PHYSICAL REVIEW D VOLUME 22, NUMBER 8 15 OCTOBER 1980

Exact Bianchi type-VIII and type-IX cosmological models with matter and electromagnetic fields

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(Received 11 February 1980)

Exact solutions of the Einstein-Maxwell equations of Bianchi types VIII and IX are derived. The solutions represent axisymmetric universes with source-free electromagnetic fields and the matter content of the models is a perfect fluid, with equation of state $p=\epsilon$.

I. INTRODUCTION

The simplest models of the expanding universe are spatially homogeneous and isotropic. There is good observational evidence that at our cosmological epoch the Universe is fairly homogeneous on large scales and has been highly isotropic since the epoch in which it became definitely transparent to radiation.[1] However, the fact that a Robertson-Walker model is a good approximation now does not imply that it has been so at the early stages of the cosmological expansion.

In recent years there has been considerable interest in spatially homogeneous, nonisotropic, cosmological models. These are the so-called Bianchi models. The existence of anisotropy in such models allows a theoretical discussion of many important effects.[2] A special class of homogeneous anisotropic models are the "magnetic" universes, endowed with a uniform primordial magnetic field. This gives rise to a preferred spatial direction and so breaks isotropy. The theory of the magnetic universe has been developed by several authors.[3-10] The interest in these models was increased by the possible discovery of an intergalactic homogeneous magnetic field of the order of 10^{-7}–10^{-8} G.[11-15]

The idea of a universe with a homogeneous magnetic field was proved to be very successful in flat (i.e., Bianchi type-I) spaces.[10] However, since Bianchi type-I models are a very special subset of spatially homogeneous models, one should consider more general situations, in order to check what implications large-scale primordial magnetic fields would have on the dynamics of the Universe. The most general sets of homogeneous models are Bianchi types VI, VII, VIII, and IX.[16] However, the basic work of Hughston and Jacobs[17] has shown that the existence of a homogeneous primordial magnetic field in our universe is limited to Bianchi types I, II, III, VI ($h=-1$), or VII ($h=0$). These results also hold for pure electric fields. Thus one is forced to consider models with both a magnetic and an electric field.

The equations for anisotropic homogeneous models for the case when an electromagnetic field is present have been considered in a number of papers.[1] In this paper we solve the Einstein-Maxwell equations for Bianchi type-VIII and type-IX models. We investigate universes containing electromagnetic fields obeying the sourceless Maxwell equations and matter, with a "stiff" equation of state. The possible relevance of the equation of state $p=\epsilon$ as regards the matter content of the Universe in its early stages has been discussed by a number of authors, since it was first proposed by Zel'dovich.[18,19] We refer to the recent paper of Barrow.[20]

Perhaps the main difference between type VIII and type IX is the sign of curvature. For type VIII this is always negative, whereas for type IX it can be positive as well as negative depending on the relations between the cosmic scale factors. For the most part we use Cartan's calculus of differential forms to obtain the components of the Ricci and the Einstein tensors and to solve the Maxwell equations.

II. DERIVATION OF THE CURVATURE

In choosing a local orthonormal basis σ^μ, we can put the metric of space-time in the form

$$ds^2=\eta_{\mu\nu}\sigma^\mu\sigma^\nu\,, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor. For a spatially homogeneous model, we take

$$\sigma^0=\omega^0=dt\,,\quad \sigma^i=R_i\omega^i\ \ (\text{no sum})\,, \tag{2}$$

where ω^i are the time-independent differential one-forms and where, because of homogeneity, the R_i are functions of t only. (Here and henceforth Latin indices assume the values 1, 2, 3, whereas Greek indices will assume the values 0, 1, 2, 3.)

