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Gravitational radiation damping of a binary system containing compact objects calculated using matched asymptotic expansions

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This paper calculates the secular changes in orbital period, semimajor axis, and eccentricity for a gravitationally bound, slow-motion system of two compact bodies, directly from the Einstein field equation, using matched asymptotic expansions. Burke previously derived the radiation damping of a weak-field, slow-motion system, also using matched asymptotic expansions. However, no previous derivations extend to systems, such as the binary pulsar PSR 1913 + 16, containing objects with *strong internal gravity*. This calculation uses distinct wave-, near-, and body-zone expansions to seek a uniformly valid, one-parameter family of approximate space-times representing a bound system of compact objects undergoing gravitational radiation reaction. As in Burke's work, matching outward gives the lowest-order near-zone and radiation fields, and then matching back inward yields near-zone resistive potentials of 5/2-post-Newtonian order, which contain the lowest-order time-odd effects of radiation. Matching inward again using my earlier technique for the problem of motion in external fields then gives the resulting deflections of the bodies from the world lines that they would otherwise follow. The secular changes in orbital parameters derived for the system of *compact* objects treated here agree with the standard formulas obeyed by *weak-field* systems.

I. INTRODUCTION

The detection of gravitational radiation from a binary system such as the binary pulsar PSR 1913 +16 is indirect in the sense that one observes, not the energy flux of the radiation, but rather the orbital-period shortening. The gravitational radiation reaction for a weak-field, slow-motion, gravitationally bound system was derived directly from the Einstein field equation (EFE) by Burke[1] in a paper that introduced the method of matched asymptotic expansions to general relativity. For the systems considered, his paper was the first to verify directly that the energy flux calculated from the usual flux integrals agrees with the rate at which resistive forces extract mechanical energy from the sources.

Burke's approach needs to be continued on several fronts: First, observational determination of the mass ratio and other parameters necessary to predict the binary pulsar's period decrease depends on a knowledge of the near-zone metric and of the motion to post-Newtonian (PN) order. Matching of the radiation fields and the PN metric terms in the near-zone expansion has not yet been carried out, and the motion up to PN order of a system containing compact objects is not yet fully understood, although D'Eath[2] has solved the problem for the special case of two black holes. Second, the overall consistency of the slow-motion expansion up to at least $\frac{5}{2}$-PN order needs to be checked. The infinities referred to by Ehlers, *et al.*[3] must be explained (see Conclusions). Third, the relation between retarded waves and a condition for the absence of incoming radiation is still being investigated.[4-7] This issue is also intimately connected to the problem of uniformity of the wave-zone expansion at large distances[8] (see Conclusions).

Leaving these first three problems to future work, I address a fourth problem: To apply Burke's method to a system containing objects with strong internal gravity, such as the binary pulsar, one needs to calculate the deflections of the objects due to the "resistive potentials" produced from matching the radiation fields back to the near zone. In Ref. 9, I first studied the more basic problem of the motion of an arbitrary object with (possibly) strong internal gravity through an unspecified, curved, matter-free region of an external spacetime. I used a matching technique based on my earlier paper[10] concerning singular perturbations on manifolds (and related to a technique used by D'Eath[11]) to show, directly from the Einstein field equations (EFE), that a body whose mass m is small compared to an external curvature reference length L moves on an approximate geodesic of the unperturbed external spacetime.

This paper combines the methods of Refs. 9 and 1 to calculate, directly from the EFE, the resistive accelerations and consequent secular changes in orbital parameters for a binary system of two compact objects. The calculation treats a system with the following properties:

(1) The ratio $v \equiv l/\lambda$ (where l is a typical separation, λ is a typical wavelength or period, and $G = c = 1$) is a small parameter.

(2) The system is gravitationally bound, so that the Newtonian interaction potentials scale with v^2.

(3) The bodies are compact enough so that tidal slowdown is negligible.

(4) Negligible direct mass loss occurs.
This paper actually derives only the first time-odd correction to the motion; this correction occurs at $\frac{5}{2}$-PN order and therefore produces deviations of order l on time scales of order λ/v^5.

Although this calculation does not depend on any definition of energy, it does agree with the usual[12-14] formulas for energy and angular momentum losses and therefore makes the usual predictions for secular changes in orbital parameters over $\frac{5}{2}$-PN time scales.

The three zones treated in this problem will require three types of asymptotic expansions. The gravitational field is weak in both the wave and near zones. However, in the wave zone both time and spatial derivatives scale with the wavelength λ, while in the near zone spatial derivatives scale instead with the typical separation l. Gravitational potentials in the near zone scale with v^2. In the two body zones, where the fields are strong, appropriate reference lengths are the masses, which scale as $v^2 l$. Body-zone perturbations change only over time scales of order m/v^3, and therefore time enters the body-zone equations only as a parameter.

The mathematical context for these expansions is based on Ref. 10. One infers the behavior of a system with small v from the asymptotic behavior of a one-parameter family of systems as $v \to 0$. Consider then a five-dimensional manifold $\tilde{M}$ that can be sliced into exact spacetimes $(S_v, \mathcal{G}_v)$, where $0<v<v_0<1$, and in which all tensors vary smoothly. I will seek a "global asymptotic approximation" on $\tilde{M}$, which in this problem means a collection of matched asymptotic expansions (in v), whose errors in approximating the exact spacetimes $(S_v, \mathcal{G}_v)$ become uniformly small as $v \to 0$.

