For the Lagrangian in (23) it has been shown[7] that

$$\tau_{kji}=G_{i[j}A_{k]}$$

or

$$\tau_{kji}=A_k\tilde{\nabla}_{[j}A_{i]}-A_j\tilde{\nabla}_{[k}A_{i]}+2S_{i[j}{}^{\sigma}A_{k]}A_\sigma .$$

Let us write for the total Lagrangian

$$\mathcal{L}=\mathcal{L}_{\text{ECSK}}+\mathcal{L}_A+\mathcal{L}_m .$$

As $\mathcal{L}$ does not contain any derivatives of torsion, the Euler-Lagrange equations obtained by variation of $K_{ij}{}^k$ are simply

$$\frac{\partial\mathcal{L}}{\partial K_{ij}{}^k}=0 . \tag{24}$$

And since

$$\sqrt{-g}\,\tau_k{}^{ji}=\frac{\partial\mathcal{L}_m}{\partial K_{ij}{}^k},$$

(24) gives

$$\frac{\partial\mathcal{L}_{\text{ECSK}}}{\partial K_{ij}{}^k}+\frac{\partial\mathcal{L}_A}{\partial K_{ij}{}^k}=-\tau_k{}^{ji}\sqrt{-g}. \tag{25}$$

From Ref. 7 we see again that

$$\frac{\partial\mathcal{L}_{\text{ECSK}}}{\partial K_{ij}{}^k}=\sqrt{-g}\,T^j{}_k{}^i ,$$

where $T_{ijk}$ is the so-called modified torsion tensor, defined as

$$T_{ijk}=S_{ijk}+g_{ik}S_{jl}{}^l-g_{jk}S_{il}{}^l . \tag{26}$$

Writing

$$\mathcal{L}_A=2a\sqrt{-g}\,\eta^{\mu\nu\lambda\beta}g_{\beta\rho}K_{\mu\sigma}{}^{\rho}K_{\nu\lambda}{}^{\sigma},$$

it is not difficult to show that

$$\frac{\partial\mathcal{L}_A}{\partial K_{ij}{}^k}=2a\sqrt{-g}\left(\eta^{i\nu\lambda}{}_kK_{\nu\lambda}{}^j+\eta^{i\nu\lambda j}K_{\nu k\lambda}\right).$$

Therefore, Eq. (25) finally gives

$$-\sqrt{-g}\,T_k{}^{ji}+\sqrt{-g}\,\tau_k{}^{ji}+2a\sqrt{-g}\left(\eta^{i\nu\lambda}{}_kK_{\nu\lambda}{}^j+\eta^{i\nu\lambda j}K_{\nu k\lambda}\right)=0$$

or

$$T_{kji}=\tau_{kji}+2a\eta^{\nu\lambda}{}_{i\sigma}K_{\nu\lambda\rho}\left(\delta_k^\sigma\delta_j^\rho-\delta_j^\sigma\delta_k^\rho\right). \tag{27}$$

As the Proca field is simply a massive Maxwell field, the field equations can be written down as

$$\nabla_\rho G_\mu{}^\rho+m^2A_\mu=0$$

or

$$\nabla_\rho B_\mu{}^\rho-2\nabla_\rho(A_\sigma S_\mu{}^{\rho\sigma})+m^2A_\mu=0 . \tag{28}$$

In order to eliminate the non-Riemannian part of Eq. (28), we must first invert Eq. (27) for the torsion. Remembering the definitions of $T_{ijk}$ and $K_{\nu\lambda\rho}$ as given in Eqs. (26) and (10), respectively, we can write Eq. (27) as

$$S_{kji}+g_{ki}S_{jl}{}^l-g_{ji}S_{kl}{}^l =\tau_{kji}+2a\eta^{\nu\lambda}{}_{i\sigma}\delta_{kj}^{\sigma\rho}\left(S_{\lambda\rho\nu}-S_{\nu\lambda\rho}-S_{\rho\nu\lambda}\right). \tag{29}$$