The one-forms σ^i, ω^i obey the relations

$$d\omega^i=-\tfrac{1}{2}C_{kl}{}^i\omega^k\wedge\omega^l\,, \tag{3a}$$

$$d\sigma^i=-\tfrac{1}{2}\gamma_{\alpha\beta}{}^i\sigma^\alpha\wedge\sigma^\beta\,, \tag{3b}$$

where the $C_{kl}{}^i$ are the structure constants, $\gamma_{\alpha\beta}{}^i$

the commutation coefficients, and $\wedge$ denotes the exterior product. The structure constants for Bianchi type VIII and type IX can be written as

$$C_{ik}{}^{l} = -\epsilon_{ikl} n_l \,, \tag{4}$$

where ϵ_{ikl} is the totally antisymmetric Levi-Civita pseudotensor and

$$\begin{aligned} &n_1 = n_2 = -n_3 = 1 \,, \quad \text{type VIII} \\ &n_1 = n_2 = n_3 = 1 \,, \quad \text{type IX} \,. \end{aligned} \tag{5}$$

The exterior derivatives of the orthonormal basis one-forms are readily found by use of Eqs. (2) and substitution of Eq. (3a):

$$\begin{aligned} &d\sigma^0 = 0 \,, \\ &d\sigma^i = H_i \sigma^0 \wedge \sigma^i + \tfrac{1}{2}\epsilon_{ikl} n_l (R_i/R_k R_l)\sigma^k \wedge \sigma^l \,, \end{aligned} \tag{6}$$

where $H_i = \dot{R}_i/R_i$ are the Hubble parameters. (A dot denotes differentiation with respect to time.)

Comparison of these equations with the relationship (3b) provides immediately the commutation coefficients

$$\gamma_{0k}{}^{k} = -H_k \,, \quad \gamma_{ik}{}^{l} = -\epsilon_{ikl} n_l (R_i/R_k R_l) \,. \tag{7}$$

These quantities enter into the formula

$$\sigma_{\mu\nu} = \tfrac{1}{2}(\gamma_{\mu\nu\alpha} + \gamma_{\mu\alpha\nu} - \gamma_{\nu\alpha\mu})\sigma^\alpha \tag{8}$$

to provide six affine connection one-forms $\sigma_{\mu\nu}$ (to lower or raise an index use $\eta_{\mu\nu}$). The results are

$$\begin{aligned} &\sigma_{0k} = -H_k \sigma^k \,, \\ &\sigma_{ik} = \tfrac{1}{2}\epsilon_{ikl}\left(\frac{n_i R_i}{R_k R_l} + \frac{n_k R_k}{R_i R_l} - \frac{n_l R_l}{R_i R_k}\right)\sigma^l \,. \end{aligned} \tag{9}$$

Equation (9) implies now

$$\begin{aligned} d\sigma_{0k} = &-(\dot{H}_k + H_k^2)\sigma^0 \wedge \sigma^k \\ &- \tfrac{1}{2}\epsilon_{klm} n_k (R_k/R_l R_m) 2H_k \sigma^l \wedge \sigma^m \,, \end{aligned} \tag{10a}$$

$$\begin{aligned} d\sigma_{ik} = \tfrac{1}{2}\epsilon_{ikl}&\left[\left(\frac{n_i R_i}{R_k R_l} - \frac{n_k R_k}{R_i R_l}\right)(H_i - H_k) \right. \\ &\left. + \frac{n_l R_l}{R_i R_k}(H_i + H_k - 2H_l)\right]\sigma^0 \wedge \sigma^l \\ &+ \tfrac{1}{2}\epsilon_{ikl}\left[\frac{n_i}{R_k^2} + \frac{n_k}{R_i^2} - n_l\left(\frac{R_l}{R_i R_k}\right)^2\right] n_l \sigma^i \wedge \sigma^k \,. \end{aligned} \tag{10b}$$

The curvature two-forms

$$\theta_{\mu\nu} = \sigma_\mu{}^{\alpha} \wedge \sigma_{\alpha\nu} + d\sigma_{\mu\nu} \tag{11}$$

can be readily computed by use of Eqs. (6), (10a), (10b), and the compatibility equation

$$0 = d\eta_{\mu\nu} = \sigma_{\mu\nu} + \sigma_{\nu\mu} \,. \tag{12}$$

Out of this calculation, one reads the individual components $R^{\mu\nu}{}_{\alpha\beta}$ of the curvature tensor by using the second Cartan equation

$$\theta^{\mu\nu} = \tfrac{1}{2} R^{\mu\nu}{}_{\alpha\beta}\sigma^\alpha \wedge \sigma^\beta \tag{13}$$

