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Motion of a small body through an external field in general relativity calculated by matched asymptotic expansions

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This paper shows that a small body with possibly strong internal gravity moves through an empty region of a curved, and not necessarily asymptotically flat, external space-time on an approximate geodesic. By "approximate geodesic," one means the following: Suppose the ratio $\epsilon \equiv m/L$, where m is the body's mass and L is a curvature reference length of the unperturbed external field, is a small parameter. Then $O(L)$ deviations from geodesic motion in the unperturbed external field vanish over times of $O(L)$, with possible $O(L)$ corrections occurring only over times of order L/ϵ or longer. The world line is here calculated directly from the Einstein field equation using a generalized method of matched asymptotic expansions based on a previous paper concerning singular perturbations on manifolds and related to a technique used by D'Eath. Aside from D'Eath's work, previous results on the motion of realistic bodies have assumed weak internal gravity, in some cases incorporating additional assumptions such as perfect fluids or high symmetry. This calculation makes no assumptions about the details of the body, such as weak fields, symmetry, the equations of state for matter, or even the presence of matter. Most previous treatments assumed asymptotic flatness of the external field. Here, it is only assumed that, in the region of interest, the external space-time is empty and free of singularities. The results extend the work of D'Eath to a more general class of objects that includes nonstationary black holes, naked singularities, and neutron stars, as well as ordinary astrophysical objects. This method can be applied to related problems, such as the motion of a charged black hole through an external gravitational and electromagnetic field. A future paper will combine this method with Burke's method of obtaining radiation reaction to calculate the orbital-period shortening of gravitationally bound, slow-motion systems, such as the binary pulsar PSR 1913 + 16, containing objects with strong internal gravity.

I. INTRODUCTION TO THE PROBLEM

The discovery of the binary pulsar PSR 1913+16 has given rise to a flurry of papers[1-10] deriving observational consequences of general relativity (and other theories) for this system. However, as argued by Ehlers *et al.*,[11] the general-relativity predictions (in particular) of orbital-period shortening due to radiation reaction all rely on a "standard" formula that has not yet been derived rigorously from general relativity. For gravitationally bound, *weak-field*, slow-motion systems, Burke[12,13] has derived the radiation reaction directly from the Einstein field equation (EFE), together with an outgoing-wave assumption, by the method of matched asymptotic expansions. His method produces, via matching to a wave-zone solution, "resistive potentials" that in the near zone cause forces of order v^5 smaller than Newtonian forces. In order to apply Burke's analysis to systems, such as the binary pulsar, containing objects with strong internal gravity, one needs to calculate the deflections of the bodies, and thus their orbital-period shortening, due to these resistive potentials.

This paper shows how the deflection of a body in an external field can also be obtained directly from the EFE. I will apply the singular-perturbation tools of Ref. 14 to show that a body whose mass m is small compared to an external curvature reference length L moves through an external spacetime on an approximate geodesic, over a time comparable to L. This calculation assumes that one is given two solutions of the EFE:

(1) The "external spacetime": an arbitrary, curved spacetime containing an empty, singularity-free region.

(2) The "body": an asymptotically flat, possibly singular spacetime whose matter is confined to a spatially bounded region and whose field approaches that of a Schwarzschild solution at large distances.

In effect, I will blend together these two given spacetimes by using them as the zeroth-order terms in (respectively) an external and an internal asymptotic expansion of an assumed "exact" one-parameter family of spacetimes. I assume that the first-order corrections are proportional to m/L and that these corrections have coordinate expansions that begin in powers. Section III details the assumptions and plan of attack.

Section II sketches the history of this problem, from the pioneering Einstein-Infeld-Hoffman (EIH) papers, through the various center-of-mass approaches to the use of singular-perturbation methods by Burke[12,13] and D'Eath.[15,16] Because of the short curvature length scale associated with its self-fields, the motion of a body with strong internal gravity cannot be derived simply from the principle of equivalence, stress-energy conservation, or the usual linearization about flat space.

In fact, no satisfactory definition of the world line of such an object has yet been given: previous definitions of the world line have only been successfully applied to objects with weak internal gravity. Most previous treatments of motion have also assumed that the external fields are asymptotically flat, a restriction that rules out some systems of astrophysical interest.

This paper extends the previous results in two directions: First, it will be shown that even objects with *strong* internal gravity move on approximate geodesics over times of order L. The calculation will be independent of the internal details of the object (such as the values of higher moments), the matter equations of state (or even the presence of matter), and the presence of event horizons or internal singularities. Second, I will calculate the motion of objects through an empty, but otherwise arbitrarily curved, region of an external spacetime whose fields are not necessarily asymptotically flat. The calculation will be independent of how these external fields are produced, and it will be unnecessary to discuss the behavior of the fields at "infinity."

To understand why this problem requires a singular-perturbation technique, it is useful to recall the example of a slow-motion, radiating system. The length scale l associated with the source dimensions (e.g., orbital radius) is small $[O(v)]$ compared to the wavelength λ of the waves. (The interaction potential Φ is also small and scales with v^2 for a gravitationally bound system.) However, the slow-motion expansion fails to be uniformly valid in the wave zone, where time and space derivatives are comparable, and there a weak-field but not slow-motion expansion is required. In the present problem, one can associate dimensionless parameters with the object by dividing its size (R_0), charge, angular momentum, higher moments, etc., by its mass m to the appropriate power. One can similarly construct dimensionless parameters from external quantities, such as curvature gradients, using a curvature reference length L. An observer at some distance comparable to L from an object with $\epsilon \equiv m/L \ll 1$ sees a weak perturbation of the external field. However, the external expansion may fail to be uniform near the object, in particular, when its mass to size ratio or *compactness* m/R_0 is comparable to unity, as for a black hole or neutron star.

For this reason, the internal region will be given its own expansion, and the internal and external expansions will be matched as described in Ref. 14. I will introduce a suitable definition of a limiting or approximate world line, and I will then show that a hypothetical nonvanishing zeroth-order acceleration of this world line would lead to violations of the EFE. I will study the system for a time of order L, and therefore this analysis will be insensitive to the small deflections from geodesic motion due to spin-curvature and higher-order couplings, which are expected to become important over time scales of order L/ϵ^n for a 2^n-pole moment. (Since $\epsilon \propto v^{1/3}$ for a bound system of comparable masses, radiation reaction does not appear at the orders considered.) However, I will not assume that these moments vanish, but only that they scale with the appropriate power of m.

One application of this method was mentioned above: calculating the response of an object with strong internal gravity to the resistive potential that Burke's radiation-reaction calculation produces. The assumption of parallel transport of spin[17] in slow-motion systems containing objects with strong internal gravity also needs to be verified, and no obstacles appear to extending the results of this paper to calculate at least the $O(1)$ spin precession. It also appears possible to derive the approximate Lorentz force law for small charged objects using similar methods. Future applications will be discussed in more detail in the conclusions.

II. A BRIEF HISTORY OF THE PROBLEM OF MOTION IN GENERAL RELATIVITY

Early treatments of the problem of motion sought to calculate the motion of ideal "point singularities" by describing only the external region of spacetime. The EIH method[18] used a slow-motion, weak-field expansion, while the fast-motion method[19] used only a weak-field expansion. Related work[20,21] did not assume weak external fields. The main shortcoming of all these methods was their use of a single expansion to describe zones having qualitatively different behavior. As a result, none of these early treatments correctly predicted radiative effects. In the EIH case, the problem was nonuniformity of the slow-motion approximation in the wave zone. In the fast-motion case, taking limits sequentially led to the neglect of terms as large as those under consideration. Later explications[22] used δ functions, despite the importance of nonlinearities. All of these expansions became nonuniform near the sources, where special rules for the addition of singular terms were needed to obtain unique equations of motion. These rules in effect restricted the physical applicability to nearly spherical and nonrotating bodies. Further discussion can be found in Goldberg[23]; implicit assumptions in EIH have been critically examined from a field-theoretic point of view by Dresden and Chen.[24]

A second line of attack sought to generalize the Newtonian idea of center of mass and to define a suitable center-of-mass world line. Dixon[25-28] defined moments of the stress-energy tensor and showed that stress-energy conservation reduces to a system of ten first-order ordinary differential equations for the time derivatives of momentum and spin. As pointed out by Ehlers and Rudolph,[29] however, Dixon's equations cannot yet be used to predict the motion of an object with significant self-gravity because of the difficulty (in a nonlinear theory) of separating fields due to external sources from self-fields. Geroch and Soo Jang[30] showed that the (suitably defined) world tube of a body contains a geodesic, *provided* the body is sufficiently small compared to a typical curvature reference length associated with the total field. However, a compact object is *not* small compared to a typical curvature reference length of the total field, because the total field includes the object's self-fields.

Since the center-of-mass approaches seem to founder on the issue of separating out the self-fields, why not account for self-energy by means of a pseudotensor? For example, one can successfully incorporate gravitational stresses with matter stresses into an approximate, post-Newtonian conservation law. However, we are treating possibly strong internal fields that do not have a weak-field expansion.

Newman and his co-workers[31-33] developed a third line of attack using the Newman-Penrose (NP) formalism. The usefulness of the approach taken by Lind *et al.*[33] is limited by one's inability to infer the observed path of a body from the equations governing quantities defined at null infinity. Another limitation is that this approach cannot describe the relative motion of individual components of an interacting system, because the system must be treated as a whole. Objects with no angular momentum or higher moments were studied in a related line of research.[31,32] By drawing an analogy to the behavior of null cones emanating from accelerated world lines in flat space, Newman and Posadas obtained certain "equations of motion." However, it is not yet possible to interpret these equations in terms of the observed path of a body moving under the influence of external fields.

Numerous disagreements have occurred concerning the energy (calculated from linearized theory) carried by gravitational waves in systems of interest. However, for the binary pulsar and most other similar sources, one observes not the energy loss, but the change in orbital period and other parameters. As yet, no self-consistent argument allows one to infer the change of a bound system's orbital period directly from an energy-loss formula. Burke's[12,13] derivation of radiation reaction requires *no* definition of gravitational energy; it seems clear that in any radiation-reaction calculation, one must discuss the gravitational field in both the radiation and near zones.

Recognizing the possibility of unexpected effects due to compactness, Borner and Rudolph[34,35] studied a static system of two compact bodies connected by a rod. Up to post-post-Newtonian order, they found no anomalous forces. However, effects of compactness may have escaped their analysis. First, only weak-field expansions were used. Second, their definitions of force and mass were somewhat arbitrary, and the presence of a rod complicated the interpretation of higher-order corrections. Third, there is no direct relation between their calculated stresses and the motion of compact objects in a dynamic, gravitationally bound system.

The first successful approach to the motion of black holes—an important class of compact objects—was D'Eath's[15] analysis of the motion of a Kerr black hole through an external universe. D'Eath's use of the method of matched asymptotic expansions correctly treated the singular nature of this problem. D'Eath's inspirational calculation depended on the details of the internal structure of a Kerr black hole and thus cannot be directly applied to the present problem. D'Eath's successful application of his method to the slow-motion interaction of two black holes[16] suggests a similar application of this calculation, as discussed in the conclusions.

III. METHOD OF ATTACK

The problem is to construct a family of approximate spacetimes representing an object of mass m propagating through an external spacetime of curvature reference length L. I seek to show that the object moves on a world line that deviates from a geodesic by distances of at most $O(\epsilon)L$ over times of order L, assuming $\epsilon \equiv m/L \ll 1$. Mathematically, the world line of a self-gravitating object has no precise meaning, and so part of my effort will have to go toward giving a suitable, self-consistent, and unambiguous definition of the world line. To show that this world line is an approximate geodesic, I will use matched asymptotic expansions to seek a uniformly valid family of approximate solutions of the EFE.