The scaling assumptions described above lead naturally to the assumed form of the asymptotic expansions. I assume as given[9] two unperturbed solutions $(B_1, \mathcal{G}^{(1)})$, $(B_2, \mathcal{G}^{(2)})$ of the EFE, each representing one of the bodies as if it were isolated. Each is assumed to be empty outside some spatially bounded region and to approach a Schwarzschild solution at large distances. These given solutions provide the zeroth-order terms in the two body-zone expansions. Terms of higher order in the small parameter v represent the mutual perturbing influences. For example, tidal perturbations of one body by the other would be expected to scale as v^6.

Zeroth-order terms in both the near- and wave-zone expansions represent Minkowski space. The near zone has a weak-field, slow-motion expansion, with Newtonian potentials proportional to v^2. The possibility of nonanalytic terms in v cannot be ruled out, but I assume that all the expansions begin in powers of v; this calculation requires only an analysis of analytic terms.

The wave zone has a weak-field expansion, with the first nonflat term entering at $O(v^3)$ (static monopole) and the first radiative term entering at $O(v^5)$. The condition of outgoing radiation is enforced in this paper by allowing only retarded potentials in the $O(v^5)$ wave-zone potentials. The purpose of the wave-zone expansion is to produce an approximation that is uniformly valid (1) out to radii scaling (in the limit $v \to 0$) as some finite multiple of the wavelength λ and (2) over retarded times of order λ/v^5. Both these limitations on uniformity have been sources of confusion in previous work and will be discussed further in the conclusions.

In the context of the near-zone expansion, it makes sense to define a $\frac{5}{2}$-PN-approximate world line. The principal goal of this paper is to calculate the time-odd part of this $\frac{5}{2}$-PN world line's deviation from the world line that would have resulted from including only the terms up to 2-PN order in the near-zone metric; this deviation contains the lowest-order secular effects due to radiation reaction. Calculating it requires a knowledge of the orbits and fields to only Newtonian order.

I first derive these orbits and fields from the EFE by matching the body- and near-zone expansions. (For body- and near-zone matching, I assume coordinate expansions, beginning in powers, of several of the perturbations.) Matching to the wave-zone expansion gives a static monopole at order v^3 and a radiating quadrupole at order v^5. (Higher moments would enter at higher orders.)

The first time-odd terms that result from matching the wave-zone expansion back to the near zone come in at $\frac{5}{2}$-PN order. Via matching to the body-zone expansions, the gradients of these $\frac{5}{2}$-PN corrections induce resistive accelerations of the bodies, just as the gradient of the Newtonian potential induces the Newtonian motion. Obtaining the resistive acceleration reduces the problem to a kinematics problem with a well-known solution.

II. COORDINATES, EXPANSIONS, AND APPROXIMATE EQUATIONS

Each of the expansions in this problem can be thought of as a "model spacetime."[10] Each model spacetime will need its own coordinate systems, asymptotic expansions, and approximate equations. The model spacetimes will later be glued together by matching.

Both the wave- and near-zone fields are expanded about Minkowski space, and therefore it is convenient to take both expansions on the same manifold R^4, but with different coordinate systems reflecting the different scaling assumptions. The body-zone fields are expanded about the assumed given solutions $(B_1, \mathcal{G}^{(1)})$, $I=1,2$, which serve as zeroth-order model spacetimes.

For the wave zone, introduce Minkowskian rectangular coordinates (t,x,y,z) and spherical coordinates (t,r,θ,ϕ) nondimensionalized by the wavelength λ. The wave-zone metric is expanded in the form

$$\bar{\mathcal{G}} \sim \left(\eta_{\mu\nu} + \sum_{n=1}^{5} v^n ({}_n h_{\mu\nu}) + \cdots\right) dx^\mu \otimes dx^\nu, \qquad (2.1)$$

where $\eta_{\mu\nu}$ is the usual Minkowski metric, and indices are raised and lowered with $\eta_{\mu\nu}$. The notation $\bar{\mathcal{G}}$ is a shorthand for the pullback of the exact metric $\mathcal{G}_v$ from the S_v to any of the model spacetimes. The ${}_n h_{\mu\nu}$ are assumed to depend functionally on the wave-zone coordinates. Since $m/\lambda = O(v^3)$, the static, monopole terms will first appear at $O(v^3)$, while the first dynamic terms will come in at $O(v^5)$. Our attention will therefore be focused on the ${}_5 h_{\mu\nu}$.

If one defines

$$\bar{h}_{\mu\nu} \equiv {}_5 h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}({}_5 h^\alpha{}_\alpha), \qquad (2.2)$$

then the EFE translates into the ordinary equations of linearized gravity, written in the "Lorentz gauge":

$$\bar{h}_{\mu\nu;\alpha}{}^{\alpha} = 0, \qquad (2.3)$$

$$\bar{h}^{\mu\nu}{}_{;\nu} = 0. \qquad (2.4)$$

Here, the semicolon refers to the flat-space covariant derivative expressed in (possibly) curvilinear coordinates.