as an identification scheme. The results are

$$R^{0k}{}_{0k} = \dot{H}_k + H_k^2 \,, \tag{14a}$$

$$\begin{aligned} R^{ik}{}_{ik} = H_i H_k &+ \frac{\delta}{2}\epsilon_{ikl}\left(\frac{n_i}{R_i^2} + \frac{n_k}{R_k^2} - \frac{n_l}{R_l^2}\right) \\ &+ \tfrac{1}{4}\epsilon_{ikl}\left[\left(\frac{R_i}{R_k R_l}\right)^2 + \left(\frac{R_k}{R_i R_l}\right) - 3\left(\frac{R_l}{R_i R_k}\right)^2\right], \\ &i \neq k \,, \quad \delta \equiv n_3 \,. \end{aligned} \tag{14b}$$

Thus we can easily calculate the Ricci tensor $R_{\mu\nu} = -R^\alpha{}_{\mu\nu\alpha}$. The nonvanishing components are

$$R_{00} = -(3\dot{H} + H_1^2 + H_2^2 + H_3^2) \,, \tag{15a}$$

$$\begin{aligned} R_{ii} = \dot{H}_i + 3HH_i + \frac{1}{2(R_1R_2R_3)^2}(R_i^4 - R_k^4 - R_l^4 \\ + 2n_i \delta R_k^2 R_l^2) \,, \end{aligned} \tag{15b}$$

where $H = \frac{1}{3}(H_1 + H_2 + H_3)$ is the average Hubble parameter and the i, k, l are in cyclic order. The Einstein tensor $G_{\mu\nu}$ is computed by

$$G_\beta{}^\delta = \epsilon^{\delta\rho\sigma\tau}\epsilon_{\beta\mu\nu\tau} R^{|\mu\nu|}{}_{|\rho\sigma|} \,, \tag{16}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the four-dimensional Levi-Civita pseudotensor with $\epsilon_{0123} = 1$. (Vertical bars around the indices mean summation extends only over $\mu < \nu$, $\rho < \sigma$.) The nonvanishing components are

$$G_{00} = \tfrac{1}{2}\sum_{i \neq j} H_i H_j + \frac{\delta}{2}\sum_k \frac{n_k}{R_k^2} - \tfrac{1}{8}\sum_{i \neq j \neq l}\left(\frac{R_i}{R_j R_l}\right)^2 , \tag{17a}$$

$$\begin{aligned} G_{ii} = \dot{H}_i - 3\dot{H} - \sum_{k,l \neq i} H_k H_l - \frac{\delta}{2}\left(\sum_l \frac{n_l}{R_l^2} - \frac{2n_i}{R_i^2}\right) \\ - \frac{1}{4}\left[\left(\frac{R_i}{R_k R_l}\right)^2 + \left(\frac{R_k}{R_i R_l}\right)^2 - 3\left(\frac{R_l}{R_i R_k}\right)^2\right]. \end{aligned} \tag{17b}$$

The three-dimensional curvature R^* is given by

$$R^* = -\frac{1}{4}\left[-4\delta\sum_l \frac{n_l}{R_l^2} + \sum_{i \neq k \neq l}\left(\frac{R_i}{R_k R_l}\right)^2\right] \tag{18}$$

as can be easily proved by setting $H_k = 0$ in Eq. (17a).

III. SOLUTION OF THE FIELD EQUATIONS

The Einstein equations considered here are

$$G_{\mu\nu} = 8\pi(E_{\mu\nu} + T_{\mu\nu}) \,, \tag{19}$$

$$E_{\mu\nu} = \frac{1}{4\pi}(F_{\mu\alpha}F_\nu{}^\alpha - \tfrac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \,, \tag{20}$$