The formalism given in Ref. 14 is well suited to this problem. One treats not a single system, but a family of systems having ϵ in the *open* interval $0<\epsilon<\epsilon_0$, each system posed on a four-dimensional manifold S. To study this family of systems, one introduces a smooth five-dimensional manifold

$$\tilde{M} = \mathcal{S} \times (0, \epsilon_0), \tag{3.1}$$

where $(0, \epsilon_0)$ is the open interval containing ϵ. The dynamics for fixed ϵ takes place on a four-dimensional slice of $\tilde{M}$ or *exact solution manifold* S_ϵ.

I assume that on each S_ϵ there exists a smooth, exact metric tensor $\mathcal{G}_\epsilon$, varying smoothly, but not necessarily analytically, with ϵ. (Other representations of the gravitational field will also be employed when convenient.) The matter tensor corresponding to each metric $\mathcal{G}_\epsilon$ will be described shortly; however, my attention will primarily be focused on a region of each exact solution manifold S_ϵ in which $\mathcal{G}_\epsilon$ satisfies the *vacuum* EFE.

Each "exact spacetime" $(S_\epsilon, \mathcal{G}_\epsilon)$ is to represent a fully nonlinear, possibly singular object propagating through a curved, and not necessarily asymptotically flat, external gravitational field. The formal task is to construct asymptotic expansions of this family of exact spacetimes, uniformly valid in the limit $\epsilon \to 0$. The practical problem is to extract the object's approximate motion from the asymptotic behavior of this family of spacetimes $(S_\epsilon, \mathcal{G}_\epsilon)$, as $\epsilon \to 0$.

The first formal step will be to give appropriate mathematical representations for the object and external spacetime (Secs. IV and V). I assume as given two exact solutions of the EFE: one representing the unperturbed object, a possibly singular solution that is empty outside a spatially bounded region; the other representing the unperturbed external spacetime, a topologically compact, matter-free, and singularity-free region of an otherwise arbitrary spacetime. The important limits to consider correspond physically to "the object's point of view" and "the external point of view." The manifolds of these given solutions will therefore serve as model manifolds,[14] and on each will be constructed the first few terms of an asymptotic expansion of the gravitational field.

The zeroth-order term in each expansion will represent the aforementioned "given" field. Terms of first order in ϵ will represent the lowest-order mutual perturbing influences. Higher-order terms will not explicitly enter this calculation. However, a slow-time variable will be needed in the internal expansion, because the time (of order L) needed to study the system is of order $1/\epsilon$ compared to the internal time scale m.

The next step will be to match these internal and external expansions. Matching on manifolds not only determines the values of unknown constants, functions, etc., that appear in asymptotic expansions of tensor fields, but it also restricts how points in one model correspond to points in another; in this problem, "points" are spacetime events. By placing restrictions on how events in the internal and external model spacetimes[14] correspond, matching will determine the motion of the object.

However, as discussed in Sec. VI, the motion of the object is only one of an infinite number of degrees of freedom in the correspondence between events in the two model spacetimes. For example, small strains can occur without affecting the zeroth-order motion. To show that these higher-order degrees of freedom do not affect the zeroth-order motion, I will show first that they can be represented as gauge transformations of the perturbations, and second that these gauge transformations do not affect any aspect of the calculation.

The most direct way to show gauge independence in this calculation is to use manifestly gauge-invariant perturbation equations whenever possible. In view of the arbitrariness implicit in the zeroth-order model spacetimes, it proves convenient to express the asymptotic expansions of the gravitational field in a Cartan representation. This representation was discussed in Ref. 36.

Because of nonlinearity, arbitrariness enters the perturbation equations in the form of unspecified coefficients. Nevertheless, by expanding in an intermediate limit process,[14] one can isolate enough of the generic behavior of the system to determine an "asymptotic world line." To justify this intermediate expansion, certain regularity assumptions are needed:

(1) The lowest-order corrections in each expansion are of order ϵ.

(2) The first-order perturbations have coordinate expansions that start off with powers.

One should be alert for hints of peculiar circumstances under which these assumptions of powers need to be modified. (In future applications involving secular effects, one should also envision the need for further modifications such as very slow variables in the external expansion.)

The last step is to analyze the terms in the intermediate expansion that depend on acceleration in order to show that the zeroth-order acceleration vanishes. This last step comprises several stages (Secs. VIII and IX). Suppose the object moves along a trial world line with nonvanishing zeroth-order acceleration $a_i(t)$. There are then nonzero first-order contributions to the internal expansion of the Einstein tensor, thus violating the vacuum EFE unless further perturbations generate an equal and opposite contribution. These further perturbations are inconsistent with the EFE (and

Bianchi identities) unless $a_i(t)=0$. Therefore, the limiting or asymptotic world line is a geodesic of the zeroth-order external gravitational field.

In the concluding section, I will discuss the results and suggest some promising generalizations and applications.

IV. INTERNAL EXPANSION

In this section, I introduce asymptotic expansions to represent an object that is slightly perturbed by the presence of external fields, and I discuss the zeroth-order terms in detail. Usually in general relativity, one assumes either weak fields, high symmetry, or a specific form for the matter tensor (e.g., perfect fluid). This calculation assumes only (1) that the zeroth-order internal model spacetime $(C, \mathcal{G}^{(C)})$ satisfies the vacuum EFE outside a spatially bounded region and (2) that it approaches a Schwarzschild solution at large distances. The model spacetime $(C, \mathcal{G}^{(C)})$ represents the object as if it were isolated from the external field. The influence of the external field will be seen in first-order perturbations, and these first-order terms will later tell us the motion of the object.

At this stage, it is not obvious how one takes the limit $\epsilon \to 0$ of an object whose mass m is much smaller than an external curvature reference length L. Recall from Sec. III that the compactness, angular momentum/mass^2, and other internal quantities have been made dimensionless by rescaling with the appropriate power of m, while external quantities, such as the curvature derivatives, have been rescaled with a power of L. The internal solution cannot tell that it is getting "smaller," except by comparison to the external length scale. The zeroth-order terms in each expansion are unaffected by the process of taking $\epsilon \to 0$. However, my scaling assumptions will be automatically incorporated in the correspondence maps between the internal and external model spacetimes.

The study of isolated objects in general relativity usually involves subtleties, such as the definitions of mass and center of mass, that must be handled delicately. I sidestep these delicate matters by studying an object whose gravitational field approaches that of an ideal monopole—a Schwarzschild solution—under a limit process $\alpha \to 0$, (R_α, Θ, Φ) fixed, where $R_\alpha \equiv \alpha R$, and $\alpha > 0$. (By analogy, the Newtonian potential of an arbitrary isolated object approaches the field of a point mass, under this limit process.) This assumption applies only to the object's zeroth-order properties, which do not include the perturbing effects of the external field.

In Schwarzschild coordinates (T, R, Θ, Φ) nondimensionalized by the mass m, the Schwarzschild metric $\hat{\mathcal{G}}$ is

$$\hat{\mathcal{G}} = -dT \otimes dT(1-2R^{-1}) + dR \otimes dR(1-2R^{-1})^{-1} + R^2(d\Theta \otimes d\Theta + \sin^2\Theta d\Phi \otimes d\Phi) . \quad (4.1)$$

Now consider a flat metric

$$\mathcal{G}^* = -dT \otimes dT + dR \otimes dR + R^2(d\Theta \otimes d\Theta + \sin^2\Theta d\Phi \otimes d\Phi) \quad (4.2)$$

on the same manifold. Rewrite each metric in terms of the auxiliary coordinates

$$T_\alpha = \alpha T , \quad R_\alpha = \alpha R \quad (4.3)$$

and consider the limit $\alpha \to 0$. The metric $\hat{\mathcal{G}}$ then has an expansion

$$\hat{\mathcal{G}} \sim \alpha^{-2}[-dT_\alpha \otimes dT_\alpha + dR_\alpha \otimes dR_\alpha + R_\alpha{}^2(d\Theta \otimes d\Theta + \sin^2\Theta d\Phi \otimes d\Phi) + 2\alpha R_\alpha{}^{-1}(dT_\alpha \otimes dT_\alpha + dR_\alpha \otimes dR_\alpha) + O(\alpha^2)] . \quad (4.4)$$

Note that the first (flat) term $\alpha^{-2}[-dT_\alpha{}^2 + dR_\alpha{}^2 + R^2(\cdots)]$ is simply $\mathcal{G}^*$. One can similarly expand a set of orthonormal frames $\hat{\omega}^\mu$ and the associated connection and curvature forms $\hat{\Omega}^\mu_\nu$ and $\hat{\mathcal{R}}^\mu_\nu$.

The first nonflat term in each expansion is down by a factor of α from the flat term. For convenience, denote the first nonflat terms in the metric, frames, connection, and curvature by $\delta\mathcal{G}^*$, $\delta\omega^*$, $\delta\Omega^*$, and $\delta\mathcal{R}^*$, respectively. The expansions of the metric, frames, connection forms, and curvature forms can then be written in abbreviated form (indices suppressed) as

$$\hat{\mathcal{G}} \sim \mathcal{G}^* + \delta\mathcal{G}^* + O(\alpha^2) , \quad (4.5)$$

$$\hat{\omega} \sim \omega^* + \delta\omega^* + O(\alpha^2), \quad (4.6)$$

$$\hat{\Omega} \sim \Omega^* + \delta\Omega^* + O(\alpha^2) , \quad (4.7)$$

$$\hat{\mathcal{R}} \sim 0 + \delta\mathcal{R}^* + O(\alpha^2) . \quad (4.8)$$

I assume that the unperturbed spacetime $(C, \mathcal{G}^{(C)})$ is close to the Schwarzschild spacetime $(R^2 \times S^2, \hat{\mathcal{G}})$ in the same sense that the Newtonian potential of an isolated object is close to the potential of an ideal Newtonian monopole. Denote the basis, connection, and curvature forms of $(C, \mathcal{G}^{(C)})$ by $\omega^{(C)}$, $\Omega^{(C)}$, and $\mathcal{R}^{(C)}$, respectively. Suppose that C has a region that can be covered by a system of coordinates (T, R, Θ, Φ). One can then define an auxiliary coordinate system $(T_\alpha, R_\alpha, \Theta, \Phi)$ exactly as for the Schwarzschild spacetime. I suppose that the frame, connection, and curvature forms have expansions (in the limit $\alpha \to 0$) that agree, to first order in α, with the Schwarzschild expressions for $\hat{\omega}$, $\hat{\Omega}$, and $\hat{\mathcal{R}}$ above:

$$\omega^{(C)} \sim \omega^* + \delta\omega^* + \delta\delta\omega^* , \quad (4.9)$$

$$\Omega^{(C)} \sim \Omega^* + \delta\Omega^* + \delta\delta\Omega^* , \tag{4.10}$$

$$\mathcal{R}^{(C)} \sim 0 + \delta\mathcal{R}^* + \delta\delta\mathcal{R}^* , \tag{4.11}$$

where the terms up to $\delta\omega^*$, $\delta\Omega^*$, and $\delta\mathcal{R}^*$ agree with their Schwarzschild counterparts. Aside from a possible overall scale factor, the terms $\delta\omega^*$, $\delta\Omega^*$, and $\delta\mathcal{R}^*$ are of order α in the limit process $\alpha \to 0$ $(T_\alpha, R_\alpha, \Theta, \Phi)$ fixed. I assume that the terms $\delta\delta\omega^*$, $\delta\delta\Omega^*$, and $\delta\delta\mathcal{R}^*$ are of order α^2 in this limit process and restrict consideration to the case in which the spacetime $(C, \mathcal{G}^{(C)})$ is matter-free outside some fixed value R_{empty} of the coordinate R.

