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Black-hole uniqueness theorems in Euclidean quantum gravity

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The Euclidean section of the classical Lorentzian black-hole solutions has been used in approximating the functional integral in the Euclidean path-integral approach to quantum gravity. In this paper the claim that classical black-hole uniqueness theorems apply to the Euclidean section is disproved. In particular, it is shown that although a Euclidean version of Israel's theorem *does* provide a type of uniqueness theorem for the Euclidean Schwarzschild solution, a Euclidean version of Robinson's theorem *does not* allow one to form conclusions about the uniqueness of the Euclidean Kerr solution. Despite the failure of uniqueness theorems, "no-hair" theorems are shown to exist. Implications are discussed. A precise mathematical statement of the Euclidean black-hole uniqueness conjecture is made and the proof left as an unsolved problem in Riemannian geometry.

I. INTRODUCTION

This paper considers the uniqueness of the Euclidean black-hole solutions[1] used in the Euclidean path-integral approach to quantum gravity.[2] "Euclidean" or "Euclidean section" will mean in this paper that the metric on a four-dimensional manifold is of positive-definite signature. "Solution" will mean that the metric is Ricci flat. For example, the Euclidean Schwarzschild solution can be written in a local coordinate chart as

$$ds^2=(1-2m/r)d\tau^2+dr^2/(1-2m/r)+r^2(d\theta^2+\sin^2\theta\, d\phi^2)\,. \tag{1.1}$$

It can be obtained from the Lorentzian Schwarzschild solution describing a nonrotating black hole of mass m,

$$ds^2=-(1-2m/r)dt^2+dr^2/(1-2m/r)+r^2(d\theta^2+\sin^2\theta\, d\phi^2) \tag{1.2}$$

by $t\to i\tau$. τ must be identified with period $8\pi m$ for the Euclidean section to be regular. θ and ϕ are the usual polar and azimuthal coordinates on a two-sphere and $r\in[2m,\infty)$. The manifold is geodesically complete and has topology $R^2\times S^2$. The Euclidean Kerr solution

$$\begin{aligned}ds^2&=(d\tau-\alpha\sin^2\theta\, d\phi)^2\Delta/\rho^2\\&\quad+[(r^2-\alpha^2)d\phi-\alpha\, d\tau]^2\sin^2\theta/\rho^2+\rho^2dr^2/\Delta+\rho^2d\theta^2,\\ \Delta&=r^2-2mr-\alpha^2,\quad \rho^2=r^2-\alpha^2\cos^2\theta\end{aligned} \tag{1.3}$$

can be obtained from the Lorentzian Kerr solution describing a rotating black hole of mass m and angular momentum ma,

$$\begin{aligned}ds^2&=-(dt-a\sin^2\theta\, d\phi)^2\Delta/\rho^2\\&\quad+[(r^2+a^2)d\phi-a\,dt]^2\sin^2\theta/\rho^2+\rho^2dr^2/\Delta+\rho^2d\theta^2,\\ \Delta&=r^2-2mr+a^2,\quad \rho^2=r^2+a^2\cos^2\theta\end{aligned} \tag{1.4}$$

by $\tau\to i\tau$, $a\to -i\alpha$. The $\{\tau,\phi\}$ plane must be identified as $\{\tau,\phi\}=\{\tau+\beta,\phi+\beta\Omega_H\}$, where $\beta=4\pi m[m+(m^2+\alpha^2)^{1/2}]/(m^2+\alpha^2)^{1/2}$ and $\Omega_H=\alpha\{2[m^2+m(m^2+\alpha^2)^{1/2}]\}^{-1}$. θ and ϕ are again the usual two-sphere coordinates and $r\in[m+(m^2+\alpha^2)^{1/2},\infty)$. The manifold has topology $R^2\times S^2$ and is geodesically complete with the metric given above.