All quantities in the near-zone expansion will depend functionally on coordinates (t, x^{*i}) or (t, r^*, θ, ϕ), where

$$x^{*i} \equiv x^i/v, \quad r^* \equiv r/v. \qquad (2.5)$$

I assume[1] the near zone has a weak-field, slow-motion expansion

$$\begin{aligned}\mathcal{G} \sim & [-1 + \tfrac{1}{2}v^2\psi + \tfrac{1}{2}v^4(b + H^i{}_i) \\ & + \tfrac{1}{2}v^5(X^i{}_i) + \cdots + \tfrac{1}{2}v^7 f]\, dt \otimes dt \\ & + [-v^3 V_j + \cdots - v^6 W_j + \cdots](dx^j \otimes dt + dt \otimes dx^j) \\ & + [\eta_{ij} + \tfrac{1}{2}v^2\psi\eta_{ij} + v^4(\tfrac{1}{2}b\eta_{ij} + H_{ij} - \tfrac{1}{2}H^k{}_k\eta_{ij}) \\ & + v^5(X_{ij} - \tfrac{1}{2}X^k{}_k\eta_{ij}) + \cdots]\, dx^i \otimes dx^j. \end{aligned} \qquad (2.6)$$

The absence of certain terms such as $O(v)$ and $O(v^3)$ in $\mathcal{G}_{tt}$ will be verified by matching to the body expansions. It is convenient to adopt the conventions

$$\mathcal{V} \equiv V^j e_j, \qquad (2.7)$$

$$\mathcal{H} \equiv H^{ij} e_i \otimes e_j, \qquad (2.8)$$

$$e_i \equiv \partial/\partial x^i. \qquad (2.9)$$

For matching, I will also need to expand the orthonormal basis one-forms,

$$\omega^t \sim [1 - \tfrac{1}{4}v^2\psi - \tfrac{1}{4}v^4(b + H^k{}_k + \tfrac{1}{8}\psi^2)]\, dt - v^3 V_i dx^i, \qquad (2.10)$$

$$\omega^j \sim \omega^j + \tfrac{1}{4}v^2\psi\omega^j + v^4[\tfrac{1}{2}H_{\hat{j}\hat{k}}\,\omega^k + \tfrac{1}{4}(b - H - \tfrac{1}{8}\psi^2)\omega^j], \qquad (2.11)$$

$$H_{\hat{j}\hat{k}}\,\omega_j \otimes \omega^k \equiv H_{jk} dx^j \otimes dx^k. \qquad (2.12)$$

The EFE, together with the gauge conditions

$$\nabla\cdot\mathcal{V} + \partial_t\psi = 0, \qquad (2.13)$$

$$\nabla\cdot\mathcal{H} + \partial_t\mathcal{V} = 0, \qquad (2.14)$$

translates into the following equations for ψ, $\mathcal{V}$, and $\mathcal{H}$:

$$\nabla^2\psi = 0, \qquad (2.15)$$

$$\nabla^2\mathcal{V} = 0, \qquad (2.16)$$

$$\nabla^2\mathcal{H} = [\tfrac{1}{4}\psi^{;j}\psi^{;k} + \tfrac{1}{2}\psi\psi^{;jk} - \eta_{jk}(\tfrac{3}{8}\nabla\psi\nabla\psi + \tfrac{1}{2}\psi\nabla^2\psi)]\, e_j \otimes e_k \equiv 16\pi\,{}^{G}\mathcal{S}. \qquad (2.17)$$

In order to translate information about the field quantities into information about the motion of the bodies, it is necessary to define n-PN-approximate world lines. I use a criterion[9] based on the requirement of uniformity for a global asymptotic approximation[10]: For Newtonian accuracy, let $\gamma_v(t)$ be any smooth world line, expressed in coordinates as $x^{*i} = r^i(t)$. [For $\frac{5}{2}$-PN accuracy, let $\tilde{\gamma}_v(t)$ be parametrized by $t, \ldots, t/v^5$ and be expressed in coordinates as $x^{*i} = r^i(t, \ldots, t/v^5)$.] Suppose in either case that the domains of validity $\mathfrak{D}_v^{(N)}$ of the near-zone expansion exclude γ_v (or $\tilde{\gamma}_v$). Define

$$w^i \equiv x^{*j} - r^i. \qquad (2.18)$$

In the Newtonian ($\frac{5}{2}$-PN) case, pick a family of events p_v ($\tilde{p}_v$) with constant coordinates (t, w^i) [$(t/v^5, w^i)$]. If p ($\tilde{p}$) is eventually in a domain of validity $\mathfrak{D}_v^{(N)}$ of the near-zone expansion up to Newtonian ($\frac{5}{2}$-PN) order for sufficiently small v, then I call γ_v ($\tilde{\gamma}_v$) a Newtonian-($\frac{5}{2}$-PN-) approximate world line, provided that the near-zone domains of validity overlap with the body-zone domains of validity. This definition means physically that the body becomes increasingly localized to its world line as $v \to 0$. Note, however, that this definition will only be sensitive to those higher-order effects that produce secular orbital changes.

