Some exact inhomogeneous cosmologies with equation of state $p=\gamma\mu$

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A new class of exact solutions of the Einstein field equations with perfect fluid source and equation of state $p=\gamma\mu$ is presented. These solutions can be interpreted as spatially inhomogeneous cosmological models. It is shown that these models do not approach spatial homogeneity at large times or near the singularity.

I. INTRODUCTION

Spatially homogeneous cosmologies have been studied in great detail over the past 15 years (e.g., Ellis and MacCallum[1] and Ryan and Shepley[2]). The real universe is not exactly spatially homogeneous, however, and may not have been even approximately so at early times. Thus in order to understand the evolution of the universe one has also to consider spatially inhomogeneous models, and over the past five years relativists have studied various aspects of such models (e.g., Szekeres,[3] Liang,[4] Centrella and Matzner,[5] and Collins and Szafron[6]; see MacCallum[7] for a recent survey).

This paper deals with exact solutions of the Einstein field equations with *irrotational* perfect fluid source, which can be interpreted as *spatially inhomogeneous* cosmologies. By "spatially inhomogeneous" we mean that the dimension of the orbits of the maximal group of local isometries is less than or equal to 2. This means in particular that the hypersurfaces orthogonal to the fluid flow are not the orbits of a local group of isometries, as in the Bianchi models.[1] Relatively few exact solutions of this type are known. The known solutions either admit a local group of isometries with two-dimensional orbits or are algebraically special.[8] A more serious restriction is that the only equations of state of the form

$$p=\gamma\mu\ ,\quad \gamma=\text{constant} \tag{1.1}$$

where μ is the density and p the pressure of the fluid, are $\gamma=0$, i.e., dust (see for example Refs. 3 and 9), and $\gamma=1$, i.e., stiff matter (see for example Refs. 10 and 11). In this note we present a simple class of spatially inhomogeneous solutions with two commuting Killing vector fields, which permit an equation of state of the form (1.1) with $0<\gamma<1$.

The solutions are given in Sec. II and their curvature singularities are described in Sec. III. In Sec. IV we discuss the intrinsic and extrinsic geometry of the hypersurfaces orthogonal to the fluid flow and compare the results to other spatially inhomogeneous solutions. Section V deals with the asymptotic behavior of the solutions, i.e., near the big bang, and at late times. We note that the solutions were derived and the various invariants were calculated using a library of programs[12] written in the algebraic computing language CAMAL.[13]

II. THE SOLUTIONS

The solutions were derived by assuming the existence of two commuting Killing vector fields (KVF's) ξ and η which are hypersurface orthogonal and orthogonal to each other and to the fluid four-velocity u. This implies the existence of local coordinates (t,x,y,z) such that[8]

$$ds^2=-e^{2k}dt^2+e^{2h}dx^2+r(fdy^2+f^{-1}dz^2)\ ,$$
$$u=e^{-k}\frac{\partial}{\partial t}\ .$$

The Killing vector fields have the form

$$\xi=\frac{\partial}{\partial y}\ ,\quad \eta=\frac{\partial}{\partial z}$$

so that the functions k,h,r,f are independent of y and z. The Einstein field equations

$$G_{ab}=8\pi[(\mu+p)u_a u_b+pg_{ab}]$$

were solved by setting $h=k$ and assuming that each metric component was a product of a function of t and a function of x.

The line element is[14]

$$ds^2=S^{-2m}C^{-2m-2}(-dt^2+dx^2)$$
$$+SC^{\alpha}(T^n dy^2+T^{-n}dz^2)\ , \tag{2.1}$$

where

$$S=\sinh 2qt\ ,$$
$$C=\cosh(2qx/\alpha)\ ,$$
$$T=\tanh qt\ ,$$
$$\alpha=-(2m+2)/(2m+1)\ ,$$

and q,m,n are constants, with $q\neq 0$, $m\neq -1$, $-\frac{1}{2}$. The fluid density and pressure are

$$8\pi\mu=q^2S^{2m}C^{2m}[(1-4m-n^2)C^2S^{-2}$$
$$-(4m+3)\gamma^{-1}]\ , \tag{2.2a}$$