Now, because of the antisymmetry of $\eta^{\nu\lambda}{}_{i\sigma}$ in $\nu\lambda$,

$$\eta^{\nu\lambda}{}_{i\sigma}S_{\lambda\rho\nu}=-\eta^{\nu\lambda}{}_{i\sigma}S_{\nu\rho\lambda},$$

and antisymmetry of $S_{\nu\rho\lambda}$ in its first two indices further implies that

$$\eta^{\nu\lambda}{}_{i\sigma}S_{\lambda\rho\nu}=\eta^{\nu\lambda}{}_{i\sigma}S_{\rho\nu\lambda} .$$

Substituting this into Eq. (27) gives

$$S_{kji}+g_{ki}S_{jl}{}^l-g_{ji}S_{kl}{}^l=\tau_{kji}-2a\eta^{\nu\lambda}{}_{i\sigma}\delta_{kj}^{\sigma\rho}S_{\nu\lambda\rho} . \tag{30}$$

At this stage we note that (setting $a=0$), the ECSK theory indeed does not predict any parity-violating effects when coupled to a massive- vector field. For the purposes of solving Eq. (27) for the torsion, we simplify Eq. (30) with the help of Eq. (26) to

$$T_{kji}=\tau_{kji}+2a\eta^{\nu\lambda}{}_{\sigma i}\delta_{kj}^{\sigma\rho}S_{\nu\lambda\rho} . \tag{31}$$

Multiplying (31) by $\eta^{kj\alpha\beta}$ gives

$$\eta^{kj\alpha\beta}T_{kji}=\tau_{kji}\eta^{kj\alpha\beta}+2a\eta^{\nu\lambda}{}_{\sigma i}\eta^{kj\alpha\beta}\delta_{kj}^{\sigma\rho}S_{\nu\lambda\rho} . \tag{32}$$

Now,

$$2\eta^{kj\alpha\beta}\delta_{kj}^{\sigma\rho}=\eta^{\sigma\rho\alpha\beta}-\eta^{\rho\sigma\alpha\beta}=2\eta^{\sigma\rho\alpha\beta}$$

and

$$2\eta^{\nu\lambda}{}_{\sigma i}\eta^{\sigma\rho\alpha\beta}=2g_{i\mu}\eta_\sigma{}^{\nu\lambda\mu}\eta^{\sigma\rho\alpha\beta}=-2g_{i\mu}\delta^{\nu\lambda\mu,\rho\alpha\beta},$$

or

$$2\eta^{\nu\lambda}{}_{\sigma i}\eta^{\sigma\rho\alpha\beta}=-2g_{i\mu}\left(g^{\nu\rho}g^{\lambda\alpha}g^{\mu\beta}-g^{\nu\rho}g^{\lambda\beta}g^{\mu\alpha}+g^{\nu\alpha}g^{\lambda\beta}g^{\mu\rho}-g^{\nu\alpha}g^{\lambda\rho}g^{\mu\beta}+g^{\nu\beta}g^{\lambda\rho}g^{\mu\alpha}-g^{\nu\beta}g^{\lambda\alpha}g^{\mu\rho}\right).$$

Therefore, we finally find that

$$2\eta^{\nu\lambda}{}_{\sigma i}\eta^{kj\alpha\beta}\delta_{kj}^{\sigma\rho}S_{\nu\lambda\rho}=2\eta^{\nu\lambda}{}_{\sigma i}\eta^{\sigma\rho\alpha\beta}S_{\nu\lambda\rho}=-2g_{i\mu}\left(S^{\rho\alpha}{}_\rho g^{\mu\beta}-S^{\rho\beta}{}_\rho g^{\mu\alpha}+S^{\alpha\beta\mu}-S^{\alpha\rho}{}_\rho g^{\mu\beta}+S^{\beta\rho}{}_\rho g^{\mu\alpha}-S^{\beta\alpha\mu}\right)$$
$$=-2g_{i\mu}\left[2\left(S^{\alpha\beta\mu}+S^{\beta}g^{\mu\alpha}-S^{\alpha}g^{\mu\beta}\right)\right]=-4T^{\alpha\beta}{}_i .$$