I choose the origin of the near-zone spatial coordinates (and, consequently, of the wave-zone spatial coordinates) to be at the "center of mass," defined here by the standard Newtonian formula for two point bodies of masses m_1 and m_2 moving along paths r_1^i and r_2^i (this definition needs to be accurate only to Newtonian order, and therefore it may differ at higher orders from formal definitions[15,16]). This choice of origin will simplify the calculation by eliminating the lowest-order dipole moments in the wave-zone expansion. (The absense of dipole terms is peculiar to theories without scalar fields. In the scalar-tensor theory, the presence of two independent "masses" means that in general one cannot simultaneously eliminate both dipole moments, and therefore dipole radiation in general does occur.)

Each of the model spacetimes representing the body-zone expansions has coordinates (T,R,Θ,Φ) and (T,X,Y,Z) as defined in Ref. 9. As in Ref. 9, the zeroth-order metric is assumed to have an expansion of the form

$$\mathcal{G}^{(I)} \sim \mathcal{G} + \delta\mathcal{G} + \delta\delta\mathcal{G}^{(I)} + \cdots, \quad I = 1, 2 \tag{2.19}$$

where

$$\begin{aligned}\mathcal{G} &\equiv \alpha^{-2}[-dT_\alpha \otimes dT_\alpha + dR_\alpha \otimes dR_\alpha \\ &\quad + R_\alpha{}^2(d\Theta \otimes d\Theta + \sin^2\Theta d\Phi \otimes d\Phi)] \\ &= O(1),\end{aligned} \tag{2.20}$$

$$\delta\mathcal{G} \equiv 2\alpha^{-1}[R_\alpha{}^{-1}dT_\alpha \otimes dT_\alpha + R_\alpha{}^{-1}dR_\alpha \otimes dR_\alpha] = O(\alpha), \tag{2.21}$$

$$\delta\delta\mathcal{G}^{(I)} = O(\alpha^2), \tag{2.22}$$

and where X_α are auxiliary coordinates defined by

$$X^\mu{}_\alpha \equiv X^\mu/\alpha, \; R_\alpha \equiv R/\alpha, \; \alpha \to 0. \tag{2.23}$$

These auxiliary expansions express the requirement that each body approach a Schwarzschild solution at large distances.

The metric is expanded about the zeroth-order term as follows:

$$\overline{\mathcal{G}} \sim \mathcal{G}^{(I)} + v^4({}_4h^{(I)}) + \cdots. \tag{2.24}$$

Newtonian and $\frac{5}{2}$-PN perturbations enter at $O(v^4)$ and $O(v^9)$, respectively. (This expansion assumes that the $\frac{1}{2}$-pre-Newtonian spin precession vanishes, an assumption that will be checked in a future paper.) The slow-motion assumption implies that the internal perturbations depend on one or more slow-time variables m/v^5, etc. It is possible that this expansion might need to be augmented by certain nonanalytic terms.[17] However, only analytic terms enter the present calculation.

III. DERIVATION OF NEWTONIAN FIELDS AND ORBITS

In this section, I match the body- and near-zone expansions and apply the EFE at Newtonian order to derive the Newtonian motion. The same method will later be used at $\frac{5}{2}$-PN order to derive the resistive acceleration.

A "calculation of a body's world line" is really a calculation of the constraints imposed by the EFE and matching on the correspondence maps[9]

$$\Psi_v^{(I)}: U_I \to N, U_I \subset B_I, \quad (X^\mu) \to (t, w^i) \tag{3.1}$$

between the body-zone model manifolds B_I and the near-zone model manifold $N = R^4$. An arbitrary correspondence map

$$\Psi_v'^{(I)}: U_I \to N \tag{3.2}$$

can be expressed as the composition

$$\Psi'^{(I)}_v = \Xi_v^{(I)} \circ \Psi_v^{(I)} \circ \Sigma_v^{(I)}, \tag{3.3}$$

where $\Psi_v^{(I)}$ is of an assumed canonical form. It is convenient to choose

$$(t, w^i) = \Psi_v^{(I)}(X) = \mu_I(v^3 T, v^2 \Lambda^i_{Ij} X^j), \tag{3.4}$$

where $\mu_I \equiv m_I/(\lambda v^3)$ and Λ^i_j is a rotation matrix that will be chosen to simplify the internal perturbations.

The Ξ_v and Σ_v will refine this initial, crude choice for $\Psi_v^{(I)}$ in order to satisfy the requirements of matching, as in D'Eath's[2] analysis of his Eq. (2.11). For example, the velocity of B_I's world line in N induces a boost of the near-zone coordinates that is not included in $\Psi_V^{(I)}$. As part of the slow-motion assumption, I expand the Ξ_v and Σ_v as follows (superscripts indicating body 1 or 2 are suppressed below):

$$\Xi_v: (t, w^j) \to (\Xi_v^t, \Xi_v^i), \tag{3.5}$$

$$\Xi_v^t \sim t + v^2 \Xi_2^t(t, w^j) + \cdots, \tag{3.6}$$

$$\Xi_v^i \sim w^i + v^2 \Xi_2^i(t, w^j) + \cdots, \tag{3.7}$$

$$\Sigma_v: (X^\mu) \to [\Sigma_v^\mu(X)], \tag{3.8}$$

$$\Sigma_v^\mu \sim X^\mu + v^4 \Sigma_4^\mu + \cdots. \tag{3.9}$$

[Expansions (3.5)–(3.7) are equivalent to D'Eath's[2] Eqs. (2.9), except that he expanded the inverse maps $\Xi_v{}^{-1}$.] The lowest-order corrections to Ξ_v and Σ_v are Newtonian, because I have used the earlier assumption that the $\frac{1}{2}$-pre-Newtonian spin precession vanishes. Terms of order v^2 in Ξ_v^t and v^4 in Σ_v can be represented by Newtonian gauge transformations (and so on for higher-order terms). The remaining Newtonian degrees of freedom are already represented by the coordinates of the world line.