The category of objects whose gravitational fields satisfy these assumptions appears to include most nonpathological asymptotically flat, stationary spacetimes satisfying the vacuum EFE outside a spatially bounded region [e.g., the Kerr, Weyl, and Tomimatsu-Sato (T-S) solutions] as well as many nonstationary ones. An object in this category need not satisfy any particular symmetry assumption, equation of state, causality condition, or topological restriction. It is likely that all compact objects of astrophysical interest meet the above assumptions.

It will be important for matching to note that the monopole part $\delta\mathcal{G}^*$ of the internal metric $\mathcal{G}^{(C)}$ determines privileged local coordinates

$$\begin{aligned} X^\mu &= (T, X, Y, Z), \\ X &= R \sin\Theta \cos\Phi , \\ Y &= R \sin\Theta \sin\Phi , \\ Z &= R \cos\Theta . \end{aligned} \tag{4.12}$$

Now consider a coordinate transformation of the form

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu , \tag{4.13}$$

where $\Lambda^\mu{}_\nu$ represents some combination of boosts and rotations. As far as the flat metric $\mathcal{G}^*$ is concerned, the X^μ and X'^μ coordinate systems are equivalent, because the frames dX^μ and dX'^μ are both orthonormal in $\mathcal{G}^*$. However, decomposing $\Lambda^\mu{}_\nu$ into a product of pure boosts and rotations, one finds that the rotations leave invariant the monopole term $\delta\mathcal{G}^*$, while the boosts change the form of $\delta\mathcal{G}^*$. In this sense, the monopole part of $\mathcal{G}^{(C)}$ distinguishes, up to rotations, an *asymptotic rest frame* dX^μ [see also Sec. 19.3 of Misner, Thorne, and Wheeler[37] (MTW)].

So far, I have been discussing the unperturbed or zeroth-order internal model spacetime. The internal expansion will need to be accurate to first order in ϵ and valid for times comparable to m/ϵ for this calculation. The metric expansion therefore takes the form

$$\bar{\mathcal{G}}^* \sim \mathcal{G}^{(C)} + \epsilon h^{(C)*}(X^\mu, \epsilon T) + O(\epsilon^2) , \tag{4.14}$$

where the slow-time argument allows the internal expansion to respond "quasistatically" to changes on the external time scale L that arise from matching. The time dependence of these slow changes enters the first-order internal equations only as a parameter, however.

The frames, connection, and curvature forms are similarly expanded in the form

$$\bar{\omega}^* \sim \omega^{(C)} + \epsilon w^{(C)*}(X^\mu, \epsilon T) + O(\epsilon^2) , \tag{4.15}$$

$$\bar{\Omega}^* \sim \Omega^{(C)} + \epsilon W^{(C)*}(X^\mu, \epsilon T) + O(\epsilon^2) , \tag{4.16}$$

$$\bar{\mathcal{R}}^* \sim \mathcal{R}^{(C)} + \epsilon P^{(C)*}(X^\mu, \epsilon T) + O(\epsilon^2) . \tag{4.17}$$

The curvature due to external fields becomes large compared to the object's own curvature at large distances. The domains of validity[14] $\mathfrak{D}_\epsilon^{(C)}$ of the internal expansion cannot be expected to cover the entire internal model spacetime. However, as $\epsilon \to 0$, the $\mathfrak{D}_\epsilon^{(C)}$ can be expected to increase in size fast enough to provide an overlap with the external expansion, which will be introduced in the following section.

V. THE EXTERNAL EXPANSION

Just as the world line of a non-test body has no precise, *a priori* meaning in general relativity, so the concept of a ray has no precise, *a priori* meaning in electrodynamics. However, one can operationally define "ray," "polarization," "amplitude," and "phase" in the context of a short-wavelength expansion of the electromagnetic field (see Chapter 22 of MTW[37] or Ref. 36). In this section, the approximate world line of a self-gravitating object is defined operationally in the context of an asymptotic expansion of the gravitational field.

The external expansion of this section complements the internal expansion of Sec. IV. While neither expansion will be uniformly valid by itself, I will match them in order to seek, in the language of Ref. 14, a "global asymptotic approximation." Under the external limit process, the size and mass of the object scale with m. I therefore represent the effect of the object on the external fields by expansions

$$\bar{\mathcal{G}}^{**} \sim \mathcal{G}^{(E)} + \epsilon h^{(E)} + O(\epsilon^2) , \tag{5.1}$$

$$\bar{\omega}^{**} \sim \omega^{(E)} + \epsilon w^{(E)} + O(\epsilon^2) , \tag{5.2}$$

$$\bar{\Omega}^{**} \sim \Omega^{(E)} + \epsilon W^{(E)} + O(\epsilon^2) , \tag{5.3}$$

$$\bar{\mathcal{R}}^{**} \sim \mathcal{R}^{(E)} + \epsilon P^{(E)} + O(\epsilon^2) . \tag{5.4}$$

The $O(\epsilon)$ terms in these external expansions will later communicate to the internal expansion, via matching, essential information concerning the calibration of clocks and rods in the overlap zones. For the remainder of this section, however, I discuss only the zeroth-order terms.

The zeroth-order model spacetime $(E, \mathcal{G}^{(E)})$ represents the external field as if the object were not present and is assumed to satisfy the following requirements: (1) E is a topologically compact manifold, (2) $\mathcal{G}^{(E)}$ is nonsingular in E, and (3) $\mathcal{G}^{(E)}$ satisfies the vacuum EFE in E. Of course, $(E, \mathcal{G}^{(E)})$ may have an extension containing matter or singularities. Assumption (1) implies physically that the size of the region under consideration scales with L as $\epsilon \to 0$, rather than, say, with L/ϵ. This calculation is therefore insensitive to secular effects that would create $O(L)$ deflections only over times of order L/ϵ. Assumption (2) [together with (1)] guarantees that the external reference length L is in fact well defined; relaxing this assumption could lead to nonuniformities. Therefore, this calculation will not predict either (i) the motion of an object just before it reaches a curvature singularity, or (ii) the motion of objects subject to nongravitational forces; such forces would obviously violate assumption (3).

In order to define an asymptotic world line, consider a smooth, timelike, *trial world line* $\gamma(\tau)$ in E. For only certain γ will there be approximate solutions of the EFE such that, as $\epsilon \to 0$, the object becomes localized to γ; these particular trial world lines will be called *asymptotic world lines*. "Localizing the object to γ" means the following: Use γ to define a region $D^{(E)}$ by

$$D^{(E)} \equiv E - \{\gamma\}. \tag{5.5}$$

Suppose the domains of validity $\mathfrak{D}_\epsilon^{(E)}$ eventually fill $D^{(E)}$; i.e.,

$$\text{union over } \epsilon \text{ of } \mathfrak{D}_\epsilon^{(E)} = D^{(E)}. \tag{5.6}$$

Then the external domains of validity $\mathfrak{D}_\epsilon^{(E)}$ eventually enter every neighborhood of γ, and the object appears asymptotically localized to γ. Now, suppose that for some trial world line γ, one can indeed construct a first-order global asymptotic approximation satisfying Eq. (5.6). This trial world line γ is then called an *asymptotic world line*.

Suppose p_0 is an initial event along an asymptotic world line $\gamma(\tau)$ and u_0 a tangent vector to $\gamma(t)$ at p_0. This calculation will show that $\gamma(t)$ is a geodesic, and thus will be uniquely determined, modulo initial conditions. This "uniqueness" does not, of course, rule out higher-order corrections to γ. The only eligible trial world lines in E are those whose accelerations are independent of ϵ; this restriction is adequately general for an analysis valid over time scales of $O(L)$. Future calculations of motion valid over times of $O(L/\epsilon)$ or longer would require a modified definition of the asymptotic world line (see the conclusions).

It will be extremely convenient to introduce a Fermi-normal coordinate system on $(E, \mathcal{G}^{(E)})$ near the trial world line $\gamma(\tau)$. This expansion will be valid at least in a neighborhood of $\gamma(\tau)$ because $\gamma(\tau)$ is smooth and E is free of singularities. In this coordinate system, the acceleration of γ appears as a readily identifiable parameter $a_i(\tau)$ in $\mathcal{G}^{(E)}$. The well-known Fermi coordinate construction and expansion are given in MTW[37] and in Ref. 38. Suppose τ is the proper time along $\gamma(\tau)$, with $\tau = 0$ at p_0, which implies that the timelike orthonormal basis vector $e_\tau^{(E)}$ satisfies

$$e_\tau^{(E)}\big|_{\tau=0} = u_0. \tag{5.7}$$

It is convenient to allow the orthonormal bases $\omega^{(E)}$, $e^{(E)}$ of forms and vectors to rotate with spin vector $S_i(\tau)$ (as defined in MTW) with respect to Fermi-Walker transport along γ.

Before proceeding further, one should clarify the meaning of the reference length L associated with $(E, \mathcal{G}^{(E)})$: Let $R^{(E)\text{Riemann}}_{\mu\nu\sigma\tau}$ be the components of $R^{(E)\text{Riemann}}$ in the coordinate basis $(d\tau, d\xi^i)$ (note that this basis is orthonormal along γ). Since these components are by assumption bounded in E, one can therefore define

$$L \equiv [\text{supremum in } E \text{ over all } \mu, \nu, \alpha, \beta \times (|R^{(E)\text{Riemann}}_{\mu\nu\alpha\beta}|)]^{-1/2} \tag{5.8}$$

unless the curvature tensor vanishes identically in E. This special case in which $(E, \mathcal{G}^{(E)})$ is flat can easily be handled separately, and it is assumed from here on that $R^{(E)\text{Riemann}}$ is nonvanishing.

Using L to scale the Fermi coordinates (τ, ξ^i) defined above, one obtains dimensionless coordinates (t, x^i) in a neighborhood of γ in E; hereafter, t will be the parameter along $\gamma(t)$. My index convention for these coordinates is

$$x^i = (x, y, z), \quad x^\mu = (t, x, y, z). \tag{5.9}$$

I express the Fermi expansion in terms of an auxiliary coordinate system

$$t_\lambda = t/\lambda, \quad x_\lambda^i = x^i/\lambda, \tag{5.10}$$

where λ is *not* an index, but a parameter in the range $0 < \lambda < \lambda_0$. When expanded in powers of λ under the limit process $\lambda \to 0$, (t_λ, x_λ^i) fixed, $\mathcal{G}^{(E)}$ becomes

$$\begin{aligned}\mathcal{G}^{(E)} \sim \lambda^2 (& dt_\lambda \otimes dt_\lambda\{1 + \lambda[2a_i(t)x_\lambda^i] + O(\lambda^2)\} \\ &+ (dt_\lambda \otimes dx_\lambda^k + dx_\lambda^k \otimes dt_\lambda)[\lambda\epsilon_{ijk}S_i(t)x_\lambda^j + O(\lambda^2)] \\ &+ \delta_{ij}dx_\lambda^i \otimes dx_\lambda^j + O(\lambda^2)dx_\lambda^i \otimes dx_\lambda^j).\end{aligned} \tag{5.11}$$

This expansion, like the internal expansion of Sec. IV in powers of α, will later be used for matching.