The claim has been made[3] that the classical black-hole theorems apply to the Euclidean section. It is straightforward to show that Israel's theorem,[4] which in essence proves that (1.2) is the unique, static asymptotically flat vacuum solution of Einstein's equations with a regular fixed-point surface of the staticity Killing vector, can be taken over to the Euclidean section essentially line for line. However, the claim is disproved by showing that the Euclidean version of Robinson's theorem[5] proving the uniqueness of the Lorentzian Kerr solution no longer works on the Euclidean section. Various implications are discussed.

This paper is divided into seven sections. The following section briefly reviews the Euclidean path-integral derivation of the thermal properties of black holes and the role that Euclidean classical solutions to Einstein's equations play in quantum gravity.[1,2] The boundary conditions relevant to the Euclidean section that Euclidean solutions must obey are stated. The physical importance of the uniqueness of the Euclidean solution is emphasized. In Sec. III the Lorentzian black-hole uniqueness theorems are summarized and their applicability to the Euclidean section noted. In Sec. IV it is shown how Israel's theorem, proving the uniqueness of the Schwarzschild solution (1.1), works on the Euclidean section while Robinson's theorem proving the uniqueness of the Lorentzian Kerr solution (1.4) has no Euclidean analog that proves the uniqueness of the Euclidean Kerr solu-

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tion (1.3).

In Sec. V we search for possible counterexamples to the uniqueness conjecture of the Euclidean Kerr solution. Apart from the Kerr solution, the only Lorentzian, stationary, asymptotically flat, axisymmetric vacuum solution for which the metric is explicitly known is the Tomimatsu-Sato family of solutions.[6] The Lorentzian Tomimatsu-Sato solutions suffer from naked ring singularities (among other things) and hence are not counterexamples to the uniqueness of the Lorentzian Kerr solution. The Euclidean section of the Tomimatsu-Sato solutions is defined and it is shown that the ring singularities disappear on the Euclidean section. However, the north and south poles of the horizon, which were nonsingular on the Lorentzian section, develop curvature singularities on the Euclidean section. Hence the Euclidean Tomimatsu-Sato solution is not a counterexample to the conjectured uniqueness of the Euclidean Kerr solution. Some Euclidean solutions cannot be obtained from Lorentzian solutions. A sufficient, though not necessary, condition is that the curvature be (anti) self-dual. By an argument paralleling an argument of Gibbons and Pope[7] ruling out the existence of asymptotically Euclidean (anti) self-dual solutions (essentially an application of Lichnerowicz's theorem[8]) we similarly rule out (anti) self-dual asymptotically flat solutions. Hence if there exist nonsingular, positive-definite-signature, asymptotically flat, stationary, axisymmetric solutions that satisfy the boundary conditions other than the Euclidean Kerr solution, then it is not (anti) self-dual, nor is it the Euclidean Tomimatsu-Sato solution. In Sec. VI a Euclidean "no-hair" theorem is proved. A Euclidean version of Carter's Lorentzian no-hair theorem[9] is not used, as this approach involves a linearized Robinson identity and suffers from the same pathologies as the nonlinear Robinson identity used in attempting to prove uniqueness. Instead, the perturbation analysis of Teukolsky,[10,11] and Wald[12] that employs the Newman-Penrose formalism is used to show that the only regular perturbations of the Euclidean Kerr solution are perturbations in m and α, the mass and angular momentum parameters. In Sec. VII a precise statement of the Euclidean uniqueness conjecture is made. The proof of a uniqueness theorem is left as an unsolved problem in classical Riemannian geometry.

II. CLASSICAL SOLUTIONS IN QUANTUM GRAVITY

The thermal properties of black holes, first derived by Hawking[13] in 1975 using a semiclassical formalism, can be recovered using the Euclidean path-integral approach to quantum gravity.[1] We follow the analysis as reviewed by Hawking in Ref. 2. The essential idea is that the partition function for a system of temperature $1/\beta$ can be represented as a functional integral over fields periodic with period β in Euclidean time:

$$Z=\int_C d[\phi]\,e^{-I[\phi]}\,. \tag{2.1}$$

Here Z is the partition function, $d[\phi]$ denotes functional integration over fields ϕ (indices to be appropriately added for spinor, vector, tensor), $I[\phi]$ is the classical action functional for ϕ on the Euclidean section, while the subscript C on the integral denotes the class of fields to be integrated, e.g., periodic in imaginary time with Dirichlet boundary conditions.