For matching, it is convenient to compare the body- and near-zone expansions under intermediate limit processes of the form $v \to 0$, x_η^μ fixed, where

$$\begin{aligned}&(x_\eta^\mu) \equiv (t/\eta, v w^i/\eta), \\ &v^3 \ll \eta \ll v \ll 1.\end{aligned} \tag{3.10}$$

If one were to try to match out the body-zone metrics using the canonical correspondence maps

$\Psi_v^{(I)}$, the $O(v^2)$ part of $\mathcal{G}$ would not be of the assumed form $\frac{1}{2}v^2\psi$ in Eq. (2.6). However, the Schwarzschild part of the body-zone metrics can be expressed in isotropic coordinates, and this transformation is equivalent to a gauge transformation of the near-zone metric [i.e., a PN correction to Eq. (3.7)]. The solution of Eq. (2.15) that matches to each of the body-zone expansions and also has the proper behavior at large r—no growing terms occur at Newtonian order—is

$$\begin{aligned}&\psi=4\mu_1|\vec{r}^*-\vec{r}_1|^{-1}+4\mu_2|\vec{r}^*-\vec{r}_2|^{-1},\\ &\vec{r}^*=(x^*,y^*,z^*)=(x^{*i}),\\ &\vec{r}_1=(r_I^i),\ I=1,2.\end{aligned}\tag{3.11}$$

Note that no terms of order v or v^3 appear in $\overline{\mathcal{G}}_{tt}$, verifying the assumption mentioned above in Eq. (2.6).

Suppose now that body 1 has a coordinate acceleration

$$a^i\equiv\frac{d^2r_1^i}{dt^2}\tag{3.12}$$

at $t=0$ (the argument could be repeated at any other time). Then if the orthonormal frames are expanded in the intermediate limit (3.10) as

$$\begin{aligned}\overline{\omega}\to\omega&+v\eta({}_4\kappa_+ + {}_4\kappa_-)+\cdots\\ &+(v^3/\eta)\delta\omega+v^4({}_4\lambda_+ + {}_4\lambda_-)+\cdots\\ &+(v^6/\eta^2)\delta\delta\omega+(v^7/\eta)_4\sigma+\cdots,\end{aligned}\tag{3.13}$$

the following perturbations are obtained at $O(v\eta)$:

$${}_4\kappa_+^t=\left(a^i-\frac{1}{4}\frac{\partial\psi}{\partial x^{*i}}\right)x_\eta^i dt_\eta=Az_\eta dt_\eta\,,\tag{3.14}$$

$${}_4\kappa_+^j=\frac{1}{4}\left[\psi(r^i)+x_\eta^i\frac{\partial\psi}{\partial x^{*i}}\right]dx_\eta^j\,,\tag{3.15}$$

$$A\equiv|\vec{a}-\tfrac{1}{4}\vec{\nabla}\psi|,\tag{3.16}$$

where the rotation matrix Λ_j^i of Eq. (3.4) has been chosen so that $a^i-\frac{1}{4}\partial\psi/\partial x^{*i}$ is in the z direction. This quantity represents a trial Newtonian acceleration of the body away from the world line that would be predicted from Newton's law.

In order to use the arguments of Ref. 9, I first remove the ${}_4\kappa_+^j$ by a gauge transformation

$$\begin{aligned}&{}_4\kappa\to{}_4\kappa+\mathcal{L}_\xi\omega,\\ &{}_4\kappa_+^t\to{}_4\kappa_+^t,\\ &{}_4\kappa_+^i\to 0.\end{aligned}\tag{3.17}$$

The frame perturbation ${}_4\kappa_+$ is now of the form κ_+ of Ref. 9.

This perturbation produces an $O(v^4)$ (under the intermediate limit) contribution

$${}_4G_{+I}=-6Ar_\eta^{-2}\cos\theta dr_\eta\otimes dr_\eta\tag{3.18}$$

to the Einstein tensor (ET). If the ET at $O(v^4)$ is to vanish, then ${}_4G_{+I}$ must be canceled by a term ${}_4G_{+II}$, generated by the $O(v^4)$ (intermediate limit) perturbation ${}_4\lambda_+$ of Eq. (3.13). The arguments of Ref. 9 applied to ${}_4\lambda_+$ show that such a term G_{+II} would violate the EFE and Bianchi identities. Therefore, the Newtonian coordinate acceleration of the world lines must agree with that predicted by the geodesic equation expanded to Newtonian order:

$$a_i(t)=\frac{1}{4}\frac{\partial\psi(t)}{\partial x^{*i}}\bigg|_{\gamma(t)}.\tag{3.19}$$