The metric $\mathcal{G}^{(E)}$ can be represented by an orthonormal frame $\omega^{(E)}$. One obvious choice for $\omega^{(E)}$ is a nearly Cartesian frame having the expansion

$$\omega^{(E)t} \sim \lambda(dt_\lambda\{1+\lambda[a_i(t)x_\lambda^i]+\cdots\} + dx_\lambda^k\{\lambda[S_i(t)x_\lambda^j\epsilon_{ijk}]+\cdots\}), \tag{5.12}$$

$$\omega^{(E)i} \sim \lambda[dx_\lambda^i + O(\lambda^2)dx_\lambda^\mu]. \tag{5.13}$$

A second choice, which will be useful for matching, is a nearly spherical basis differing only in the spatial triad and having an expansion

$$\omega^{(E)r} \sim \lambda[dr_\lambda + O(\lambda^2)dx_\lambda^\mu], \tag{5.14}$$

$$\omega^{(E)\theta} \sim \lambda[r_\lambda d\theta + O(\lambda^2)dx_\lambda^\mu], \tag{5.15}$$

$$\omega^{(E)\phi} \sim \lambda[r_\lambda \sin\theta d\phi + O(\lambda^2)dx_\lambda^\mu], \tag{5.16}$$

where

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta, \\ r_\lambda &= r/\lambda. \end{aligned} \tag{5.17}$$

Henceforth, the notation $\omega^{(E)}$ refers to this spherical basis, unless otherwise stated. The connection and curvature forms corresponding to $\omega^{(E)}$ have expansions

$$\Omega^{(E)} \sim \Omega + \lambda\nu + O(\lambda^2), \tag{5.18}$$

$$\mathcal{R}^{(E)} \sim O(\lambda^2), \tag{5.19}$$

where the $O(\lambda)$ connection forms ν follow routinely from the Cartan equations. Note that the explicit values of $\mathcal{R}^{(E)}|_\gamma$ (the curvature forms $\mathcal{R}^{(E)}$ evaluated at γ) will not enter this calculation.

So far, I have emphasized the zeroth-order terms in the internal and external expansions. Each zeroth-order term has itself been expanded in powers of an as yet artificial small parameter. Each of these secondary expansions has the effect of a power-series expansion in the appropriate distance variable. The artificial parameters α and λ will shortly be identified with functions of ϵ that approach 0 with ϵ. These functions arise naturally in the intermediate limit process to be used in matching. One can, of course, use any convenient limit process to study the family of spacetimes $(S_\epsilon, \mathcal{G}_\epsilon)$, because[14] a global asymptotic approximation must be uniform under *all* limit processes.

VI. THE GEOMETRY OF MATCHING

In the next two sections, I describe the process by which the two asymptotic expansions are matched to construct a one-parameter family of approximate spacetimes. Matching not only determines the gravitational perturbations of each model due to the other, but it also constrains which events in C can correspond to events in E. These constraints will eventually determine how the object moves.

To discuss the geometry of matching requires some essential formal ideas.[14] The purpose of the two model spacetimes C and E is to approximate the assumed family of exact spacetimes $(S_\epsilon, \mathcal{G}_\epsilon)$. C and E are linked to the S_ϵ by identification maps

$$\Phi_\epsilon^{(C)}: \mathcal{D}_\epsilon^{(C)} \subset C \to S_\epsilon, \tag{6.1}$$

$$\Phi_\epsilon^{(E)}: \mathcal{D}_\epsilon^{(E)} \subset D^{(E)} \to S_\epsilon, \tag{6.2}$$

where the $\mathcal{D}_\epsilon^{(C)}$ and $\mathcal{D}_\epsilon^{(E)}$ are the "domains of validity" of the expansions on C and E, respectively. The formal requirement for uniformity (i.e., a global asymptotic approximation) is that the union of the jurisdictions

$$\mathcal{L}_\epsilon^{(C)} = \Phi_\epsilon^{(C)}(\mathcal{D}_\epsilon^{(C)}), \tag{6.3}$$

$$\mathcal{L}_\epsilon^{(E)} = \Phi_\epsilon^{(E)}(\mathcal{D}_\epsilon^{(E)}) \tag{6.4}$$

covers the S_ϵ. In the regions O_ϵ of intersection or "overlap" of the $\mathcal{L}_\epsilon^{(C)}$, $\mathcal{L}_\epsilon^{(E)}$, the two expansions should be close to each other, and I assume the existence of an intermediate limit expansion for matching.

Fortunately, matching does not require one to construct explicit identification maps, but rather "correspondence maps"

$$\Psi_\epsilon: \mathcal{U}_\epsilon^{(C)} \xrightarrow{\text{diffeomorphisms}} \mathcal{U}_\epsilon^{(E)}, \tag{6.5}$$

where the open sets $\mathcal{U}_\epsilon^{(C)}$ and $\mathcal{U}_\epsilon^{(E)}$ are the inverse images (under the identification maps $\Phi_\epsilon^{(C)}$ and $\Phi_\epsilon^{(E)}$, respectively) of the overlap zones O_ϵ. These correspondence maps are defined from the identification maps $\Phi_\epsilon^{(C)}$, $\Phi_\epsilon^{(E)}$ by

$$\Psi_\epsilon = [(\Phi_\epsilon^{(E)})^{-1} \circ \Phi_\epsilon^{(C)}]\,|_{\mathcal{U}_\epsilon}, \tag{6.6}$$

where the notation $(\Phi_\epsilon^{(E)})^{-1} \circ \Phi_\epsilon^{(C)}$ means that $(\Phi_\epsilon^{(E)})^{-1}$ is applied after $\Phi_\epsilon^{(C)}$.

To represent a particular family of correspondence mappings Ψ_ϵ, one can write the external coordinates x^μ as functions of the internal coordinates X^μ:

$$x^\mu = \Psi_\epsilon^\mu(X), \quad X = (T, X, Y, Z). \tag{6.7}$$

Although the coordinates X^μ and x^μ need not cover C and E, respectively, the regions $\mathcal{U}_\epsilon^{(C)}$ and $\mathcal{U}_\epsilon^{(E)}$ will be covered by the X^μ and x^μ systems, provided ϵ_0 is chosen sufficiently small.

A family of correspondence maps is admissible if it localizes the object to the trial world line γ as $\epsilon \to 0$. For example one admissible map is given by

$$x^\mu = \Psi_\epsilon^\mu(X) = \epsilon X^\mu. \tag{6.8}$$

However, clearly an infinite number of different correspondence maps are also admissible under this standard. Some of the degrees of freedom in these correspondence maps will be shown to represent effects that would only be significant beyond the order considered here; the remaining ones are discussed separately in later sections.

To analyze these degrees of freedom, I first rewrite an arbitrary family Ψ'_ϵ of correspondence maps in terms of the canonical family Ψ_ϵ [Eq. (6.8)] and reexpress the condition that any admissible Ψ'_ϵ should preserve the trial world line γ. Let

$$\Psi'_\epsilon : \mathfrak{U}_\epsilon^{(C)} \to \mathfrak{U}_\epsilon^{(E)} \tag{6.9}$$

be any family of correspondence maps. One can choose a diffeomorphism

$$\Xi_\epsilon : E \to E \tag{6.10}$$

such that, for each ϵ, Ψ'_ϵ is obtained from Ψ_ϵ by composition on the left:

$$\Psi'_\epsilon = \Xi_\epsilon \circ \Psi_\epsilon . \tag{6.11}$$

The diffeomorphisms Ξ_ϵ must clearly satisfy

$$\Xi_\epsilon \big|_{\mathfrak{U}_\epsilon^{(E)}} = \Psi'_\epsilon \circ \Psi_\epsilon^{-1} . \tag{6.12}$$

In terms of coordinates x^μ, relations (6.10)–(6.12) become

$$x^\mu = \Psi'^\mu_\epsilon(X) = \Xi^\mu_\epsilon[\Psi_\epsilon(X)] = \Xi^\mu_\epsilon(\epsilon X) . \tag{6.13}$$

Similarly, one can choose a family of diffeomorphisms

$$\Sigma_\epsilon : C \to C \tag{6.14}$$

such that

$$\Psi'_\epsilon = \Psi_\epsilon \circ \Sigma_\epsilon \tag{6.15}$$

(composition on the right). The Σ_ϵ must satisfy

$$\Sigma_\epsilon \big|_{\mathfrak{U}_\epsilon^{(C)}} = \Psi_\epsilon^{-1} \circ \Psi'_\epsilon . \tag{6.16}$$

In terms of coordinates, relations (6.14)–(6.16) become

$$x^\mu = \Psi'^\mu_\epsilon(X) = \Psi^\mu_\epsilon(\Sigma_\epsilon(X)) = \epsilon \Sigma^\mu_\epsilon(X) . \tag{6.17}$$

One can also obtain the Ψ'_ϵ from the Ψ_ϵ by composing diffeomorphisms with the Ψ_ϵ on *both* left and right, i.e.,

$$\Psi'_\epsilon = \Xi_\epsilon \circ \Psi_\epsilon \circ \Sigma_\epsilon . \tag{6.18}$$

The available freedom allows one to compensate for any change in the Ξ_ϵ by a counterbalancing change in the Σ_ϵ, and vice versa.

The condition that γ and the initial event p_0 be preserved translates into a condition on the Ξ_ϵ as follows:

$$\lim_{\epsilon \to 0} \Xi_\epsilon(\gamma) = \gamma . \tag{6.19}$$

That is, if

$$\{\pi_n\} = \{\Xi_{\epsilon_0}(\pi_0)\} \tag{6.20}$$

is a sequence of events, where $\{\epsilon_n\}$ is a monotonically decreasing sequence and π_0 is an event on γ, then $\{\pi_n\}$ must converge to π_0. In terms of coordinates, Eq. (6.19) says

$$\lim_{\epsilon \to 0} (\Xi^t_\epsilon(t, 0)) = t , \tag{6.21}$$

$$\lim_{\epsilon \to 0} (\Xi^i_\epsilon(t, 0)) = 0 . \tag{6.22}$$

Next, I assume expansions of the mappings Ξ_ϵ and Σ_ϵ:

$$\Xi^\mu_\epsilon \sim \Xi^\mu_0(x) + \epsilon \Xi^\mu_1(x) + \cdots , \tag{6.23}$$

$$\Sigma^\mu_\epsilon \sim \Sigma^\mu_0(X, \epsilon T) + \epsilon \Sigma^\mu_1(X, \epsilon T) + \cdots . \tag{6.24}$$

The first-order and higher-order terms $\epsilon \Xi^\mu_1$, $\epsilon \Sigma^\mu_1$ in each expansion can be thought of as generated by vector fields on C and E, respectively. In turn, these vector fields have the effect of gauge transformations of the perturbations. For example, a mapping Ξ_ϵ defined by

$$x^\mu \to x^\mu + \epsilon \Xi^\mu(x) \tag{6.25}$$

would induce a gauge transformation

$$w^{(E)\mu} \to w^{(E)\mu} + \mathcal{L}_{\Xi_1^\nu \partial/\partial x^\nu}(\omega^{(E)\mu}) \tag{6.26}$$

in the first-order external frame perturbations. This equivalence between gauge transformations and higher-order terms in Ξ_ϵ and Σ_ϵ allows one to account for all but a few of the degrees of freedom in Ξ_ϵ and Σ_ϵ by gauge invariance of the perturbation equations.

The zeroth-order terms Ξ^μ_ϵ and Σ^μ_0 in each expansion represent finite motions of E and C, respectively. However, the available freedom in Ξ_ϵ and Σ_ϵ permits simplification of Ξ_0 and Σ_0. It is easiest to simplify these zeroth-order terms under the intermediate limit process (ILP) to be used for matching. Recall the auxiliary outer coordinates x_λ of Eq. (6.11). The choice $\lambda = \eta(\epsilon)$, where $\eta(\epsilon)$ is a smooth, monotonic, positive function satisfying

$$\lim_{\epsilon \to 0} \eta = 0, \quad \lim_{\epsilon \to 0} \epsilon/\eta = 0 \tag{6.27}$$

(e.g., $\eta = \epsilon^{2/3}$), defines an intermediate limit process by

$$\epsilon \to 0, \; x^\mu_\eta = \frac{x^\mu}{\eta(\epsilon)} \text{fixed} \quad (\eta \text{ is not an index}). \tag{6.28}$$

Under the canonical correspondence mappings Ψ_ϵ, this intermediate limit process can be expressed as $\epsilon \to 0$, $(\epsilon/\eta)X^\mu$ fixed, equivalent to the choice $\alpha = \epsilon/\eta$ for the auxiliary internal coordinates X^ν_α (where again, α is not an index). This intermediate limit process takes the external x^μ coordinates toward zero while taking the internal X^ν coordinates toward infinity.