The appropriate action for gravity is

$$I=\frac{1}{16\pi G}\int R\sqrt{g}\,d^4x+\frac{1}{8\pi G}\int K\sqrt{h}\,d^3x+C_0\,, \tag{2.2}$$

where G is Newton's constant in natural units, R is the Ricci scalar, h is the determinant of the induced metric h_{ab} on the boundary, K is the trace of the second fundamental form of the boundary, and C_0 is a constant adjusted to make the action of flat space vanish. The integral is over all asymptotically flat metrics, periodic in Euclidean time, which fill in an $S^2\times S^1$ boundary at infinity. The $S^2\times S^1$ boundary is chosen to represent a large spherical "box" S^2 bounding three-space, cross the periodically identified Euclidean time axis, S^1.

It is impossible to evaluate the functional exactly and hence a steepest-descents approximation is employed. That is, one expands the action about a classical solution of the field equations,

$$\left.\frac{\delta I}{\delta g_{ab}}\right|_{g_{ab}^{\text{classical}}}=0$$

and integrates over fluctuations away from this solution. Hence,

$$g_{ab}=g_{ab}^{\text{classical}}+\tilde{g}_{ab} \tag{2.3}$$

and

$$I[g]=I_0[g^{\text{classical}}]+I_2[\tilde{g}_{ab}]+\cdots\ . \tag{2.4}$$

$I_2[\tilde{g}_{ab}]$ is quadratic in the fluctuation $\tilde{g}_{ab}$ and has the form $\int\tilde{g}_{ab}O^{abcd}\tilde{g}_{cd}\sqrt{g}\,d^4x$, where O^{abcd} is a second-order differential operator in the "background" metric g_{ab}. Truncation of the expansion at quadratic order is called the "one-loop expansion" and leads to an expression for $\ln Z$ of the form

$$\ln Z=-I[g_{ab}^{\text{classical}}]+\ln\left\{\int d[\tilde{g}_{ab}]\,e^{-I_2[\tilde{g}_{ab}]}\right\}, \tag{2.5}$$

where I is the contribution of classical background fields to $\ln Z$ and the second term (the "one-loop" term) represents the effect of quantum fluctuations about the background fields. Evaluation of the second term involves the determinant of the operator O^{abcd}. A convenient definition of $\det O^{abcd}$ is the ζ-function definition of Singer.[14] Hawking[15] has employed this definition to calculate one-loop terms. Gibbons and Perry[16] have investigated the one-loop term in detail. It should be noted that more than one background field (classical solution) may satisfy the boundary conditions, and in this event there are contributions to $\ln Z$ of the form (2.5) for each classical background field.

One background field satisfying the boundary conditions of asymptotic flatness, $S^2\times S^1$ boundary, and periodicity β in Euclidean time is flat space

$$ds^2=d\tau^2+dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{2.6}$$

with τ identified with period β. The action (2.2) of flat space is zero. In the limit of a very large spherical box S^2 with radius r_0 tending to infinity, the one-loop term can be evaluated exactly[15] as $4\pi r_0{}^3/135\beta^3$. The interpretation is that this is the contribution to the partion function for thermal gravitons on a flat-space background. Another background field satisfying the boundary conditions is the Euclidean Schwarzschild solution,

$$\begin{aligned}ds^2=(1-2m/r)d\tau^2+\frac{dr^2}{1-2m/r}\\ +r^2(d\theta^2+\sin^2\theta\, d\phi^2)\,,\end{aligned} \tag{2.7}$$

where regularity requires $\tau=\tau+\beta$, $\beta=8\pi m$. This has action $I=4\pi m^2$ and a one-loop term[16]

$$\frac{106}{45}\ln\left(\frac{\beta}{\beta_0}\right)\frac{4\pi r_0{}^3}{135\beta^3}$$

for $r_0\gg\beta$ i.e., for a box size large compared to the black hole. β_0 is related to the one-loop renormalization parameter.