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + \eta_{\mu\nu} p \,, \tag{21}$$

where $E_{\mu\nu}$ is the electromagnetic stress-energy tensor, $F_{\mu\nu}$ is the electromagnetic field tensor, $T_{\mu\nu}$ is the energy-momentum tensor, u^μ is the velocity four-vector, and ϵ and p are, respectively, the density and pressure of the fluid. The perfect-fluid matter is characterized by the equation of state

$$p=(\gamma-1)\epsilon\,,\quad 1\leq\gamma\leq 2\,. \tag{22}$$

The source-free Maxwell equations are

$$dF=0\,,\quad d{}^*F=0\,, \tag{23}$$

where the two-form F represents the electromagnetic field and $^*F$ is its dual. In the basis $\sigma^\alpha\wedge\sigma^\beta$ we have

$$F=E_i\sigma^i\wedge\sigma^0+\tfrac{1}{2}B_i\epsilon_{ijk}\sigma^j\wedge\sigma^k\,, \tag{24}$$

$$^*F=-B_i\sigma^i\wedge\sigma^0+\tfrac{1}{2}E_i\epsilon_{ijk}\sigma^j\wedge\sigma^k\,. \tag{25}$$

Owing to homogeneity, the electric field E_i and the magnetic field B_i depend only on t. By using Eqs. (2), (3a), and (4) the sourceless Maxwell equations (23) become

$$E_iR_in_i=\partial_t(B_iR_jR_k)\,,\quad B_iR_in_i=-\partial_t(E_iR_jR_k)\,, \tag{26}$$

where $\partial_t=\partial/\partial t$. It is convenient to introduce the variables t_i by $dt_i=n_i(R_i/R_jR_k)dt$. Then Eqs. (26) take the form

$$E_iR_jR_k=\partial t_i(B_iR_jR_k)\,,\quad B_iR_jR_k=-\partial t_i(E_iR_jR_k)\,, \tag{27}$$

with the solutions

$$E_i=\frac{a_i}{R_jR_k}\cos(t_i+\tau_i)\,,\qquad B_i=\frac{a_i}{R_jR_k}\sin(t_i+\tau_i)\,, \tag{28}$$

where a_i, τ_i are constants. Because the Einstein tensor is diagonal, the electromagnetic stress-energy tensor must be diagonal too. The off-diagonal components of Eq. (20) are

$$F_{02}F_{12}+F_{03}F_{13}=0\,,\quad F_{01}F_{12}-F_{03}F_{23}=0\,,$$
$$F_{01}F_{13}+F_{02}F_{23}=0\,,\quad F_{01}F_{02}-F_{13}F_{23}=0\,,$$
$$F_{01}F_{03}+F_{12}F_{23}=0\,,\quad F_{02}F_{03}-F_{12}F_{13}=0\,,$$

which lead to three possible cases:

(i) $F_{02}=F_{03}=F_{12}=F_{13}=0\,,\quad F_{01},F_{23}\neq 0\,,$

(ii) $F_{01}=F_{03}=F_{12}=F_{23}=0\,,\quad F_{02},F_{13}\neq 0\,,$

(iii) $F_{01}=F_{02}=F_{13}=F_{23}=0\,,\quad F_{03},F_{12}\neq 0\,.$

Without loss of generality, we may consider only case (iii) : $F_{30}=E_3=E$, $F_{12}=B_3=B$. We note that the electric and the magnetic fields must be parallel and point in the direction of the σ_3 axis. From (28) it follows that

$$E=\frac{a}{R_1R_2}\cos(t_3+\tau_3)\,,\quad B=\frac{a}{R_1R_2}\sin(t_3+\tau_3)\,, \tag{29}$$

where $a=a_3$. The nonvanishing components of the electromagnetic stress-energy tensor are

$$E_{00}=E_{11}=E_{22}=-E_{33}=\frac{1}{8\pi}(E^2+B^2)=\frac{1}{8\pi}\left(\frac{a}{R_1R_2}\right)^2\,. \tag{30}$$

We note that the trace $E=\eta^{\mu\nu}E_{\mu\nu}$ of the electromagnetic stress-energy tensor vanishes.

We now turn to the Einstein equations. Because of our last remark Eqs. (19)–(21) reduce to the simple form

$$R_{\mu\nu}=8\pi(T_{\mu\nu}+E_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}T_\lambda{}^\lambda)\,. \tag{31}$$