I assume that the two zeroth-order terms Ξ_0^μ and Σ_0^μ themselves have coordinate expansions that begin in powers . In terms of the above intermediate limit process, Σ_0^μ then has an expansion

$$\Xi_0^\mu \sim C^\mu + \eta d^\mu_\nu x^\nu_\eta + \eta^2 e^\mu_{jk} x^j_\eta x^k_\eta + \cdots , \tag{6.29}$$

where C^μ, $d^\mu_\nu, \ldots$ are functions of the time t. The first term C^μ must vanish to preserve the location of the trial world line $\gamma(t)$ and the initial event p_0, as follows from Eqs. (6.21) and (6.22). The succeeding terms can be absorbed into Σ_ϵ. The effect of the linear term d^μ_ν can be accounted for by a linear term in Σ_0^μ, to be examined shortly, and therefore without loss of generality,

$$d^\mu_\nu = \delta^\mu_\nu . \tag{6.30}$$

The next term $\eta^2 e^\mu{}_{jk} x^j_\eta x^k_\eta$ is equivalent to a revision of the term $\epsilon\Sigma_1^\mu$ in Σ_ϵ^μ. Similarly, order η^3, $\eta^4, \ldots$ terms in expansion (6.29) for Ξ_ϵ^μ can be replaced by higher-order $[O(\epsilon^2),\ O(\epsilon^3), \ldots]$ terms in a revised Σ_ϵ^μ expansion. Thus, no generality is lost by taking

$$\Xi_0^\mu(x) = x^\mu . \tag{6.31}$$

My intermediate expansion of the zeroth-order internal transformation Σ_0^μ is

$$\Sigma_0^\mu \sim (\eta/\epsilon) D^\mu{}_\nu(\epsilon T) X^\nu_{\epsilon/\eta} + 1[E^\mu_{jk}(\epsilon T) X^j_{\epsilon/\eta} X^k_{\epsilon/\eta} R^{-2}] + (\epsilon/\eta)[F^\mu{}_{jkl} X^j X^k X^l_{\epsilon/\eta} R^{-3}] + \cdots . \tag{6.32}$$

$D^\mu_\nu(0)$ can be accounted for by the combination of a boost $\Lambda^\mu{}_\nu(\vec{\alpha})$ (where $\vec{\alpha}$ is the rapidity) of the internal coordinates and a trivial coordinate rotation. However, as shown explicitly in Ref. 36, $\vec{\alpha} \neq 0$ would lead to nonuniformity of the external expansion. Time dependence in $D^\mu{}_\nu$ is accounted for already by $a_i(t)$ and $S_i(t)$.

The remaining terms in Σ_0^μ are equivalent to first- and higher-order gauge transformations of the external perturbations. One is then left with

$$\Sigma_0^\mu(X) = X^\mu . \tag{6.33}$$

Thus, adequate generality is maintained by using the canonical correspondence maps of Eq. (6.7), provided one uses gauge-invariant equations.

VII. INTERMEDIATE LIMIT-PROCESS EXPANSIONS

The internal and external expansions used so far constitute two ways of ordering terms, each way consistent with its own privileged limit process. The intermediate expansions to be introduced in this section preserve both orderings and provide a format for comparing the two principal expansions. The usual matching procedure would be to calculate explicit internal and external perturbations containing unknown constants, functions of integration, etc., to be determined by matching. However, the presence of arbitrary coefficients in the perturbation equations (due to the general nature of this calculation) forces one to invert the usual procedure by first expanding in an intermediate limit process and only then analyzing the solutions. The justification for these intermediate expansions depends on my assumption that the first-order internal and external perturbations have expansions that begin with powers of the appropriate distance variable.

The two principal expansions exist on different manifolds. Applying the push-forward mappings $\Psi_{\epsilon *}$ defined by the correspondence maps Ψ_ϵ allows a comparison and thus matching. It is convenient to rescale the exact and approximate quantities as described in Table I and to refer to the order of a given term in either expansion by its size relative to the flat-space term in that expansion.

The auxiliary internal and external zeroth-order expansions of Eqs. (4.9) and (5.11) provide terms of order $\{1, \epsilon/\eta, \epsilon^2/\eta^2,\}$ and $\{1, \eta, \eta^2, \ldots\}$, respectively, in the intermediate expansions; the relevant terms for this calculation are defined in Tables II and III. Note that arbitrariness due to the object's spin and higher moments first enters at $O(\epsilon^2/\eta^2)$, while arbitrariness due to the external curvature first enters at $O(\eta^2)$.

TABLE I. Notational conventions for exact quantities: barred quantities are exact. Quantities without asterisk superscripts are rescaled so that they are manifestly of order unity under the intermediate limit process. Here $\psi_\epsilon *$ is the push-forward mapping associated with the "corresponding map" ψ_ϵ. All quantities depend implicitly on ϵ.

Name of quantity	Rescaled for ILP	Relation to exact quantity	Relation to intermediate quantity
Metric	$\bar{\mathcal{G}}$	$=\eta^{-2}\bar{\mathcal{G}}^{**}$	$=\epsilon^2\eta^{-2}\psi_\epsilon *(\bar{\mathcal{G}}^*)$
Frame	$\bar{\omega}$	$=\eta^{-1}\bar{\omega}^{**}$	$=\epsilon\eta^{-1}\psi_\epsilon *(\bar{\omega}^*)$
Connection	$\bar{\Omega}$	$=\bar{\Omega}^{**}$	$=\psi_\epsilon *(\bar{\Omega}^*)$
Curvature	$\bar{\mathcal{R}}$	$=\bar{\mathcal{R}}^{**}$	$=\psi_\epsilon *(\bar{\mathcal{R}}^*)$

TABLE II. Order unity, ϵ/η, ϵ^2/η^2 quantities in the intermediate limit process.

$\mathcal{G} \equiv \epsilon^2\eta^{-2}\psi_\epsilon*(\mathcal{G}*)$	
$\omega \equiv \epsilon\eta^{-1}\psi_\epsilon*(\omega*)$	order unity (flat space)
$\Omega \equiv \psi_\epsilon*(\Omega*)$	
$\mathcal{R} \equiv \psi_\epsilon*(\mathcal{R}*) = 0$	
$\delta\mathcal{G} \equiv \epsilon^2\eta^{-2}\psi_\epsilon*(\delta\mathcal{G}*)$	
$\delta\omega \equiv \epsilon\eta^{-1}\psi_\epsilon*(\delta\omega*)$	
$\delta\Omega \equiv \psi_\epsilon*(\delta\Omega*)$	order ϵ/η (monopole part of internal geometry)
$\delta\mathcal{R} \equiv \psi_\epsilon*(\delta\mathcal{R}*)$	
$\delta\delta\mathcal{G} \equiv \epsilon^2\eta^{-2}\psi_\epsilon*(\delta\delta\mathcal{G}*)$	
$\delta\delta\omega \equiv \epsilon\eta^{-1}\psi_\epsilon*(\delta\delta\omega*)$	order ϵ^2/η^2 (higher moments, such as spin dependence, of internal geometry; first instance of "arbitrarinesss" in object)
$\delta\delta\Omega \equiv \psi_\epsilon*(\delta\delta\Omega*)$	
$\delta\delta\mathcal{R} \equiv \psi_\epsilon*(\delta\delta\mathcal{R}*)$	

The first-order terms in the internal and external expansions also provide important terms $(\lambda_\pm, \sigma, \ldots)$ in the intermediate expansion. The intermediate frame, connection, and curvature expansions become

$$\begin{aligned}\bar{\omega} \sim\ & \omega + \eta(\kappa_+ + \kappa_-) + O(\eta^2)\\ & +(\epsilon/\eta)\delta\omega + \epsilon(\lambda_+ + \lambda_-) + O(\epsilon\eta)\\ & +(\epsilon^2/\eta^2)\delta\delta\omega + (\epsilon^2/\eta)(\sigma) + O(\epsilon^2)\,,\end{aligned} \tag{7.1}$$

$$\begin{aligned}\bar{\Omega} \sim\ & \Omega + \eta(\nu_+ + \nu_-) + O(\eta^2)\\ & +(\epsilon/\eta)\delta\Omega + \epsilon(\mu_+ + \mu_-) + O(\epsilon\eta)\\ & +(\epsilon^2/\eta^2)\delta\delta\Omega + (\epsilon^2/\eta)(\rho) + O(\epsilon^2)\,,\end{aligned} \tag{7.2}$$

$$\begin{aligned}\bar{\mathcal{R}} \sim\ & 0 + \eta(0) + O(\eta^2)\\ & +(\epsilon/\eta)\delta\mathcal{R} + \epsilon(s_+ + s_-) + O(\epsilon\eta)\\ & +(\epsilon^2/\eta^2)\delta\delta\mathcal{R} + (\epsilon^2/\eta)(q) + O(\epsilon^2)\,,\end{aligned} \tag{7.3}$$

where

$$\begin{aligned}\omega^t &= \epsilon/\eta\psi_\epsilon*(dT) = dt_\eta\,,\\ \omega^r &= \epsilon/\eta\psi_\epsilon*(dR) = dr_\eta\,,\\ \omega^\theta &= \epsilon/\eta\psi_\epsilon*(\omega^{\Theta*}) = r_\eta d\theta\,,\\ \omega^\phi &= \epsilon/\eta\psi_\epsilon*(\omega^{\Phi*}) = r_\eta \sin\theta\, d\phi\,,\end{aligned} \tag{7.4}$$

$$\begin{aligned}\delta\omega^t &= (-)r_\eta^{-1}dt_\eta\,,\\ \delta\omega^r &= r_\eta^{-1}dr_\eta\,,\\ \delta\omega^\theta &= \delta\omega^\phi = 0\,,\end{aligned} \tag{7.5}$$

$$\begin{aligned}\delta\Omega^t_r &= r_\eta^{-2}dt_\eta\,,\\ \delta\Omega^r_\theta &= r_\eta^{-1}d\theta\,,\\ \delta\Omega^r_\phi &= r_\eta^{-1}\sin\theta\, d\phi\,,\end{aligned} \tag{7.6}$$

$$\begin{aligned}\delta\mathcal{R}^i_\phi &= (-)r_\eta^{-3}dt_\eta \wedge r_\eta \sin\theta d\phi\,,\\ \delta\mathcal{R}^r_\theta &= (-)r_\eta^{-3}dr_\eta \wedge r_\eta d\theta\,,\\ \delta\mathcal{R}^t_\theta &= (-)r_\eta^{-3}dt_\eta \wedge r_\eta d\theta\,,\\ \delta\mathcal{R}^r_\phi &= (-)r_\eta^{-3}dr_\eta \wedge r_\eta \sin\theta\, d\phi\,,\\ \delta\mathcal{R}^t_r &= 2r_\eta^{-3}dt_\eta \wedge dr_\eta\,,\\ \delta\mathcal{R}^\theta_\phi &= 2r_\eta^{-3}r_\eta d\theta \wedge r_\eta \sin\theta d\phi\,,\end{aligned} \tag{7.7}$$

TABLE III. $O(\eta)$ basis and connection forms defined under intermediate limit process.