Substituting this result back into Eq. (32) gives

$$\eta^{kj\alpha\beta}T_{kji} + 4ag^{k\alpha}g^{j\beta}T_{kji} = \tau_{kji}\eta^{kj\alpha\beta}. \tag{33}$$

Multiplying (33) by $\eta_{\alpha\beta\rho\sigma}$ gives

$$\eta_{\alpha\beta\rho\sigma}\eta^{kj\alpha\beta}T_{kji} + 4ag^{k\alpha}g^{j\beta}\eta_{\alpha\beta\rho\sigma}T_{kji} = \tau_{kji}\eta^{kj\alpha\beta}\eta_{\alpha\beta\rho\sigma},$$

$$2\tau_{kji}\delta^{kj}_{\rho\sigma} = 2T_{kji}\delta^{kj}_{\rho\sigma} - 4a\eta^{kj}{}_{\rho\sigma}T_{kji},$$

$$\tau_{\rho\sigma i} = T_{\rho\sigma i} - 2a\eta^{kj}{}_{\rho\sigma}T_{kji},$$

or

$$T^{\alpha\beta}{}_{i} - 2a\eta^{kj\alpha\beta}T_{kji} = \tau^{\alpha\beta}{}_{i}. \tag{34}$$

From Eq. (33) we see that

$$2a\eta^{kj\alpha\beta}T_{kji} + 8a^2g^{k\alpha}g^{j\beta}T_{kji} = 2a\tau_{kji}\eta^{kj\alpha\beta}.$$

Substituting this into Eq. (34),

$$T^{\alpha\beta}{}_{i} + 8a^2T^{\alpha\beta}{}_{i} - 2a\tau_{kji}\eta^{kj\alpha\beta} = \tau^{\alpha\beta}{}_{i}$$

or

$$T_{kji}(1+8a^2) = \tau_{\alpha\beta i}(\delta^{\alpha}_{k}\delta^{\beta}_{j} + 2a\eta^{kj\alpha\beta}).$$

We have, therefore,

$$S_{kji} + g_{ki}S_{jl}{}^{l} - g_{ji}S_{kl}{}^{l} = \frac{\tau_{\alpha\beta i}}{(1+8a^2)}(\delta^{\alpha}_{k}\delta^{\beta}_{j} + 2a\eta^{kj\alpha\beta}). \tag{35}$$

Tracing over indices $i$ and $j$ yields

$$S_k = -\frac{1}{2(1+8a^2)}[\tau_k + 2a\tau_{\alpha\beta i}\eta_k{}^{i\alpha\beta}].$$

Substitution of $S_k$ into Eq. (35) finally gives the result that

$$\begin{aligned} S_{kji} = \frac{1}{1+8a^2}[&\tau_{\alpha\beta i}(\delta^{\alpha}_{k}\delta^{\beta}_{j} + 2a\eta^{\alpha\beta}{}_{kj}) \\ &- \tfrac{1}{2}g_{ij}(\tau_k + 2a\eta_k{}^{\gamma\alpha\beta}\tau_{\alpha\beta\gamma}) \\ &+ \tfrac{1}{2}g_{ik}(\tau_j + 2a\eta_j{}^{\gamma\alpha\beta}\tau_{\alpha\beta\gamma})], \end{aligned} \tag{36}$$

where we have used the abbreviations

$$K_j \equiv K_{jl}{}^{l} \text{ and } K^j = g^{jl}K_l$$

for the traces of the torsion and spin angular momentum tensors.

It is clear that upon substitution of the expression for torsion given in Eq. (36) into Eq. (28) we shall indeed have parity-violating interaction terms, which would not be present in the usual ECSK theory—thus demonstrating that the new Lagrangian proposed in this paper predicts parity-violating effects not present in the ECSK framework. In Sec. VII we conclude with a brief discussion and possible further work along these lines.

## VII. DISCUSSION

We have shown that the general structure of metric-torsion theories of gravitation predicts the presence of a parity-violating interaction mediated by torsion. This appears to us to be the most distinctive feature of this theory which could serve to distinguish it from the pure Einstein case.