It is convenient to label the near-zone coordinates so that the Keplerian orbits lie in the x-y plane; they then satisfy[13]

$$(x_1,y_1,z_1)=(d_1\cos\phi_1,d_1\sin\phi_1,0),\tag{3.20}$$

$$(x_2,y_2,z_2)=(-d_2\cos\phi_1,-d_2\sin\phi_1,0),\tag{3.21}$$

$$d_1=\frac{\mu_2}{\mu_1+\mu_2}d,\tag{3.22}$$

$$d_2=\frac{\mu_1}{\mu_1+\mu_2}d,\tag{3.23}$$

$$d=\frac{a(1-e^2)}{1+e\cos\phi_1},\tag{3.24}$$

$$\frac{d\phi_1}{dt}=v\frac{[(\mu_1+\mu_2)a(1-e^2)]^{1/2}}{d^2},\tag{3.25}$$

where a is the (nondimensionalized) semimajor axis, ϕ_1 is the angle of body 1 from periastron, and e is the eccentricity. Over Newtonian time scales, it suffices to consider constant a, e, and period. The secular effects of radiation reaction, which become important over time scales of order λ/v^5, can be accounted for by allowing the parameters a and e to depend on the slow-time parameter t/v^5. While other slow changes, such as the perihelion precession, come in at lower orders, radiation reaction produces the lowest-order changes in a and e.

IV. RADIATION AND RESISTIVE POTENTIALS

In this section, I match the near-zone and wave-zone expansions to compute the lowest-order radiation and resistive potentials. The lowest-order radiation is a sum of $L=2$ multipoles. The lowest-order, time-odd potentials in the near zone generated by matching are of $\frac{5}{2}$-PN order. Two subtleties of the matching should be noted.

(1) Although the gravitational stresses ${}^G s$ are important as sources of radiation, it will not be necessary to compute the $\mathcal{H}$ (and $\mathcal{V}$) fields explicitly, because the $\hbar_{ij}$ (and $\hbar_{tj}$) fields to which they would match can be determined from the gauge conditions (2.13) and (2.14) and from the

form (2.6) of the near-zone expansion.

(2) The resistive effects generated by straightforward matching would come in at $\frac{5}{2}$-PN order in the scalar, vector, and tensor potentials. In order to simplify later computation of the resistive deflections, I will use a gauge transformation (similar to that of Ref. 1) to transform all the $\frac{5}{2}$-PN effects into the scalar potential f.

The Newtonian motion calculated in the preceding section determines the ψ field over Newtonian time scales:

$$\psi=4\mu_1[(d_1\cos\phi_1-x^*)^2+(d_1\sin\phi_1-y^*)^2+z^{*2}]^{-1/2}+4\mu_2[(-d_2\cos\phi_1-x^*)^2+(-d_2\sin\phi_1-y^*)^2+z^{*2}]^{-1/2}. \tag{4.1}$$

The static, monopole part of ψ matches routinely to a static monopole in ${}_3h_{\mu\nu}$. The dipole part of ψ vanishes as explained above. The lowest-order, time-dependent part of ψ has a quadrupole dependence

$$\psi\to 2(\pi/30)^{1/2}Qr^{-3}[-\tfrac{1}{2}\sqrt{6}\,Y_{20}+\tfrac{3}{2}\exp(-2i\phi_1)Y_{22}+\tfrac{3}{2}\exp(2i\phi_1)Y_{2-2}], \tag{4.2}$$

$$Q\equiv\frac{4\mu_1\mu_2}{\mu_1+\mu_2}\left[\frac{a(1-e^2)}{1+e\cos\phi_1}\right]^2 \tag{4.3}$$

and matches to the potential $\bar{h}_{tt}$ defined in Eq. (2.2).

Outgoing wave solutions of Eqs. (2.3) and (2.4) having electric parity and $L=2$ are linear combinations of

$$\bar{h}_{ij}=(\tfrac{15}{2})^{1/2}[F''_M(t-r)/r](\mathcal{T}_{20M})_{ij}, \tag{4.4}$$

$$\bar{h}_{tj}=-(\tfrac{5}{2})^{1/2}[F''_M(t-r)/r+F'_M(t-r)/r^2](\mathcal{Y}_{21M})_j, \tag{4.5}$$

$$\bar{h}_{tt}=[F''_M(t-r)/r+3F'_M(t-r)/r^2+3F_M(t-r)/r^3]Y_{2M}, \tag{4.6}$$

and

$$\tilde{h}_{ij}=(\tfrac{35}{12})^{1/2}\{F_M{}^{(2)}(t-r)\}_4(\mathcal{T}_{24M})_{ij}, \tag{4.7}$$

$$\tilde{h}_{tj}=(\tfrac{5}{3})^{1/2}\{F_M{}^{(2)}(t-r)\}_3(\mathcal{Y}_{23M})_j, \tag{4.8}$$

$$\tilde{h}_{tt}=[F''_M(t-r)/r+3F'_M(t-r)/r^2+3F_M(t-r)/r^3]\,\mathcal{Y}_{2M}, \tag{4.9}$$

where $\{F_M{}^{(2)}\}_4$ and $\{F_M{}^{(2)}\}_3$ are defined by

$$\{F^{(2)}\}_3\equiv F''_M/r+6F'_M/r^2+15F_M/r^3+15F_M{}^{(-1)}/r^4, \tag{4.10}$$

$$\{F^{(2)}\}_4\equiv F''_M/r+10F'_M/r^2+45F_M/r^3+105F_M{}^{(-1)}/r^4+105F_M{}^{(-2)}/r^5. \tag{4.11}$$