$\kappa^t_+ \equiv a_i(t)x^i dt$	
$\kappa^j_+ \equiv 0$	$O(\eta)$ frame corrections
$\kappa^t_- \equiv \epsilon_{ijk}S_i(t)x^j_\eta dx^k_\eta$	
$\kappa^j_- \equiv 0$	
$\nu^T_{+R} \equiv a\cos\theta dt_\eta$	$O(\eta)$ connection form corrections (note: these are written for the case $a_z = a$, $a_x = a_y = 0$)
$\nu^T_{+\theta} \equiv -a\sin\theta dt_\eta$	

TABLE IV. Summary of notation for intermediate expansions.

	Basis	Connection	Curvature	Order
Exact quantity	$\overline{\omega}$	$\overline{\Omega}$	$\overline{\mathfrak{R}}$	$O(1)$
Flat space	ω	Ω	$\mathfrak{R}$	$O(1)$
Monopole correction	$\delta\omega$	$\delta\Omega$	$\delta\mathfrak{R}$	$O(\epsilon/\eta)$
Higher moments	$\delta\delta\omega$	$\delta\delta\Omega$	$\delta\delta\mathfrak{R}$	$O(\epsilon^2/\eta^2)$
Acceleration	κ_+	ν_+	0	$O(\eta)$
Rotation	κ_-	ν_-	0	$O(\eta)$
Outer curvature	unspecified	$\rightarrow$	$\rightarrow$	$O(\eta^2)$
Even mode	λ_+	μ_+	s_+	$O(\epsilon)$
Odd mode	λ_-	μ_-	s_-	$O(\epsilon)$
Coupled mode	σ	ρ	q	$O(\epsilon^2/\eta)$

κ and ν are given in Table III, and the remaining quantities have not yet been specified. The $O(\epsilon)$ frame, connection, and curvature terms, respectively $\lambda_\pm$, $\mu_\pm$, and $s_\pm$, represent modes due to acceleration (even parity) and rotation of the object's asymptotic rest frame with respect to Fermi-Walker transport (odd parity). The $O(\epsilon^2/\eta)$ terms σ, ρ, and q represent higher-order effects of indefinite parity.

Table IV summarizes the intermediate expansions and groups the terms according to their physical interpretation.

VIII. ANALYSIS OF PERTURBATION EQUATIONS UP TO $O(\epsilon^2/\eta)$

The remaining degrees of freedom are parametrized by the acceleration $a_i(t)$ and the rotation $S_i(t)$. As we saw in Sec. VII, these parameters appear in the $O(\eta)$ frames $\kappa_\pm$ and connection forms $\nu_\pm$, where "+" quantities depend on acceleration and "−" quantities depend on rotation. In the next two sections, I examine the vacuum EFE (the overlap zone in which matching takes place is matter-free) and BI at $O(\eta)$, $O(\epsilon)$, and $O(\epsilon^2/\eta)$ to calculate $a_i(t)$. As the calculation progresses through these orders, the equations begin to include contributions from higher moments (e.g., $\delta\delta\omega$) of the object's zeroth-order field. However, the result $a_i(t)=0$ is independent of the specific values of these higher moments.

To show that $a_i(t)=0$ all along γ, I show that nonvanishing at any event on γ leads to a contradiction. Suppose then that p_0 is an event along γ at which $a_i(t)\neq 0$. For convenience, relabel the coordinates so that $t=0$ at p_0 and $a_i(0)$ is in the z direction, i.e.,

$$a_x(0)=0,\quad a_y(0)=0,\quad a_z(0)=a. \tag{8.1}$$

Unless otherwise stated, all quantities are hereafter evaluated at $t=0$.

To maintain a certain amount of physical intuition, I will describe the successive stages of this calculation by referring to perturbations of backgrounds having successively more structure (e.g.: flat, ω; Schwarzschild, $\delta\omega$; higher moments, $\delta\delta\omega$). Note that $O(\epsilon^2/\eta)$ perturbations are $O(\epsilon^2)$ under the external limit process (and thus *not* described by the first-order external expansion), while only $O(\epsilon)$ under the internal limit process (and thus described by the first-order internal expansion).

The purpose of this section is to obtain a set of equations that are independent of the object's higher moments. These equations will then be used in Sec. IX to show that $a_i(t)=0$ from relatively simple equations that were originally written to describe perturbations of an *exact* Schwarzschild background.

In the coordinates described above, the nonzero, $O(\eta)$, even-parity basis and connection forms are

$$\kappa_+^t = ar_\eta \cos\theta dt_\eta, \tag{8.2}$$

$$\nu_{+r}^t = a\cos\theta dt_\eta,\quad \nu_{+\theta}^t=(-)a\sin\theta dt_\eta. \tag{8.3}$$

The $O(\eta)$ curvature forms vanish as they must, and thus the EFE and BI (Bianchi identities) are satisfied. (More generally, orders $\{1,\eta,\eta^2,\ldots\}$ and $\{1,\epsilon,\epsilon^2/\eta^2,\ldots\}$ are automatically satisfied due to my assumption that the zeroth-order external and internal model spacetimes are given solutions of the EFE.)

Two kinds of terms contribute to the Einstein tensor (abbreviated ET) at $O(\epsilon)$: Type-I contributions result from products $O(\epsilon/\eta)$ monopole terms with $O(\eta)$ acceleration (or rotation) terms. Type-II contributions result from products of $O(\epsilon)$ and $O(1)$ terms and from differentiation of $O(\epsilon)$ terms. Type-I terms are a direct consequence of the $O(\eta)$ perturbations; type-II terms will be chosen, if possible, to cancel the effect of type-I contributions to the ET and thus to satisfy the EFE. From this discussion, one can see why even and odd modes do not mix at $O(\epsilon)$: for both types I and II, all $O(\epsilon)$ equations are indistinguishable from those that would occur if the object were exactly Schwarzschild, and first-order perturbations of definite parity on a Schwarzschild background decouple.

I adopt the following notational conventions: $Q_\pm$ means that Q_+ is the even-parity part and Q_- the odd-parity part of a quantity Q, i.e.,

$$Q = Q_+ + Q_- . \tag{8.4}$$

Terms such as μ and s are further subdivided into type I and II:

$$\mu_\pm = \mu_{\pm\mathrm{I}} + \mu_{+\mathrm{II}} , \tag{8.5}$$

$$s_+ = s_{+\mathrm{I}} + s_{+\mathrm{II}} \tag{8.6}$$

as discussed above, unless the distinction is already clear: e.g., κ_+ is clearly a type-I quantity.

The $O(\epsilon)$ part of the first Cartan structure equation is

$$\mu_\pm \wedge \omega = (-)[\delta\Omega\wedge\kappa_\pm + \nu_\pm\wedge\delta\omega] - [d\lambda_\pm + \Omega\wedge\lambda_\pm] . \tag{8.7}$$

With the subdivision into type-I and –II terms, one obtains two equations

$$\mu_{\pm\mathrm{I}}\wedge\omega = (-)[\delta\Omega\wedge\kappa_\pm + \nu_\pm\wedge\delta\omega] , \tag{8.8}$$

$$\mu_{\pm\mathrm{II}}\wedge\omega = (-)[d\lambda_\pm + \Omega\wedge\lambda_\pm] , \tag{8.9}$$

each of which can be solved uniquely for the desired quantity $\mu_{\pm\mathrm{I}}$ or $\mu_{+\mathrm{II}}$. Since κ_+ and ν_+ are known, one finds that the only nonvanishing, $O(\epsilon)$, even-parity, type-I connection form is

$$\mu^{\ t}_{+\mathrm{I}r} = (-) r_\eta^{-1} a \cos\theta dt_\eta . \tag{8.10}$$

The $O(\epsilon)$ part of the second structure equation can similarly be divided into type-I and –II contributions to $s_\pm$:

$$s_{\pm\mathrm{I}} = \delta\Omega\wedge\nu_\pm + \nu_\pm\wedge\delta\Omega + d\mu_{\pm I} + \Omega\wedge\mu_{\pm\mathrm{I}} + \mu_{\pm\mathrm{I}}\wedge\Omega , \tag{8.11}$$

$$s_{\pm\mathrm{II}} = d\mu_{\pm\mathrm{II}} + \Omega\wedge\mu_{\pm\mathrm{II}} + \mu_{\pm\mathrm{II}}\wedge\Omega . \tag{8.12}$$

The nonzero, even-parity, type-I curvature forms at $O(\epsilon)$ are then

$$\begin{aligned} s^{t}_{+\mathrm{I}r} &= a r_\eta^{-2}\cos\theta dr_\eta\wedge dt_\eta , \\ s^{t}_{+\mathrm{I}\theta} &= 2 a r_\eta^{-2}\cos\theta dt_\eta\wedge r_\eta d\theta , \\ s^{t}_{+\mathrm{I}\phi} &= 2 a r_\eta^{-2}\cos\theta dt_\eta\wedge r_\eta \sin\theta d\phi . \end{aligned} \tag{8.13}$$

From these quantities, one can compute the even-parity, type-I Ricci and Einstein tensors:

$$\mathrm{Ricci}_{+\mathrm{I}} = 3 a r_\eta^{-2}\cos\theta[-dt_\eta\otimes dt_\eta - dr_\eta\otimes dr_\eta + r_\eta^{2}(dt\otimes d\theta + \sin^2\theta d\phi\otimes d\phi)] , \tag{8.14}$$

$$G_{+\mathrm{I}} = (-) a r_\eta^{-2}\cos\theta dr_\eta\otimes dr_\eta . \tag{8.15}$$

Note that these even-parity terms do indeed contribute to the $O(\epsilon)$ ET.

As an internal check, type-I terms should satisfy the contracted BI at $O(\epsilon)$, since if type-I terms vanished, $G_{+\mathrm{I}}$ would be the complete $O(\epsilon)$ ET. Since $G_{+\mathrm{I}}$ is already $O(\epsilon)$, the contracted BI for $G_{+\mathrm{I}}$ becomes

$$\nabla\cdot G_{+\mathrm{I}} = 0 , \tag{8.16}$$

where $\nabla\cdot$ is a flat-space divergence; this equation is easily seen to be satisfied by substitution of Eq. (8.15).

At this point, there is a type-I contribution to the ET; to satisfy the EFE, of course, the total Einstein tensor at $O(\epsilon)$, $G_{+\mathrm{I}} + G_{+\mathrm{II}}$ must vanish. The as yet uncalculated term $G_{+\mathrm{II}}$ comes from the $O(\epsilon)$ frame perturbations λ_+ and from the type-II terms generated by λ_+. Since the operator that would produce $G_{+\mathrm{II}}$ from λ_+ is essentially a flat-space D'Alembertian, it is clear that such a λ_+ can be found. (In fact, a class of λ_+ can be found, differing from each other by terms that are pure gauge with respect to a flat background.) No contradiction to the hypothesis $a_i(t) \neq 0$ is yet evident.

However, λ_+ contributes to the $O(\epsilon^2/\eta)$ ET and BI, and therefore I next analyze the $O(\epsilon^2/\eta)$ structure equations, EFE, and BI. Note in this analysis that one cannot simply choose a particular λ_+ whose ET contribution $G_{+\mathrm{II}}$ cancels $G_{+\mathrm{I}}$: At $O(\epsilon^2/\eta)$, the Schwarzschild part of the background is important for contributions arising from λ_+, and terms that are pure gauge in a flat background are *not* pure gauge in a Schwartzschild background.