Given the partition function one can evaluate relevant thermodynamic quantities such as energy and entropy in the usual fashion

$$\langle E\rangle=-\frac{\partial}{\partial\beta}\ln Z\,, \tag{2.8a}$$

$$S=\beta\langle E\rangle+\ln Z\,. \tag{2.8b}$$

Applying this to the contribution to $\ln Z$ from the classical action of the Schwarzschild solution yields

$$S=4\pi m^2=A/4\,, \tag{2.9}$$

where A is the area of the "event horizon," $r=2m$. Hence the classical background contribution to the partition function yields a temperature

$$T=\frac{1}{\beta}=\frac{1}{8\pi m}$$

and an entropy $S=4\pi m^2$. These are precisely the expressions for the temperature and entropy of a nonrotating black hole that Hawking[13] first obtained in 1975 by completely different methods.

One can calculate the (unstable) equilibrium states of a black hole and thermal gravitons in a large box by including the one-loop terms in the expression for $\ln Z$. Maximization of the entropy with fixed energy leads to the conclusion that if the volume V of the box satisfies

$$E^5<\frac{\pi^2}{15}(8354.5)V\,, \tag{2.10}$$

then the most probable state of the system is flat space with thermal gravitons, while if the inequality is not satisfied the most probable state is a black hole (Schwarzschild solution) in equilibrium with thermal gravitons.

One can also consider the partition function for grand canonical ensembles in which a chemical potential is associated with a conserved quantity. For example, one can consider a system at a temperature $T=1/\beta$ and a given (conserved) angular momentum J with associated chemical potential, Ω, where Ω is the angular velocity. The partition function would then be given by a functional integral over all fields with $(t,r,\theta,\phi)=(t+\beta,r,\theta,\phi+\beta\Omega)$. The Euclidean Kerr solution (1.3) would then be a classical background solution around which one could expand the action in a one-loop calculation analogous to the above. It is a major point of this paper that although a type of uniqueness theorem exists for the Euclidean Schwarzschild solution via a Euclidean Israel theorem, the Euclidean version of Robinson's theorem does not allow one to draw similar conclusions concerning the Euclidean Kerr solution.

It is clear from the analysis just reviewed that the Euclidean black-hole solutions, both Schwarzschild and Kerr, play a key role in approximating the functional integrals occurring in quantum gravity, and connect in a fundamental way to the thermal properties of black holes discovered by Hawking.[13] It would be valuable to prove that the Euclidean Schwarzschild solution (1.1) is the unique nonsingular extremal of the action (2.2) (i.e., Ricci flat metric) satisfying

(i) asymptotic flatness,
(ii) an $S^2\times S^1$ boundary at infinity,
(iii) an identification of the coordinates $(t,\theta,\phi)=(t+\beta,\theta,\phi)$ on the boundary, and
(iv) nontrivial topology,

and similarly that the Euclidean Kerr solution is the unique, nonsingular, Ricci flat solution satisfying

(i) asymptotic flatness,
(ii) an $S^2 \times S^1$ boundary at infinity,
(iii) an identification of the coordinates $(t, \theta, \phi) = (t+\beta, \phi+\Omega\beta)$ on the boundary, and
(iv) nontrivial topology.

The requirement of nontrivial topology excludes flat space from being a counterexample. The class of metrics contributing to the functional integral are not necessarily Ricci flat and should satisfy requirements (i), (ii), and (iii). The conjectures above are made more precise in Sec. VII.

If the Euclidean black-hole solutions are not unique then there exists at least one other Euclidean solution, satisfying the conditions above, which would necessarily have to be included in the steepest-descents approximation of the functional integral. This would mean there exists the possibility of a third phase, in addition to the Euclidean black-hole solutions and flat space, contributing to the analysis of the possible states of a gravitational field in a box. One might call such a solution a new Euclidean black-hole solution. This new Euclidean black-hole solution would either not admit a Lorentzian section, or if a Lorentzian section exists, it would violate the conditions of a Lorentzian black-hole solution by being, for example, singular or perhaps not asymptotically flat. Hence the new Euclidean black-hole solution would play a role somewhat analogous to the instantons of Yang-Mills theory, insomuch as the Lorentzian sections of such solutions are not physical objects, although they do have a physical effect by making a large contribution to the functional integral in the quantization of the theory.