A solution $h_{\mu\nu}$ of the form (4.4)–(4.6) may be transformed into a solution of the form (4.7)–(4.9) by a gauge transformation that preserves the Lorentz gauge condition (2.4). Under this gauge transformation, the time-time component $\bar{h}_{tt}$ undergoes the change

$$\tilde{h}_{tt}=6\bar{h}_{tt}. \tag{4.12}$$

In matching the ψ, $\mathcal{V}$, and $\mathcal{K}$ fields, I choose the form (4.4)–(4.6), because otherwise the h_{tj} and h_{ij} fields would match to near-zone potentials of order v and unity, respectively, and such potentials would violate the form of the near-zone expansion (2.6). Matching thus determines the functions F_M, leading to the solutions

$$F_0=-\tfrac{1}{3}(4\pi/5)^{1/2}Q, \tag{4.13}$$

$$F_{\mp2}=(4\pi/30)^{1/2}Q\exp(\pm2i\phi_1), \tag{4.14}$$

$$F_{\mp1}=0. \tag{4.15}$$

Note that matching could in principle be carried to higher order in the ${}_nh_{\mu\nu}$ with radiation from higher multipoles and post-Newtonian terms becoming important. However, higher-order terms in the wave-zone expansion do not apparently affect the lowest-order resistive potentials, although this point should be checked in future work.

One can rewrite the solutions (4.4)–(4.6) in near-zone coordinates for matching, with the aid of their Taylor-series expansions for small r. The lowest-order terms that depend on the outgoing wave condition are

$$\bar{h}_{ij}\to-\tfrac{1}{3}F_M{}^{(3)}(t)(\mathcal{T}_{20M})_{ij}, \tag{4.16}$$

$$\bar{h}_{tj}\to(r/8)F_M{}^{(4)}(t)(\mathcal{Y}_{21M})_j, \tag{4.17}$$

$$\bar{h}_{tt}\to-(r^2/15)F_M{}^{(5)}(t)Y_{2M}. \tag{4.18}$$

These time-odd terms match to the near-zone potentials X_{ij}, W_j, and f of Eq. (2.6), which obey the equations

$$\nabla^2(X_{ij})=0, \tag{4.19}$$

$$\nabla^2(W_j)=0, \tag{4.20}$$

$$\nabla^2 f=X_{ij}\psi_{,ij}. \tag{4.21}$$

X_{ij} and W_j are given by Eqs. (4.16) and (4.17), whereas f is given by

$$f=-r^{*2}/15\,F_M{}^{(5)}(t)Y_{2M}-\tfrac{1}{2}X_{ij}\chi_{,ij}, \tag{4.22}$$

where $\nabla^2\chi=-2\psi$.

It is convenient to gauge transform these resistive potentials to make X_{ij} and W_j vanish. However, as pointed out by Walker and Will,[18,19] one must be careful to include the effect of this gauge transformation on the Newtonian potential ψ of Eq. (2.6). It can be shown that this effect precisely cancels the second term in (4.22). The gauge-transformed scalar potential, denoted f_R, becomes

$$f_R = -\tfrac{2}{5} r^{*2} F_M{}^{(5)}(t) Y_{2M}\,. \tag{4.23}$$

The resistive potential f_R now contains all the information needed to calculate the lowest-order effects of radiation reaction on the orbits.

V. RESISTIVE DEFLECTION AND SECULAR CHANGES

In order to verify the standard formulas for secular changes in the semimajor axis a, period P, and eccentricity e, I match f_R with the two body-zone expansions. Although of much higher order than the Newtonian potential ψ, the resistive potential f_R also causes the bodies to deflect by the analog of Newton's law for a pointlike body moving through an external potential. The derivation of changes in a, e, and P is therefore reduced to a previously solved problem in kinematics.

In principle, matching at $\frac{5}{2}$-PN order involves terms up to Ξ_7^t, Ξ_5^i, and Σ_9^μ in Eqs. (3.6), (3.7), and (3.9). However, terms beyond order unity in these expansions actually have no effect on this calculation. The $\frac{5}{2}$-PN coordinate strains Ξ_7^t, Ξ_5^i, and Σ_9^μ can be absorbed into gauge transformations. The remaining lower-order strains would presumably be determined in a more complete calculation by matching. Whatever the precise values of these strains, though, their effect on f_R would not be seen until beyond $O(v^9)$ in the body-zone expansions.