In the local inertial frame determined by (2), an observer comoving with the fluid is assumed to have four-velocity $u^\alpha=\delta_0^\alpha$. The field equations (31) reduce to the following independent equations:

$$R_{00}=\left(\frac{a}{R_1R_2}\right)^2+4\pi(\epsilon+3p)\,, \tag{32a}$$

$$R_{11}=\left(\frac{a}{R_1R_2}\right)^2+4\pi(\epsilon-p)=R_{22}\,, \tag{32b}$$

$$R_{33}=-\left(\frac{a}{R_1R_2}\right)^2+4\pi(\epsilon-p)\,. \tag{32c}$$

We take the stiff ($\gamma=2$) equation of matter. The conservation law for the energy-momentum tensor $T_{\mu\nu}$,

$$T^k{}_{i;k}=0\,, \tag{33}$$

gives for $\gamma=2$

$$\epsilon=\frac{1}{8\pi}\left(\frac{b}{R_1R_2R_3}\right)^2\,,\quad b=\text{const}\,. \tag{34}$$

It is convenient to solve the equations

$$G_{00}=\left(\frac{a}{R_1R_2}\right)^2+\left(\frac{b}{R_1R_2R_3}\right)^2\,,\quad R_{22}+R_{33}=0 \tag{35}$$

for an axisymmetric model. It can easily be seen that Eqs. (17b) and (19) do not turn into each other under any permutation of the indices i, j, k for type VIII, whereas for type IX the intrinsic geometry of three-space does not privilege any direction of space. For type VIII we can equate only R_1 with R_2 obtaining a symmetry about the third axis. Here we set $R\equiv R_1=R_2$, $S\equiv R_3$, so that Eqs. (35) take the form

$$\left(\frac{\dot R}{R}\right)^{\cdot}+\left(\frac{\dot S}{S}\right)^{\cdot}+\left(2\frac{\dot R}{R}+\frac{\dot S}{S}\right)\left(\frac{\dot R}{R}+\frac{\dot S}{S}\right)+\frac{\delta}{R^2}=0\,, \tag{36a}$$

$$\left(\frac{\dot R}{R}\right)^2+2\frac{\dot R}{R}\frac{\dot S}{S}-\frac{1}{4R^4}(S^2-4\delta R^2)=\frac{a^2}{R^4}+\frac{b^2}{R^4S^2}\,. \tag{36b}$$

Introducing a new time coordinate $dt'=Sdt$ we obtain from (36a)

$$\frac{d^2(RS)^2}{dt'^2}+2\delta=0\,, \tag{37}$$

with the general solution

$$(RS)^2=-\delta(t'-t_1')(t'-t_2')\,, \tag{38}$$

where t_1', t_2' are constants of integration. It follows then that

$$(t_2'-t_1')^2=4\delta(RS)^2+4(R^2\dot S+R\dot RS)^2\,. \tag{39}$$

By setting $4q^2\equiv(t_2'-t_1')^2$ and introducing another time coordinate $dt=SR^2d\tau$, we obtain

$$\frac{d}{d\tau}\ln(RS)^2=2(q^2-4\delta(RS)^2)^{1/2} \tag{40}$$

with the solutions

$$(RS)^2=q^2\sinh^{-2}(q\tau+u)\,,\quad \text{type VIII} \tag{41a}$$

$$(RS)^2=q^2\cosh^{-2}(q\tau+u)\,,\quad \text{type IX}\,. \tag{41b}$$

We can manipulate Eq. (36b) to take the form

$$\frac{dS^2}{d\tau}=S^2[4(q^2-b^2-a^2S^2)-S^4]^{1/2} \tag{42}$$

with the solution

$$S^2=2(q^2-b^2)\{a^2+(q^2-b^2+a^4)^{1/2}\cosh[2(q^2-b^2)^{1/2}\tau+v]\}\,,\quad \text{types VIII, IX}\,, \tag{43}$$

where u and v are constants of integration. When we define

$$\lambda(\tau)\equiv a^2+(q^2-b^2+a^4)^{1/2}\cosh[2(q^2-b^2)^{1/2}\tau+v]\,, \tag{44}$$

the solutions assume the final form

$$R^2=\tfrac{1}{2}\lambda(\tau)\frac{q^2\sinh^{-2}(q\tau+u)}{q^2-b^2}\,,\quad \text{type VIII} \tag{45a}$$

$$R^2=\tfrac{1}{2}\lambda(\tau)\frac{q^2\cosh^{-2}(q\tau+u)}{q^2-b^2}\,,\quad \text{type IX} \tag{45b}$$