III. CLASSICAL BLACK-HOLE UNIQUENESS THEOREMS

The classical black-hole uniqueness theorems, proving the uniqueness of the Schwarzschild solution and Kerr solution for (vacuum) nonrotating and rotating black holes, respectively, can be roughly separated into two qualitatively distinct parts. The first part consists of the assumption of stationarity (the existence of a Killing vector which is timelike near infinity) and the use of global analysis and the causal structure of spacetime to conclude that a stationary black hole must be either axisymmetric or static. This result is known as the strong rigidity theorem.[17] The second part has two subsections. If the black hole is static, i.e., if the stationarity Killing vector is hypersurface orthogonal, then Israel's theorem[4] essentially proves that the Schwarzschild solution (1.2) is the unique, nonsingular, asymptotically flat, static solution which contains an event horizon. If the black hole is rotating, i.e., if the stationarity Killing vector is not hypersurface orthogonal, then Robinson's theorem[5] essentially proves that the Kerr solution (1.4) is the unique, nonsingular, asymptotically flat, axisymmetric rotating solution which contains an event horizon. The assumption of stationarity is essentially a statement that one is considering the final state of a black hole (formed, for example, by a star's collapse) after all the dynamics has been resolved; while the physical reason behind the conclusion of axisymmetry given rotation is that a rotating nonaxisymmetric black hole would eventually become axisymmetric by gravitationally radiating away its asymmetries.

Both Robinson's and Israel's theorems do not explicitly involve causal structure even though the phrase "event horizon" appeared in their description above. This is because the strong rigidity theorem implies that the event horizon is also a Killing horizon. That is, the null geodesic generators of the horizon coincide on the horizon with a generator of an isometry of the full four-dimensional spacetime, with the Killing vector of the isometry becoming null on the horizon. It is this geometric property of the event horizon, rather than its causal properties, which is utilized in the Israel-Robinson theorems. Since it is only these theorems that do not explicitly involve causal analysis we will not review any of the other theorems leading to the conclusion of black-hole uniqueness, as they would not be applicable on the Euclidean section. Furthermore, since it is quite clear that Israel's theorem applys to the Euclidean section virtually line for line, we will not review it in detail, but merely point out enough of the structure of the proof to enable the reader to quickly verify the veracity of this statement. We will review Robinson's theorem in somewhat more detail.

Israel's theorem concerns a static spacetime manifold. In a local coordinate chart the metric can be written in a form such that the hypersurface orthogonal Killing vector $\xi = \partial/\partial t$ is manifest:

$$ds^2 = -V^2 dt^2 + g_{\alpha\beta}(x^1, x^2, x^3)dx^\alpha dx^\beta, \quad V = V(x^1, x^2, x^3), \tag{3.1}$$

where Greek indices run from 1 to 3, and capital Latin indices are reserved for the range 0 to 3. Let Σ denote the hypersurface t = constant and consider the class of static fields such that

(a) Σ is regular, empty, noncompact and "asymptotically flat," i.e.,

$$g_{\alpha\beta} = \delta_{\alpha\beta} + O(1/r), \quad \partial_\gamma g_{\alpha\beta} = O(r^{-2}),$$
$$V = (-g_{00})^{1/2} = 1 - m/r + \eta, \quad m = \text{const},$$
$$\eta = O(r^{-2}), \quad \partial_\alpha \eta = O(r^{-3}), \quad \partial_\alpha \partial_\beta \eta = O(r^{-4})$$

in the limit $r=(\delta_{\alpha\beta}x^\alpha x^\beta)^{1/2}\to\infty$ (in suitable coordinates);

(b) the surfaces $V=$ constant, $t=$ constant are regular, simply connected closed two surfaces;

(c) the four-dimensional invariant $R_{ABCD}R^{ABCD}$ is bounded on Σ;

(d) the intrinsic geometry (characterized by ${}^{(2)}R$) of the two-spaces $V=C$ approaches a limit as $C\to 0^+$, corresponding to a regular compact two-surface.