Let $\gamma(t,\ldots,t/v^5)$ be the $\frac{5}{2}$-PN world line that body 1 would follow in the absence of the resistive potentials f_R. (For example, one could solve the problem again using a half-advanced, half-retarded boundary condition.) One expects γ to deviate from geodesic motion due to couplings such as those analyzed in Ref. 2. Let

$$\overline{g} \sim g^{(I)} + v^4\,{}_4h^{(I)} + \cdots + v^9\,{}_9h^{(I)} + \cdots \tag{5.1}$$

be the body-zone expansion for this hypothetical problem. Let $\gamma_R(t,\ldots,t/v^5)$ be the $\frac{5}{2}$-PN world line that body 1 follows with f_R taken into account. Suppose the acceleration $v^5 a^i$ of γ_R with respect to γ differs from the value $\frac{1}{4}\partial f_R/\partial x^{*i}$ by an amount

$$k^i = a^i - \frac{1}{4}\frac{\partial f_R}{\partial x^{*i}}\,. \tag{5.2}$$

Then matching generates an additional $O(v^9)$ correction ${}_9h^{(I)}{}_R$ to ${}_9h^{(I)}$.

This correction ${}_9h^{(I)}{}_R$ then satisfies the same equations as the earlier Newtonian perturbations ${}_4h^{(I)}$ of Sec. II. By a similar argument, with ${}_9\kappa_R$, ${}_9\lambda_R,\ldots$ in place of ${}_4\kappa$, ${}_4\lambda,\ldots$ one finds that the EFE can only be satisfied if

$$k^i = 0\,. \tag{5.3}$$

Thus, the resistive potential f_R causes the bodies to deflect according to the acceleration law

$$a^i = \frac{1}{4}\frac{\partial f_R}{\partial x^{*i}}\,. \tag{5.4}$$

The problem of calculating the secular changes in a, e, and P—which become significant over $\frac{5}{2}$-PN time scales—has now been reduced to kinematics. A convenient approach is to use an indirect argument: Translate the acceleration law (5.4) into a "force" law by taking the Newtonian law as a definition

$$F_j^i = v^5 \mu_I a_I^i\,, \tag{5.5}$$

and similarly take Newtonian definitions for mechanical energy and angular momentum. The rates of energy and angular momentum loss in a Newtonian system of course satisfy

$$\left\langle \frac{dE}{dt} \right\rangle = \sum_{j,I} \left\langle F_I^j \frac{dr_I^j}{dt} \right\rangle, \tag{5.6}$$

$$\left\langle \frac{dM^i}{dt} \right\rangle = \sum_{j,k,I} \langle r_I^j F_I^k \rangle \epsilon_{ijk}\,, \tag{5.7}$$

and these quantities are related to the semimajor axis and eccentricity by the usual Keplerian expressions. Substitution of Eqs. (4.23), (5.7), and (5.8) into Eqs. (5.6) and (5.7) gives the same expressions for the time-averaged changes of energy and angular momentum as were derived in Refs. 12 and 14 using the linearized theory. The rates of change of the semimajor axis, eccentricity, and period therefore obey the standard formulas given in Refs. 12, 14, and 20.

Note that the mechanical energy and angular momentum defined above are not assumed to be the same as the energy and angular momentum calculated from integral conservation laws, but are merely convenient, slowly varying parameters of the motion. However, their rates of change do in fact agree with those given by the usual pseudotensor fluxes.

VI. CONCLUSIONS

This calculation has treated a binary system containing two compact objects with negligible mass loss. If the system obeys my mathematical assumptions, then the rates of change of orbital period, semimajor axis, and eccentricity due to radiation reaction obey the same formulas that were derived previously using the linearized theory. Whereas the previous derivations were only valid in the case of objects whose internal gravity is weak, this calculation has shown that the standard formulas remain valid even for objects whose internal gravity is strong, as in the binary pulsar.

These formulas and those given by Burke[3] and

MTW (Ref. 20) will be uniformly valid only over time scales of order (orbital period/v^5). Over still longer time scales, radiation reaction from higher-order terms in the wave-zone expansion—due to higher moments, post-Newtonian sources, etc.—eventually will become important. While the lowest-order radiation reaction will dominate the higher-order terms over observational time scales, there is no *a priori* justification for ignoring the higher-order terms when studying the entire past history of a system, as was done in Ref. 4.

Since the Newtonian orbits and potentials determine the lowest-order resistive effects, I have ignored the PN through 2-PN (time-even) corrections to the near-zone fields and motion in calculating the radiation reaction. The PN fields and motion are used in interpreting observations of the binary pulsar to compute a mass ratio. These terms have never been calculated self-consistently for systems of the type studied here. In particular, one must show that the PN terms match to the wave-zone expansion. The higher-order terms ($\frac{3}{2}$-PN, etc.) in the near-zone should also be checked as part of an overall program to show self-consistency of the slow-motion approximation. Infinities that occur in some treatments[21-23] will need to be understood, perhaps, as suggested in Ref. 17 by introducing nonanalytic terms into the near-zone expansion.

Further nonuniformities will occur in the limit of large radius (at some fixed retarded time), where the Minkowskian characteristics make large phase errors; i.e., the radiation is redshifted due to, for example, the mass of the system. A proper treatment of these phase shifts using straining techniques requires a study of higher-order terms in the wave-zone expansion, because these terms eventually become sensitive to nonlinearity. [Nonlinear source terms first enter at $O(v^6)$ and become dynamically important at $O(v^8)$.] This issue and its relation to the retardation boundary condition will be discussed in a future paper.[8]

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