In the $O(\epsilon)$ equations, terms arising from κ were denoted as type-I terms and terms arising from λ as type-II terms. Now, because σ will have to be chosen (in principle) to cancel the $O(\epsilon^2/\eta)$ ET due to κ and λ, third-generation or type-III terms also arise. At $O(\epsilon^2/\eta)$, these three types of terms differ in their sensitivity to the object's moments:

(1) Type-I terms include products of $O(\epsilon^2/\eta)$ moments of the object $(\delta\delta\omega, \cdots)$ with $O(\eta)$ (acceleration) terms such as κ_+.

(2) Type-II terms result from products between monopole part of the object's field $(\delta\omega, \cdots)$ with $O(\epsilon)$ corrections such as λ_+.

(3) Type-III terms are sensitive only to the flat part of the background and arise from $O(\epsilon^2/\eta)$ corrections such as σ.

The first structure equation at $O(\epsilon^2/\eta)$ gives the following three equations:

$$\rho_{\mathrm{I}} \wedge \omega = (-)[\mu_{\mathrm{I}}\wedge\delta\omega + \nu\wedge\delta\delta\omega + \delta\delta\Omega\wedge\kappa] , \tag{8.17}$$

$$\rho_{\mathrm{II}}\wedge\omega = (-)[\mu_{\mathrm{II}}\wedge\delta\omega + \delta\Omega\wedge\lambda] , \tag{8.18}$$

$$\rho_{III} \wedge \omega = (-)[d\sigma + \Omega \wedge \sigma]\,. \qquad (8.19)$$

Note that Eqs. (8.17)—(8.19) are progressively less sensitive to the object's moments. The second equation [(8.18)] involves quantities $\mu_{\pm II}$ and $\lambda_\pm$ of definite parity and does not mix these parities; it is convenient to define $\rho_{\pm II}$ satisfying

$$\rho_{\pm II} \wedge \omega = (-)[\mu_{\pm II} \wedge \delta\omega + \delta\Omega \wedge \lambda_\pm]\,. \qquad (8.20)$$

One can also separate the second structure equation at $O(\epsilon^2/\eta)$ into type-I, -II, and -III parts and the type-II terms into ± parity to obtain

$$q_I = d\rho_I + \rho_{\pm I} \wedge \Omega + \Omega \wedge \rho_{\pm I} + \nu \wedge \delta\delta\Omega + \delta\delta\Omega \wedge \nu\,, \qquad (8.21)$$

$$q_{II} = d\rho_{\pm II} + \rho_{\pm II} \wedge \Omega + \Omega \wedge \rho_{\pm II} + \mu_{\pm II} \wedge \delta\Omega + \delta\Omega \wedge \mu_{\pm II}\,, \qquad (8.22)$$

$$q_{III} = d\rho_{III} + \rho_{III} \wedge \Omega + \Omega \wedge \rho_{III}\,. \qquad (8.23)$$

For the $O(\epsilon^2/\eta)$ part of the BI, abbreviated as

$$dR = R \wedge \Omega - \Omega \wedge R\,, \qquad (8.24)$$

the type-I, type-II (±), and type-III equations read

$$dq_I = q_I \wedge \Omega - \Omega \wedge q_I + \delta\delta\mathcal{R} \wedge \nu - \nu \wedge \delta\delta\mathcal{R}\,, \qquad (8.25)$$

$$dq_{\pm II} = q_{\pm II} \wedge \Omega - \Omega \wedge q_{\pm II} + \delta\mathcal{R} \wedge \mu_{\pm II} + \mu_{\pm II} \wedge \delta\mathcal{R} + s_{\pm II} \wedge \delta\Omega - \delta\Omega \wedge s_{\pm II}\,, \qquad (8.26)$$

$$dq_{\pm III} = q_{\pm III} \wedge \Omega - \Omega \wedge q_{\pm III}\,. \qquad (8.27)$$

Similarly, it is routine to separate the $O(\epsilon^2/\eta)$ Einstein tensor and contracted Bianchi identities into Type-I, -II (±), and -III parts.

The following argument shows that type-II+ terms satisfy the contracted BI. If type-II and -III terms were absent, the type-I terms would themselves satisfy the contracted BI, because type-I terms are the direct result of κ. Therefore, the sum of type-II and -III terms must satisfy the $O(\epsilon^2/\eta)$ contracted BI. Now type-III contributions to the $O(\epsilon^2/\eta)$ ET result from a flat-space D'Alembertian, just as did type-II contributions to the $O(\epsilon)$ ET. Thus, for the same reason that type-II $O(\epsilon)$ contracted BI terms vanish identically [Eq. (8.16)], so do type-III $O(\epsilon^2/\eta)$ terms. Therefore, type-II terms must *themselves* satisfy the contracted BI up to $O(\epsilon^2/\eta)$. Finally, since the effective background for type-II terms in $O(\epsilon^2/\eta)$ equations preserves parity, the type-II, even- and odd-parity terms *separately* satisfy the $O(\epsilon^2/\eta)$ contracted BI.

To summarize the most useful results of this section:

(1) Type-II, even-parity terms themselves satisfy the contracted BI to $O(\epsilon^2/\eta)$.

(2) The effective background for type-II terms in all $O(\epsilon)$ and $O(\epsilon^2/\eta)$ equations is insensitive to the object's higher moments.

IX. COMPLETION OF THE CALCULATION USING METRIC PERTURBATIONS ON AN EXACT SCHWARZSCHILD BACKGROUND

The arguments of the preceding section show that type-II(+) terms obey the first-order perturbation equations up to $O(\epsilon^2/\eta)$ that would occur on an *exactly* Schwarzschild background. Terms of higher order than $O(\epsilon^2/\eta)$ could depend on the object's higher moments; only the equations up to $O(\epsilon^2/\eta)$ will be needed.

Metric perturbations of Schwarzschild have been analyzed in a convenient form by Thorne and Compolattoro[39] (referred to as TC below). In this section, I apply their analysis by working with the $O(\epsilon)$ metric corrections corresponding to the $O(\epsilon)$ frame corrections λ_+. The perturbation equations up to $O(\epsilon^2/\eta)$ reduce to an overdetermined system of algebraic equations that can only be satisfied if the zeroth-order acceleration vanishes.

As in Sec. IV, I denote exact Schwarzschild quantities with carets (e.g., $\hat{\mathcal{G}}$). It is convenient here to work in the internal model spacetime C. The metric $\hat{\mathcal{G}}$ is, in terms of the internal coordinate system (T, R, Θ, Φ),

$$\hat{\mathcal{G}} = -(1-2/R)dT\otimes dT + (1-2/R)^{-1}dR\otimes dR + R^2(d\Theta\otimes d\Theta + \sin^2\Theta d\Phi\otimes d\Phi)\,. \qquad (9.1)$$

TC showed that the most general, even-parity, $l=1$, first-order metric perturbations of Schwarzschild can be written in the form

$$\bar{\mathcal{G}}^* \sim \hat{\mathcal{G}} + \epsilon\mathcal{K} + \cdots\,, \qquad (9.2)$$

where

$$\mathcal{K} = \{-(1-2/R)H_0 dT\otimes dT - H_1(dT\otimes dR + dR\otimes dT) - (1-2/R)^{-1}H_2 dR\otimes dR\}Y_{1M} + r^2(K-G)(d\Theta\otimes d\Theta + \sin^2\Theta d\Phi\otimes d\Phi) - (f_0 dT + f_1 dR)\otimes(\Psi^1_{M\Theta}d\Theta + \Psi^1_{M\Theta}d\Phi)\,,$$

$$\Psi^1_M \equiv \text{vector harmonics}\,, \qquad (9.3)$$

$H_0, H_1, H_2, K, G, f_0, f_1$ functions of (R, T).

Because of the form of the $O(\epsilon)$ ET contribution

$$G_{+II} = 6r_\eta^{-2}a\cos\theta\, dr_\eta\otimes dr_\eta\,, \qquad (9.4)$$

one only needs to consider $l=1$ perturbations in this scheme. TC showed that the gauge may be chosen to annul f_0, f_1, and $K-G$, thus simplifying $\mathcal{K}$ to

$$\mathcal{K} = \{-(1-2/R)^{-1}H_0 dT\otimes dT - H_1(dT\otimes dR + dR\otimes dT) - (1-2/R)^{-1}H_2 dR\otimes dR\}Y_{1M}\,. \qquad (9.5)$$

Since all possible $O(\epsilon)$ frame corrections λ_+ generating the correct ET contribution $G_{+\mathrm{II}}$ must be considered, one must regard the particular $\mathcal{H}$ satisfying TC's gauge conditions as merely one representative of a class of possible $\mathcal{H}$. If the zeroth-order internal metric $\mathcal{G}^{(C)}$ were an *exact* Schwarzschild metric $\hat{\mathcal{G}}$, then the effect of a gauge transformation of $\mathcal{H}$ on the $O(\epsilon)|_{\mathrm{internal}}$ ET would vanish, since the zeroth-order ET vanishes; similarly, the $O(\epsilon)|_{\mathrm{internal}}$ BI would be unaffected. One could in that case regard $\mathcal{H}$ as representing the *entire* class of $O(\epsilon)|_{\mathrm{internal}}$ metric perturbations.

However, since the zeroth-order internal metric $\mathcal{G}^{(C)}$ in general differs from $\hat{\mathcal{G}}$, the result of a gauge transformation on the actual background $\mathcal{G}^{(C)}$ differs from the result that would have been obtained on the purely Schwarzschild background $\hat{\mathcal{G}}$. One can represent an arbitrary gauge transformation in the form

$$\mathcal{H}\to\mathcal{H}+\mathcal{L}_\xi\mathcal{G}^{(C)}\,. \tag{9.6}$$

If the object's metric were exactly Schwarzschild, ξ would produce instead the effect

$$\mathcal{H}\to\mathcal{H}+\mathcal{L}_\xi\hat{\mathcal{G}}\,. \tag{9.7}$$

Suppose $\xi=\xi_1$ is chosen to annul unwanted functions h_0, h_1, and $K-G$ in $\mathcal{H}$. Then the actual effect of ξ_1 is to generate additional unwanted terms of the form

$$\mathcal{H}'=\mathcal{L}_\xi[\mathcal{G}^{(C)}-\mathcal{G}] \tag{9.8}$$

in $\mathcal{H}$. It is these extra terms $\mathcal{H}'$ that I now discuss.

One is interested only in $\mathcal{H}'$ of $O(\epsilon)$ or smaller, because λ_+ itself is only $O(\epsilon)$. One needs the largest post-Schwarzschild contributions to the ET and BI due to $\mathcal{H}'$. These contributions involve products of higher moments with $\mathcal{H}'$. The analysis of Sec. VIII shows that no type-II terms contain products with the object's higher moments up to $O(\epsilon^2/\eta)$. The first instance of such products occurs at $O(\epsilon^3/\eta^2)$. Thus, $\mathcal{H}'$ has no effect on the ET and BI until $O(\epsilon^3/\eta^2)$. Conclusions about $\mathcal{H}$ based on orders ϵ and ϵ^2/η will be unaffected by the difference between $\mathcal{G}^{(C)}$ and $\hat{\mathcal{G}}$.