In this note we have restricted our attention to two cases only. Firstly, in the vacuum case we have shown that the effects of the additional term vanish even if we allow the simple form of dynamic "vacuum torsion" we motivated. We have obtained the field equations for this theory and will consider some special solutions elsewhere. Secondly, in order to illustrate the fact that our Lagrangian may give rise to new parity-violating effects we considered coupling to a Proca field. In this case we found that indeed parity-violating effects arise where the ECSK theory would not predict them.

It is interesting to consider what modifications, if any, would arise in the theory of supergravity recently worked out if we include, apart from $R$, the $\epsilon R$ term we have motivated in the gravity Lagrangian.

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## APPENDIX

We give below the most general form for $\mathcal{L}_K$ mentioned in Sec. V. The $G_i$ are arbitrary parameters and we have only given the terms for the ordinary Einstein-Cartan theory. Allowing the additional term $\mathcal{L}_A$ the situation can only get more complicated:

$$\mathcal{L}_K = \sum_{i=1}^{16}\frac{1}{16\pi G_i}Q_i,$$

with

$$Q_1 = K^{\alpha\beta\lambda}{}_{;\sigma}K_{\lambda\alpha\beta}{}^{;\sigma}, \quad Q_9 = K^{\beta\sigma\lambda}{}_{;\alpha}K^{\alpha}{}_{\sigma\lambda;\beta},$$

$$Q_2 = K_{\alpha}{}^{\alpha\lambda}{}_{;\sigma}K_{\beta\lambda}{}^{\beta;\sigma}, \quad Q_{10} = K^{\beta\sigma\lambda}{}_{;\alpha}K_{\sigma}{}^{\alpha}{}_{\lambda;\beta},$$

$$Q_3 = K^{\alpha\beta\lambda}{}_{;\sigma}K_{\alpha\beta\lambda}{}^{;\sigma}, \quad Q_{11} = K^{\sigma\alpha\lambda}{}_{;\alpha}K_{\sigma}{}^{\beta}{}_{\lambda;\beta},$$

$$Q_4 = K^{\alpha\beta\sigma}{}_{;\alpha}K_{\lambda\sigma}{}^{\lambda}{}_{;\beta}, \quad Q_{12} = K^{\sigma\beta\lambda}{}_{;\alpha}K_{\sigma}{}^{\alpha}{}_{\lambda;\beta},$$

$$Q_5 = K^{\beta\alpha\sigma}{}_{;\alpha}K_{\lambda\sigma}{}^{\lambda}{}_{;\beta}, \quad Q_{13} = K^{\sigma\alpha\lambda}{}_{;\alpha}K_{\lambda}{}^{\beta}{}_{\sigma;\beta},$$

$$Q_6 = K^{\sigma\alpha\beta}{}_{;\alpha}K_{\lambda\sigma}{}^{\lambda}{}_{;\beta}, \quad Q_{14} = K^{\sigma\beta\lambda}{}_{;\alpha}K_{\lambda}{}^{\alpha}{}_{\sigma;\beta},$$

$$Q_7 = K^{\alpha\sigma\lambda}{}_{;\alpha}K^{\beta}{}_{\sigma\lambda;\beta}, \quad Q_{15} = K^{\sigma\alpha}{}_{\sigma;\alpha}K^{\lambda\beta}{}_{\lambda;\beta},$$

$$Q_8 = K^{\alpha\sigma\lambda}{}_{;\alpha}K_{\sigma}{}^{\beta}{}_{\lambda;\beta}, \quad Q_{16} = K^{\sigma\beta}{}_{\sigma;\alpha}K^{\lambda\alpha}{}_{\lambda;\beta}.$$

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[9]$R_{\mu\nu\alpha\beta}$ denotes the curvature tensor constructed from the full nonsymmetric connection. The analogous quantity constructed from the Levi-Civita connection shall be distinguished by a tilde; thus, $\tilde{R}_{\mu\nu\alpha\beta}$. The same notation shall be employed for other quantities also.

[10]S. Hojman, M. Rosenbaum, M. P. Ryan, and L. C. Shepley, Phys. Rev. D 17, 3141 (1978); C. Mukku and W. A. Sayed, Phys. Lett. 82B, 382 (1979).

[11]We use the conventions of Ref. 7.