Israel's theorem states that the only static spacetime satisfying (a), (b), (c), and (d) is the Schwarzschild solution.

The proof utilizes four main equations, two of which are identities involving geometrical constructs of the $t=$ constant, $V=$ constant two-dimensional surfaces, while the third is the following expression for the four-dimensional invariant $R_{ABCD}R^{ABCD}$ in terms of V and the geometry of the two surfaces just mentioned:

$$\tfrac{1}{8}R_{ABCD}R^{ABCD}=(V\rho)^{-2}[K_{ab}K^{ab}+2\rho^{-2}\rho_{;a}\rho^{;a} + \rho^{-4}(\partial\rho/\partial V)^2]\,, \qquad (3.2)$$

where $\rho=(g^{\alpha\beta}\partial_\alpha V\partial_\beta V)^{-1/2}$, K_{ab} is the second fundamental form of the $t=$ constant, $V=$ constant two-surfaces, and lower-case Latin letters refer to coordinates in the two-surfaces. The fourth is the following equation obtained by projecting the field equations into the two-dimensional surfaces:

$$\frac{\partial}{\partial V}[(\sqrt{g})/\rho]=0\,, \qquad (3.3)$$

where g is the 2×2 determinant of the metric g_{ab} on the two-surfaces. Applying conditions (a)–(d) to equations (3.2) and (3.3) leads to the conclusion that

$$S_0/\rho_0=4\pi m\,, \qquad (3.4)$$

where S_0 is the area of the two-surface $t=$ constant, $V=C$, and the subscript zero indicates evaluation as $C\to 0^+$. Manipulation of the identities previously mentioned leads to

$$\rho_0 \geq 4m\,, \quad S_0 \geq \pi\rho_0^{\,2} \qquad (3.5)$$

with equality if and only if

$$\partial_a\rho=0=\rho(K_{ab}-\tfrac{1}{2}g_{ab}K) \qquad (3.6)$$

everywhere on Σ. (3.5) is inconsistent with (3.4) unless equality holds, and then spherical symmetry follows immediately from (3.6). Birkhoff's theorem applied to this spherically symmetric, static manifold immediately proves that the unique solution is the Schwarzschild solution. Condition (b) has been removed by Robinson.[18]

Robinson's theorem utilizes work by Carter,[19] who shows that the domain of communication (the "exterior") of a stationary (rotating), axisymmetric black hole can be covered globally (apart from trivial coordinate singularities) by a coordinate system t, ϕ, λ, μ in which the metric takes the form

$$ds^2=-V\,dt^2+W\,d\phi\,dt+X\,d\phi^2 + U[d\lambda^2/(\lambda^2-c^2)+d\mu^2/(1-\mu^2)]\,, \qquad (3.7)$$

where V,W,X,U are functions only of λ and μ. The ignorable coordinates have ranges $-\infty<t<\infty$, $0<\phi<2\pi$, and the two-dimensional subspace

$$ds^2=d\lambda^2/(\lambda^2-c^2)+d\mu^2/(1-\mu^2) \qquad (3.8)$$

is covered by a standard ellipsoidal coordinate system with $\lambda\in[c,\infty]$ and $\mu\in[-1,1]$. It is convenient to introduce the "twist potential" Y by requiring

$$(1-\mu^2)Y_{,\mu}=XW_{,\lambda}-WX_{,\lambda}\,, \quad -(\lambda^2-c^2)Y_{,\lambda}=XW_{,\mu}-WX_{,\mu}\,, \qquad (3.9)$$

where a comma denotes partial differentiation. Carter[19] has shown that if X, Y satisfy certain conditions on the axisymmetry axis and horizon then the four-dimensional geometry will be regular. These conditions are as follows: As $\mu\to\pm 1$, X and Y are well-behaved functions of λ and μ with