Since one is only interested in perturbations that vary on the external time scale, one can assume a time dependence for H_0, H_1, and H_2 of the form $H_0(R,\epsilon T)$, $H_1(R,\epsilon T)$, and $H_2(R,\epsilon T)$. The functions H_0, H_1, and H_2 are expanded in the intermediate limit process in the manner of Sec. VII:

$$H_0\sim h_0(\epsilon T)+\epsilon/\eta r_\eta^{-1}j_0(\epsilon T)+\cdots\,, \tag{9.9}$$

$$H_1\sim h_1(\epsilon T)+\epsilon/\eta r_\eta^{-1}j_1(\epsilon T)+\cdots\,, \tag{9.10}$$

$$H_2\sim h_2(\epsilon T)+\epsilon/\eta r_\eta^{-1}j_2(\epsilon T)+\cdots\,. \tag{9.11}$$

The relation between h_0, h_1, h_2, and λ_+ is

$$\begin{aligned}\omega\otimes\lambda_++\lambda_+\otimes\omega=[&-h_0(dt_\eta\otimes dt_\eta)\\&-h_1(dt_\eta\otimes dr_\eta+dr_\eta\otimes dt_\eta)\\&-h_2(dr_\eta\otimes dr_\eta)]Y_{1M}\,.\end{aligned} \tag{9.12}$$

Note that h_0, h_1, and h_2 are $O(\epsilon)$ quantities.

Equations (D2) of TC give the ET at $O(\epsilon)|_{\mathrm{internal}}$ in terms of H_0, H_1, and H_2. In order to analyze the leading terms in the intermediate limit, a few preliminary observations are needed:

(1) Because of the slow time dependence of H_0, H_1, and H_2, time derivatives such as $(\partial/\partial T)\times[H_1(R,\epsilon T)]$ reduce the size of a term by a factor of ϵ.

(2) Because the leading terms (h_0, h_1, and h_2) in the intermediate limit expansions for H_0, H_1, and H_2 have no radial dependence, a derivative $\partial/\partial R$ raises the order of a term by a factor of ϵ^2/η^2.

Thus, while multiplication by R^{-1} raises the order by *one* factor of ϵ/η, differentiation with respect to R raises the order by *two* factors of ϵ/η.

As the above observations suggest, the leading terms in the intermediate expansion of the $O(\epsilon)|_{\mathrm{internal}}$ ET components due to λ_+ come from undifferentiated terms. The leading terms are of $O(\epsilon)$, and they are within the precision of the present analysis. In terms of the $O(\epsilon)$ metric functions h_0, h_1, and h_2, the $O(\epsilon)$, type-II, even-parity ET is

$$\begin{aligned}G_{+\mathrm{II}}=r_\eta^{-2}\{&[(h_0-h_2)dr_\eta\otimes dr_\eta+2h_2dt_\eta\otimes dt_\eta+h_1(dt_\eta\otimes dr_\eta+dr_\eta\otimes dt_\eta)+\tfrac{1}{2}(h_0-h_2)(\omega^\theta\otimes\omega^\theta+\omega^\phi\otimes\omega^\phi)]Y_{1M}\\&-\tfrac{1}{2}(h_0-h_2)[\Psi^1_{M\theta}(\omega^\theta\otimes dr_\eta+dr_\eta\otimes\omega^\theta)+\Psi^1_{M\phi}(\omega^\phi\otimes dr_\eta+dr_\eta\otimes\omega^\phi)]\}\,.\end{aligned} \tag{9.13}$$

The $O(\epsilon)$ EFE requires that

$$G_{+\mathrm{II}}=a[6r_\eta^{-2}\cos\theta dr_\eta\otimes dr_\eta]\,. \tag{9.14}$$

Clearly, only $M=0$ harmonics enter. Using a normalization such that $Y_{10}=\cos\theta$, one obtains

$$h_1=0\,, \tag{9.15}$$

$$h_2=0\,, \tag{9.16}$$

$$h_0-h_2=6a\,, \tag{9.17}$$

$$h_0-h_2=0\,. \tag{9.18}$$

This overdetermined system can only be satisfied if $a=0$.

Since the event ρ_0 was arbitrarily chosen, this result shows that the zeroth-order acceleration $a_i(t)$ vanishes at every event along the trial world line $\gamma(t)$.

X. CONCLUSIONS

The results of this calculation predict the motion of a body whose mass m is small compared to an external curvature reference length L. I have applied the singular-perturbation formulation of Ref. 14 to define an asymptotic world line $\gamma(t)$, which represents the zeroth-order world line of the object, when measured over time scales of order L. I have shown by matching that, to $O(\epsilon)$ in both the internal and external asymptotic expansions, the EFE can be satisfied only if $\gamma(t)$ is a geodesic of the zeroth-order or unperturbed external metric.

Formally, this calculation has produced a one-parameter family of approximate spacetimes obeying perturbation equations and internal uniformity criteria (such as matching) derived from the requirement that this family be "close to" some appropriate exact family of spacetimes $(S_\epsilon, \mathcal{G}_\epsilon)$. Uniformly valid expansions imply that the errors are uniformly small, even within the fully nonlinear, possibly singular object and even in the curved external spacetime. Any attempt to construct approximate spacetimes representing objects with zeroth-order acceleration leads to violation of the first-order (in the internal expansion) EFE and BI.

An interpretation of this calculation in terms of motion rests on the singular-perturbation framework of Ref. 14 and on my definition (Sec. V) of an asymptotic world line. Specifically, the zeroth-order acceleration $a_i(t)$ vanishes at every event along the trial world line $\gamma(t)$. Thus, a trial world line *can* be an asymptotic world line only if it is a geodesic of the zeroth-order external metric $\underset{\sim}{\mathcal{G}}^{(E)}$. To conclude that such a $\gamma(t)$ *must* be an asymptotic world line, one would need to solve the EFE to $O(\epsilon)$ in the internal expansion, whereas this paper has not considered the odd-parity part of the $O(\epsilon)$ and $O(\epsilon^2/\eta)$ equations, which involve the zeroth-order rotation $S_i(t)$. However, it is obvious that $S_i(t)=0$ gives *one* solution of the internal perturbation equations at $O(\epsilon)$, since in that case the first-order internal frame correction $w^{(C)*}$, which satisfies a sourceless linear equation, vanishes. Whether there can be solutions with $S_i(t) \neq 0$ is a question for future work. In any case, $\gamma(t)$ satisfies my definition of an asymptotic world line if and only if its zeroth-order acceleration $a_i(t)$ vanishes.

The object's asymptotic rest frame, as defined in Sec. IV from the Schwarzschild part of its zeroth-order metric $\mathcal{G}^{(E)}$, is unboosted to zeroth order ($\vec{\alpha}=0$) with respect to the frame of an observer moving along the asymptotic world line $\gamma(t)$. Therefore, if an object is placed at an initial event p_0 with its preferred time direction parallel to a given external velocity vector u_0, the subsequent motion of the object (in the limit $\epsilon \to 0$) will be along the particular geodesic that passes through p_0 with velocity u_0.

From these results it is reasonable to infer that a body of mass m moves through an empty and singularity-free region of an external spacetime of curvature reference length L along an approximate geodesic over times of order L, provided $\epsilon \equiv m/L$ is small compared to unity. Deviations from geodesic motion over times of order L vanish up to possible corrections of $o(L)$, e.g., $O(\epsilon)L$. In order to measure such $O(\epsilon)$ deviations from geodesicity, one would have to observe the system over times of $O(L/\epsilon)$.

Note that the motion is approximately geodesic with respect to the *unperturbed external metric* $\mathcal{G}^{(E)}$. Thus, the present method effectively separates out the self-field of a body whose internal gravity is too strong to permit the drastic assumption of a test body. This natural separation provides one of the main advantages over the center-of-mass approach to motion (cf. discussion of Sec. II).

These results of this calculation are independent of the object's higher moments: One does not have to assume that these moments vanish, but only that they scale with the appropriate power of the mass m. These results are also independent of the external spacetime's curvature gradients: One does not need to assume that these gradients vanish, but only that they scale with the appropriate power of L. Since the unperturbed object was specified only up to the Schwarzschild part of its field, it was unnecessary to discuss its internal composition, causal structure, etc. Thus, the results apply to black holes (e.g., Kerr), singularities without event horizons (e.g., Weyl solutions), and neutron stars with arbitrary equations of state, as well as to "ordinary" astrophysical objects. Since the unperturbed external spacetime was characterized only by the curvature in E, it was unnecessary to discuss either the sources of the external fields or the fields "at infinity." The results apply even when the external field is not asymptotically flat and even when no definition of energy-momentum is available.

As emphasized in Ref. 14, one must always be on the alert for hints (such as the failure of matching) of unanticipated subtleties in singular-perturbation problems. In this calculation, there was little opportunity for matching to fail because of my regularity assumptions concerning the asymptotic expansions, e.g., the assumption that the first-order external and internal perturbations have coordinate expansions that begin in powers. The possibility of "nonanalytic" behavior, such as the appearance of logarithmic terms in the perturbations, cannot in general be ruled out. Such terms occur, for example, in Kaplun's[40] resolution of the Stokes paradox in fluid mechanics (see also further examples in the standard singular-perturbation references citied in Ref. 14). One topic for future research is to look for hints of where these assumptions of powers need to be modified, particularly in higher-order calculations (see below).

It is possible that, despite the success of matching, the asymptotic expansions constructed here may fail to give a uniform approximation to any exact family of spacetimes. In the absence of an applicable "linearization stability" proof or an exact solution, one canot assert that the solutions of perturbation equations are indeed approximations to exact solutions. However, matching is the most reliable method available and has had enormous success in fluid mechanics and other fields in which it can be tested by experiments, numerical solutions, and exact calculations. One can safely conclude that the results of this paper are consistent to zeroth order with geodesic motion and inconsistent with accelerated motion.

Several straightforward extensions of this work seem feasible. A calculation of the spin-transport law up to post-Newtonian order for objects with strong internal gravity is necessary to justify the assumption of parallel transport of an object's spin vector in Ref. 17, as mentioned in Sec. I. In order to calculate the zeroth-order rotation $S_i(t)$ of the object's rest frame with respect to parallel transport along the asymptotic world line, one should examine the effect of odd-parity perturbations such as κ_-, which contain $S_i(t)$. Post-Newtonian corrections to the rotation are of course beyond the precision of the present zeroth-order analysis. The problem needs a different ordering of terms appropriate to a slow-motion, gravitationally bound system of comparable masses, as in Ref. 16.

Another extension is to study secular effects (occurring over times of $O(L/\epsilon)$ or longer), which depend on corrections of $O(\epsilon)$ and higher to the acceleration. To obtain $O(\epsilon)$ acceleration corrections, one needs to examine the $O(\epsilon^2)|_{\text{internal}}$ EFE and BI. As mentioned in Sec. I, one expects such corrections to arise due to a coupling between the object's spin and the odd-parity part of the unperturbed external curvature tensor. These $O(\epsilon)$ and smaller deviations from geodesic motion can best be discussed within the context of an approximation using multiple time scales.

Generalizations of the present method include:

(1) obtaining the Lorentz force law for the zeroth-order motion of an electrically or magnetically charged object through external gravitational and electromagnetic fields.

(2) studying the motion of sources with internal structure through external fields in various gauge theories. (A problem of this type has been studied by Manton.[41])

(3) deriving the motion of compact objects in alternate theories of gravity.

An important application of the present method is to a calculation of the decrease in orbital period due to the emission of gravitational radiation by a binary system containing a compact object. As noted in Sec. I, it would be possible to extend Burke's[13] method, if only one could determine the deflection of a compact object due to the time-odd, "resistive potentials" at 5/2-PN (post-Newtonian) order that Burke's calculation produces. The much larger but time-even effects up to PPN order also need to be studied. Since the predictions of period shortening in the binary pulsar depend themselves on a correct formula for the motion up to PN order, PN orbital corrections should be calculated explicitly; it appears possible to calculate these corrections for systems containing compact objects by generalizing D'Eath's[16] work on the slow-motion interaction of Kerr black holes. The 2-PN terms should be checked for finiteness; possible nonanalytic behavior (such as terms of order $v^n \ln v$) has recently been suggested.[42] However, only the Newtonian orbits are needed to calculate time-odd orbital corrections to the PPN motion due to radiation reaction.

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