$$8\pi p = q^2 S^{2m} C^{2m}[(1-4m-n^2)C^2 S^{-2} - (4m+3)] \, , \tag{2.2b}$$

where

$$\gamma = (m+1)/(m-1) \, . \tag{2.3}$$

The fluid four-velocity is

$$u = S^m C^{m+1} \frac{\partial}{\partial t} \, .$$

The coordinate ranges are taken to be

$$0 < t < \infty \, , \quad -\infty < x, y, z < \infty \, . \tag{2.4}$$

It follows from Eqs. (2.2) that the pressure and density satisfy

$$0 \leq p \leq \mu \tag{2.5}$$

over the whole spacetime if and only if the constants m, n satisfy

$$1 - 4m - n^2 \geq 0 \, , \quad m < -1 \tag{2.6}$$

or

$$4m + 3 = 0 \, , \quad n^2 < 4 \, . \tag{2.7}$$

In particular, *if equality holds in the first of the inequalities (2.6), we obtain an equation of state of the form (1.1),* with γ, as given by (2.3), subject to $0 < \gamma < 1$. If (2.7) holds, we have stiff matter, i.e., $p = \mu > 0$. There are no dust solutions.

In general, these solutions only admit a two-parameter Abelian group of local isometries, generated by the KVF's $\xi = \partial/\partial y$ and $\eta = \partial/\partial z$. A third KVF $\zeta = y\partial/\partial x - x\partial/\partial y$ exists if and only if $n = 0$, and in this case the solutions are locally rotationally symmetric (LRS), in class II, with $K = 0$, in the paper of Stewart and Ellis.[15] There are no other possible KVF's.[16] Thus *all members of the class are spatially inhomogeneous.* We note that there is a discrete isometry defined by $x \to -x$. We have also verified that the solutions are not self-similar.[17]

III. CURVATURE SINGULARITIES

With a view to studying the singularities of the solutions, we now give the curvature scalars constructed from the Weyl tensor. In order to do this, it is convenient to introduce a null tetrad and calculate the Newman-Penrose[18] complex null tetrad components ψ_A of the Weyl tensor. The natural orthonormal frame defined by the line element (2.1) is $e_{(0)} = u$ and $e_{(1)}, e_{(2)}, e_{(3)}$ parallel to the vector fields $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$, respectively. In terms of this frame we define a null tetrad by

$$l = \frac{1}{\sqrt{2}}[e_{(0)} + e_{(1)}] \, , \quad n = \frac{1}{\sqrt{2}}[e_{(0)} - e_{(1)}] \, ,$$

$$m = \frac{1}{\sqrt{2}}[e_{(2)} + i e_{(3)}] \, .$$

The nonzero ψ_A are given by

$$\begin{aligned} \psi_0 &= -2mnq^2 X^2 S^{-1}[\coth(2qt) - \tanh(2qx/\alpha)] \, , \\ \psi_4 &= -2mnq^2 X^2 S^{-1}[\coth(2qt) + \tanh(2qx/\alpha)] \, , \\ \psi_2 &= \tfrac{1}{3} q^2 X^2 [(2m/\alpha) C^{-2} + (n^2 - 2m - 1) S^{-2}] \, , \end{aligned} \tag{3.1}$$

where

$$X = S^m C^{m+1} \, .$$

It can be shown, using the formulas in Ref. 19, that the Weyl tensor is in general of Petrov type I, and is of Petrov type D if and only if $mn = 0$. If $m = 0$, the present solution is equivalent to a solution found by Allnutt[20] in a systematic search for type-D perfect-fluid solutions.

The Weyl curvature scalars can be expressed[21] in terms of the ψ_A, for example,

$$C_{abcd} C^{abcd} = 16(3\psi_2{}^2 + \psi_0 \psi_4) \, , \quad C_{abcd} C^{*abcd} = 0 \, ,$$

since in this case the ψ_A are real. The required information about the singularities can, however, be obtained directly from ψ_2 and the product $\psi_0\psi_4$. The Ricci curvature scalars are constructed from μ and p, and hence their behavior can be inferred from the expressions (2.2) for μ and p.