$$X=O(1-\mu^2)\,, \quad X^{-1}X_{,\mu}=-2\mu(1-\mu^2)^{-1}+O(1)\,, \quad Y_{,\lambda}=O((1-\mu^2)^2),\ \ Y_{,\mu}=O(1-\mu^2)\,, \qquad (3.10)$$

and as $\lambda\to c$, X and Y are well-behaved functions with

$$X=O(1),\ \ X^{-1}=O(1)\,, \quad Y_{,\lambda}=O(1),\ \ Y_{,\mu}=O(1)\,. \qquad (3.11)$$

Asymptotic flatness requires that as $\lambda^{-1}\to 0$, Y and $\lambda^{-2}X$ are well-behaved functions of λ^{-1} and μ with

$$\lambda^{-2}X=(1-\mu^2)[1+O(\lambda^{-1})]\,, \quad Y=2J\mu(3-\mu^2)+O(\lambda^{-1})\,, \qquad (3.12)$$

where J is the angular momentum measured in the asymptotically flat region.

For a metric of the form (3.7) Ernst[20] has shown that Einstein's equations reduce to the expressions, $E(X,Y)=F(X,Y)=0$, where

$$E(X,Y)=\nabla\cdot(\rho X^{-2}\nabla X)+\rho X^{-3}(|\nabla X|^2+|\nabla Y|^2)=0\,, \quad F(X,Y)=\nabla\cdot(\rho X^{-2}\nabla Y)=0\,. \qquad (3.13)$$

Here $\rho=(\lambda^2-c^2)^{1/2}(1-\mu^2)^{1/2}$ and ∇ denotes the covariant derivative with respect to the metric (3.8). V and U are determined from X, Y by quadrature.

The key part of Robinson's proof is the identity[5]

$$\begin{aligned}
&(X_1X_2)^{-1}(Y_2-Y_1)[X_1^{2}F(X_1,Y_1)-X_2^{2}F(X_2,Y_2)]+\tfrac{1}{2}X_2^{-1}[(Y_2-Y_1)^2+X_2^{2}-X_1^{2}]E(X_1,Y_1)\\
&\quad+\tfrac{1}{2}X_1^{-1}[(Y_2-Y_1)^2+X_1^{2}-X_2^{2}]E(X_2,Y_2)+\tfrac{1}{2}\nabla\cdot\left[\rho\nabla\left(\frac{(X_2-X_1)^2+(Y_2-Y_1)^2}{X_2X_1}\right)\right]\\
&\qquad=\rho(2X_2X_1)^{-1}\left|X_1^{-1}(Y_2-Y_1)\nabla Y_1-\nabla X_1+X_2^{-1}X_1\nabla X_2\right|^2\\
&\qquad\quad+\rho(2X_2X_1)^{-1}\left|X_2^{-1}(Y_2-Y_1)\nabla Y_2+\nabla X_2-X_1^{-1}X_2\nabla X_1\right|^2\\
&\qquad\quad+\rho(4X_1X_2)^{-1}\left|(X_2+X_1)(X_2^{-1}\nabla Y_2-X_1^{-1}\nabla Y_1)-(Y_2-Y_1)(X_2^{-1}\nabla X_2+X_1^{-1}\nabla X_1)\right|^2\\
&\qquad\quad+\rho(4X_1X_2)^{-1}\left|(X_2-X_1)(X_1^{-1}\nabla Y_1+X_2^{-1}\nabla Y_2)-(Y_2-Y_1)(X_1^{-1}\nabla X_1+X_2^{-1}\nabla X_2\right|^2. \qquad (3.14)
\end{aligned}$$

For fixed parameters c and J there is an associated Kerr solution (1.4) with $c^2=m^2-a^2$ and $J=am$. Suppose that (X_1, Y_1) corresponds to this Kerr solution and (X_2, Y_2) corresponds to a hypothetical second black-hole solution satisfying the boundary conditions. Integration of (3.14) over the two-dimensional manifold (3.8) leads to a boundary integral on the left-hand side of the identity which vanishes by the boundary conditions (3.10), (3.11), (3.12). The integrand of the right-hand side is a sum of four positive-definite terms each of which must now necessarily vanish. Simple manipulation of the resulting first-order partial differential equations soon leads to the conclusion that $Y_2=Y_1$ and $X_2=X_1$, i.e., that the Kerr solution (1.4) is the unique stationary, axisymmetric solution satisfying the boundary conditions.