$$S^2=2(q^2-b^2)\lambda^{-1}(\tau)\,,\quad \text{types VIII, IX}\,. \tag{45c}$$

The differential one-forms ω^i can be parametrized by the Euler angles (ϕ,θ,ψ):

$$\left.\begin{aligned}\omega^1&=-\sin\psi\,d\theta+\cos\psi\cosh\theta\,d\phi\,,\\ \omega^2&=\cos\psi\,d\theta+\sin\psi\cosh\theta\,d\phi\,,\quad \text{type VIII}\\ \omega^3&=d\psi+\sinh\theta\,d\phi\,,\end{aligned}\right. \tag{46a}$$

$$\left.\begin{aligned}\omega^1&=-\sin\psi\,d\theta+\cos\psi\sin\theta\,d\phi\,,\\ \omega^2&=\cos\psi\,d\theta+\sin\psi\sin\theta\,d\phi\,,\quad \text{type IX}\\ \omega^3&=d\psi+\cos\theta\,d\phi\,.\end{aligned}\right. \tag{46b}$$

If we define

$$f(\theta)\equiv\begin{cases}\cosh\theta\,, & \text{type VIII}\\ \sin\theta\,, & \text{type IX}\end{cases} \tag{47a}$$

$$g(\theta)\equiv\begin{cases}\sinh\theta\,, & \text{type VIII}\\ \cos\theta\,, & \text{type IX}\,,\end{cases} \tag{47b}$$

the metrics can be written as

$$\begin{aligned}ds^2=&-d\tau^2+R^2[d\theta^2+f^2(\theta)d\phi^2]\\ &+S^2[d\psi+g(\theta)d\phi]^2\,.\end{aligned} \tag{48}$$

We finally derive the geodesics of these spacetimes. We will follow the notation of Carter[21] fairly closely. The equations of motion can be derived from the Hamiltonian

$$H=\tfrac{1}{2}g^{\mu\nu}p_\mu p_\nu\,, \tag{49}$$

where the momenta p_μ are defined by

$$p_\mu=g_{\mu\nu}\frac{dx^\nu}{d\lambda}\,, \tag{50}$$

and λ is an affine parameter. The Hamiltonians for the geodesic equations of the metrics (48) are given by

$$H=\tfrac{1}{2}\left\{-p_\tau^{\,2}+\frac{p_\theta^{\,2}}{R^2}+\frac{[p_\phi-g(\theta)p_\psi]^2}{R^2f^2(\theta)}+\frac{p_\psi^{\,2}}{S^2}\right\}. \tag{51}$$

The Hamilton-Jacobi equation is

$$\partial W/\partial\lambda+H=0\,, \tag{52}$$

and we find that the action W separates in the form

$$W=-\tfrac{1}{2}\epsilon\lambda+\alpha\phi+\beta\psi+\int^\theta\Theta^{1/2}d\theta+\int^\tau T^{1/2}d\tau\,, \tag{53}$$

where

$$\Theta\equiv K-\frac{[\alpha-\beta g(\theta)]^2}{f^2(\theta)}\,, \tag{54}$$

$$T\equiv R^2(K-R^2\epsilon)+\frac{\beta^2R^4}{S^2}\,, \tag{55}$$

and ϵ, α, β are constants and K is Carter's "fourth constant" of motion. The equations of motion are solved by the quadratures

$$\int^\theta\Theta^{-1/2}d\theta=-\int^\tau T^{-1/2}d\tau\,, \tag{56}$$

$$\lambda=-\int^\tau T^{-1/2}R^2d\tau\,, \tag{57}$$

$$\phi = \int^{\theta} \frac{\alpha - \beta g(\theta)}{\theta^{1/2} f^2(\theta)} d\theta , \quad (58)$$

$$\psi = -\int^{\theta} \frac{g(\theta)[\alpha - \beta g(\theta)]}{\theta^{1/2} f^2(\theta)} d\theta - \int^{\tau} \frac{\beta R^2}{T^{1/2} S^2} d\tau . \quad (59)$$

The cosmological implications of these geodesics and the generalization for the motion of a charged test particle will be discussed in a future paper.