[12]For convenience we define the pseudotensors $\eta^{\mu\nu\lambda\sigma}$ and $\eta_{\mu\nu\lambda\sigma}$ from the usual Levi-Civita (pseudo-) tensor densities $\epsilon^{\mu\nu\lambda\sigma}$ and $\epsilon_{\mu\nu\lambda\sigma}$ as

$$\eta^{\mu\nu\lambda\sigma}=\frac{1}{\sqrt{-g}}\,\epsilon^{\mu\nu\lambda\sigma},\quad \eta_{\mu\nu\lambda\sigma}=\sqrt{-g}\,\epsilon_{\mu\nu\lambda\sigma}.$$

They satisfy the following properties:

$$\eta^{\mu\nu\lambda\sigma}\eta_{\mu\alpha\beta\gamma}=-\delta^{\nu\lambda\sigma}_{\alpha\beta\gamma},$$

$$\eta^{\mu\nu\lambda\sigma}\eta_{\mu\nu\beta\gamma}=-2\delta^{\lambda\sigma}_{\beta\gamma},$$

$$\eta^{\mu\nu\lambda\sigma}\eta_{\mu\nu\lambda\gamma}=-6\delta^{\sigma}_{\gamma},$$

and

$$\eta^{\mu\nu\lambda\sigma}\eta_{\mu\nu\lambda\sigma}=-24,$$

where the tensor $\delta^{\mu\nu\lambda\cdots}_{\alpha\beta\gamma\cdots}$ is a generalized Kronecker symbol obeying the following rules: If $\mu,\nu,\lambda,\ldots$ are all different and $\alpha,\beta,\gamma,\ldots$ are obtained from them by a certain permutation, then it is equal to $+1$ or $-1$ depending on whether the permutation $^{\mu\nu\lambda\cdots}_{\alpha\beta\gamma\cdots}$ is even or odd; in the remaining cases it is equal to zero.

[13]See, for example, Erwin Schrödinger, *Space-Time Structure* (Cambridge University Press, London, 1963), p. 50.

[14]For later purposes we define $a=1/16\pi G_P$. For the most part we shall set $1/16\pi G_N$ equal to unity.

[15]J. A. Schouten, *Ricci Calculus*, 2nd edition (Springer, Berlin, 1954).

[16]An alternative and more general derivation is to invert the second algebraic equation relating spin to torsion. This immediately gives the result that no spin implies no torsion.

[17]After completing this work we became aware of a recent article of Michael Hovak and Pemeter Krupka [Int. J. Theor. Phys. 17, 543 (1978)] in which these authors consider the problem of finding all first-order-invariant Einstein-Cartan structures. They consider Lagrangians containing terms linear or quadratic in the following objects: $g_{ij}$, $g_{ij;k}$, $T^i_{jk}$, $S^i_{jk}$ (the latter two are in their notation the antisymmetric and symmetric parts of the connection), $R^i{}_{jkl}$, and $T^i{}_{jk;l}$. Assuming compatibility they deduce that for a four-dimensional theory there exist at most 194 such functionally independent generally invariant Lagrangians. If we allow for the use of the tensor density $\epsilon_{ijkl}$, not considered by them, the number of such independent Lagrangian structures would increase further. In view of this the new action we propose (by restricting to linear $R$ theories) seems to us to be a reasonable one.

[18]We may here point out that (21) is not the most general form for $K$ that we can write if we allow the use of $\epsilon_{ijkl}$. In fact, it is possible (while still only introducing one-index fields) to consider the following choice for $K$:

$$K_{\alpha\beta\gamma}=\phi_\beta g_{\alpha\gamma}-\phi_\gamma g_{\alpha\beta}+\epsilon_{\alpha\beta\gamma\delta}\Psi^\delta,$$

where we have introduced a pseudoscalar field $\Psi$ ($\Psi^\delta$ being $\Psi^{,\delta}$) which, like $\phi$, would be a dynamical field once we incorporate this type of $K$ into our Lagrangian. For simplicity, however, we do not consider this choice in the present note.

[19]In this paper we do not consider the modified minimal coupling for gauge fields proposed in Ref. 10.