We now discuss the singularities of the solutions. There are two classes of solutions which satisfy the physically reasonable requirement (2.5), and they are defined by the restrictions (2.6) and (2.7) on the parameters m and n. In both cases the metric is regular [i.e., the g_{ij} are of class C^3 and $\det(g_{ij}) \neq 0$] over the coordinate ranges (2.4), but the Ricci and Weyl curvature scalars become infinite as $t \to 0^+$. Now t does not measure time elapsed along the fluid flow lines since $g_{00} \neq -1$. However, the integral

$$\int_0^{t_0} S^{-m} C^{-m-1} dt$$

which measures the time elapsed from $t = 0$ to $t = t_0$ is finite for $m < -1$ or $4m + 3 = 0$, for any fixed value of x. Thus *both classes of solutions have a big-bang singularity in the finite past.*

A significant difference emerges when one considers the dependence on the spatial variable x. For (2.6) the curvature scalars are bounded on the set $t \geq t_0$, for any $t_0 > 0$, i.e., away from the big-bang singularity, while for (2.7) the curvature scalars become infinite as $x \to \pm\infty$, *for any t.* One should, however, consider whether $x \to \pm\infty$ corresponds to infinite spatial distance in terms of the metric. It follows from (2.1) that the x-coordinate curves, i.e., the curves $t, y, z=$ const, have infinite length if and only if $m<-1$, which is satisfied in case (2.6) but not (2.7). Thus when (2.7) holds, i.e., when $p=\mu$, *a curvature singularity can be reached in a finite distance along spacelike curves.* We have not been able to establish whether there are incomplete geodesics which terminate at these curvature singularities.

IV. INTRINSIC AND EXTRINSIC GEOMETRY OF THE SLICES

The spatial inhomogeneity of an irrotational cosmological model can be classified by means of the intrinsic and extrinsic geometry of the spacelike hypersurfaces (slices) orthogonal to the fluid flow.[22] We now discuss the solutions from this point of view. Firstly, the slices are conformally flat, as can be seen by inspection of the line element.

Thus the Cotton-York tensor of the metric induced on the slices is zero.[22] The slices are not flat, however. We give the components of the induced Ricci tensor on the slices relative to the natural orthonormal frame defined in Sec. III. The nonzero components are

$$\begin{aligned} R^*_{11} &= 4q^2[m + C^{-2}(-2m^2+1)/(2m+2)]X^2 , \\ R^*_{22} &= R^*_{33} \\ &= 2q^2[(m-\tfrac{1}{2}) + C^{-2}(-2m^2+m+2)/(2m+2)]X^2 , \end{aligned} \tag{4.1}$$

where

$$X = S^m C^{m+1} .$$

Thus as regards the intrinsic geometry of the slices, these solutions are analogous to the Szekeres[3] inhomogeneous solutions—the slices are conformally flat and the induced Ricci tensor has an eigenvalue of multiplicity 2.

As regards the extrinsic geometry of the slices or, equivalently, the kinematical quantites of the fluid, we find that the nonzero frame components of the expansion tensor are

$$\begin{aligned} \theta_{11} &= -2mqX \coth 2qt , \\ \theta_{22}+\theta_{33} &= 2qX \coth 2qt , \\ \theta_{22}-\theta_{33} &= 2nqS^{-1}X . \end{aligned} \tag{4.2}$$

It follows that the expansion scalar is given by

$$\theta = 2q(1-m)X \coth 2qt ,$$

which is positive on the whole spacetime if we assume (2.6) or (2.7) to hold. Thus *the fluid is expanding everywhere.* The fluid congruence has a nonzero acceleration vector, with nonzero frame components

$$\dot{u}_1 = (2m+1)qX \tanh(2qx/\alpha) . \tag{4.3}$$

Equations (4.1) and (4.2) imply that the $\theta_{\alpha\beta}$ and $R^*_{\alpha\beta}$ have a common eigenframe, which is Fermi propagated along the fluid flow lines (as can be verified),[23] so that the present solutions again display similarity with the Szekeres solutions. One difference, however, is that in the Szekeres solutions the expansion tensor has a repeated eigenvalue, while in the present solutions this occurs if and only if $n=0$, as follows from Eq. (4.2). A more important difference, however, is that in the present solutions, *the acceleration of the fluid is always nonzero,* as follows from Eq. (4.3), since $m \neq -\frac{1}{2}$.