The solutions (45a)–(45c) are new. For $b=0$ they are, with a change of time variable, the same as one of those of Cahen and Defrise.[22] The solutions (45b) and (45c) reduce for $b=0$ to a generalization of Brill's[23] electromagnetic universe and are a special case of the solutions of Batakis and Cohen,[24] who considered in addition to the electromagnetic field a scalar field, obeying the Klein-Gordon equations. For $a=0$ we obtain the solutions sketched by Maartens and Nel.[25] We point out that further solutions for Bianchi type VIII and type IX in the presence of matter and an electromagnetic field have been obtained by Ozsvath[26] and by Soares and Assad.[27]

Note added in proof. Dr. M. A. H. MacCallum kindly pointed out to me that the solutions (45a)–(45c) may agree with those of V. A. Ruban, in Report No. 412 of the Leningrad Institute of Nuclear Physics, B. P. Konstantinova, 1978 (unpublished). However, in the meantime I have been able to extend my calculations on electromagnetic Bianchi types II, VIII, and IX cosmologies to include a source term corresponding to a scalar field obeying the Klein-Gordon equation, which will be published in a forthcoming paper.

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[1] For details and references see, e.g., M. A. H. MacCallum, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, England, 1979), p. 533; also, M. A. H. MacCallum, in *Cargèse Lectures in Physics, Vol. 6,* edited by E. Schatzman (Gordon and Breach, New York, 1973), p. 61.

[2] M. P. Ryan, Jr. and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton Univ. Press, Princeton, New Jersey, 1975).

[3] A. G. Doroshkevich, Astrofizika 1, 225 (1965) [Astrophysics 1, 138 (1965)].

[4] Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. 48, 986 (1965) [Sov. Phys.-JETP 21, 656 (1965)].

[5] I. S. Shikin, Dok. Akad. Nauk SSSR 171, 73 (1966) [Sov. Phys. Dokl. 11, 944 (1967)].

[6] I. S. Shikin, Dok. Akad. Nauk SSSR 176, 7048 (1967) [Sov. Phys. Dokl. 12, 950 (1968)].

[7] K. S. Thorne, Astrophys. J. 148, 51 (1967).

[8] K. C. Jacobs, Astrophys. J. 155, 379 (1969).

[9] J. P. Vajk and P. G. Eltgroth, J. Math. Phys. 11, 2212 (1970).

[10] Ya. B. Zel'dovich and I. D. Novikov, *Stroyenie i ēvoliutsiya Vselennoi* (Nauka, Moscow, 1975).

[11] K. Kawabata, M. Fujimoto, Y. Sofue, and M. Fukui, Publ. Astron. Soc. Japan 21, 293 (1969).

[12] Y. Sofue, M. Fujimoto, and K. Kawabata, Publ. Astron. Soc. Japan 20, 368 (1969).

[13] M. Reinhardt and M. A. F. Thiel, Astrophys. Lett. 7, 101 (1970).

[14] M. Reinhardt, Astron. Astrophys. 19, 104 (1972).

[15] V. M. Kolobov, M. Reinhardt, and V. N. Sazonov, Astrophys. Lett. 17, 183 (1976).

[16] C. B. Collins and S. W. Hawking, Astrophys. J. 180, 317 (1973).

[17] L. P. Hughston and K. C. Jacobs, Astrophys. J. 160, 147 (1970).

[18] Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. 41, 1609 (1961) [Sov. Phys.-JETP 14, 1143 (1962)].

[19] Ya. B. Zel'dovich, Mon. Not. R. Astron. Soc. 160, 1 (1970).

[20] J. D. Barrow, Nature 272, 211 (1978).

[21] B. Carter, Phys. Rev. 174, 1559 (1968).

[22] M. Cahen and L. Defrise, Commun. Math. Phys. 11, 56 (1968).

[23] D. Brill, Phys. Rev. 133, B845 (1964).

[24] N. Batakis and J. M. Cohen, Ann. Phys. (N.Y.) 73, 578 (1972).

[25] R. Maartens and S. D. Nel, Commun. Math. Phys. 59, 273 (1978).

[26] I. Ozsvath, in *Essays in Honour of V. Hlavaty* (Indiana Univ. Press, Bloomington, Indiana, 1966).

[27] I. D. Soares and M. J. D. Assad, Phys. Lett. 66A, 359 (1978).