The metric induced on the slices is sufficiently simple to enable one to study its geodesics in detail. The curves $y, z=$ const are geodesics, and it is easily seen that they have infinite length if and only if $m<-1$. Thus if $m \geq -1$, in particular, for the solutions with $p=\mu$, the slices are geodesically incomplete and cannot be extended since the curvature scalars of the slices become infinite along the incomplete geodesics, in both the positive and negative directions. Thus in this case the slices are of finite extent in the x direction. On the other hand, when $m<-1$, it can be shown[24] that all geodesics of the slices have infinite length, so that the slices are geodesically complete. In this case, the curvature scalars of the induced metric are bounded on each slice.

V. ASYMPTOTIC BEHAVIOR

In this section we discuss whether or not these spatially inhomogeneous models can be approximated by a (nontilted)[25] spatially homogeneous model, either as $t\to\infty$ or as $t\to 0^+$, i.e., whether or not they are "asymptotically" spatially homogeneous. We are working with a spacetime which admits a global coordinate system in which the line element has the form

$$ds^2 = g_{00}dt^2 + g_{\alpha\beta}dx^\alpha dx^\beta , \tag{5.1}$$

where the hypersurfaces $t=$ const are orthogonal to the fluid flow, the x^α, $\alpha=1,2,3$, are constant along the fluid trajectories, and hence $g_{00}<0$. In order that (5.1) admit a three-parameter group of isometries with orbits $t=$ const, it is necessary that $\partial g_{00}/\partial x^\alpha = 0$, i.e., that g_{00} be constant on the slices. Thus in order that a spatially inhomogeneous model with line element of the form (5.1) be asymptotically spatially homogeneous, it is necessary that g_{00} be "asymptotically constant" on the slices. We make this precise as follows. Firstly it is necessary that the positive-valued function $(-g_{00})$ be bounded above on each slice $t=$ const. This being assumed, we consider the quantities

$$M(t) = \sup_{t=\text{const}} (-g_{00}),$$

$$m(t) = \inf_{t=\text{const}} (-g_{00}).$$

These quantities are not invariants, since they are altered by a coordinate transformation of the form $t \to f(t)$, with $f'(t) > 0$, which represents the freedom in choice of the t coordinate. However, their ratio is invariant. It is thus reasonable to require as our criterion for g_{00} to be asymptotically constant that

$$\lim \frac{m(t)}{M(t)} = 1 \tag{5.2}$$

as t approaches the limiting value. It is immediately clear that this necessary condition cannot be satisfied as $t \to +\infty$ or $t \to 0^+$ by the spatially inhomogeneous solutions presented in this paper, since if (2.6) holds $(-g_{00})$ is unbounded, while if (2.7) holds, $m(t) = 0$. We thus conclude that *these solutions are not asymptotically spatially homogeneous as* $t \to +\infty$ *or* $t \to 0^+$. Indeed this behavior of g_{00} means that we cannot regard the solutions as being *approximately* spatially homogeneous at any time.

VI. CONCLUSION

We have presented a new class of spatially inhomogeneous irrotational cosmological models, containing two essential parameters. If the parameters are suitably restricted, the fluid pressure and density are related by an equation of state of the form $p = \gamma\mu$, with $0 < \gamma < 1$, and p and μ positive over the whole spacetime. To the best of the authors' knowledge, these are the first exact inhomogeneous models with such an equation of state—the other exact solutions with an equation of state have $p = 0$ or $p = \mu$ (see Ref. 8 for details). When (2.6) holds, in particular, when $p = \gamma\mu$, the only curvature singularities occur at $t = 0$, giving rise to an initial "big-bang" singularity. In this case, the pressure, density, and indeed all scalars are bounded away from the initial singularity, i.e., in any subset $t \geq t_0$, where $t_0 > 0$. This means that the pressure and density are bounded on each slice. Despite this fact, however, the discussion of Sec. V shows that the solutions cannot be regarded as being a small perturbation of a spatially homogeneous model.

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