IV. EUCLIDEAN BLACK-HOLE UNIQUENESS THEOREMS

The first part of the classical black-hole uniqueness theorems described in Sec. III, that which assumes a locally timelike Killing vector and utilizes spacetime casual structure, is clearly inapplicable to the Euclidean section for two reasons. First, there is no reason for assuming the existence of a Killing vector as one wishes to include in the functional integral all positive-definite metrics satisfying the first three of the four conditions listed previously, i.e.,

(i) asymptotic flatness,
(ii) an $S^2\times S^1$ boundary at infinity,
(iii) an identification of the metric

$$(t,r,\theta,\phi)=(t+\beta,r,\theta,\phi),$$

or

$$(t,r,\theta,\phi)=(t+\beta,r,\theta,\phi+\Omega\beta),$$

depending on the physical situation chosen, and hence the extremal metric need not *ab initio* have a Killing vector.

Secondly, there is no casual structure on the Euclidean section. However, one might hope that the second part of the classical uniqueness theorems, the Israel[4] and Robinson[5] theorems, would allow one to draw a more restricted conclusion concerning the extremal metric in the class of metrics satisfying conditions (i), (ii), and (iii) and furthermore possessing either a hypersurface orthogonal Killing vector (Euclidean analog of staticity); or a nonhypersurface orthogonal Killing vector (Euclidean analog of stationarity) that commutes with a second Killing vector generating the action of SO(2) (Euclidean analog of axisymmetry).

A positive-definite metric possessing a hypersurface orthogonal Killing vector ∂_t can be obtained from (3.1) by $t\to it$,

$$\begin{aligned}
&ds^2=V^2dt^2+g_{\alpha\beta}(x^1,x^2,x^3)\,dx^\alpha dx^\beta,\\
&V=V(x^1,x^2,x^3).
\end{aligned}\qquad (4.1)$$

It is clear that Israel's theorem can be transcribed to the Euclidean section essentially line for line because, as described in Sec. III, much of the analysis involves the two-geometry $V=$ constant, $t=$ constant. The part explicity involving the four-geometry and hence the metric signature, for example Eq. (3.2), remains unchanged independent of whether the signature is +2 or +4. The surface $V=0^+$ is the fixed-point locus of the Killing vector ∂_t or a "bolt" in the parlance of Ref. 3, and therefore the manifold has a Euler characteristic, $\chi=2$, by the fixed-point theorems. The Euclidean version of Israel's theorem therefore proves that the unique, nonsingular, Ricci flat, positive-definite metric satisfying the conditions of

(i) asymptotic flatness,
(ii) an $S^2\times S^1$ boundary at infinity,
(iii) an identification of the metric

$$(t,r,\theta,\phi)=(t+\beta,r,\theta,\phi), \qquad (4.2)$$

(iv) two-dimensional fixed-point locus of hypersurface orthogonal Killing vector (staticity + nontrivial topology),

is the Euclidean Schwarzschild solution (1.1) where $\beta=8\pi m$.

It is natural to expect a similar Euclidean analog of Robinson's theorem, however, we will now show that there are grave difficulties with the analogy. A positive-definite, axisymmetric, "stationary" (nonhypersurface orthogonal Killing vector) metric is obtained from (3.7) by $t \to it$ and $W \to -iW$. This procedure was used in going from the Lorentzian Kerr metric (1.4) to the Euclidean Kerr metric (1.3), i.e., $t \to it$ and $a \to -i\alpha$. It is important to realize that one should not merely put $V \to -V$ in (3.7). Equation (3.9) implies that $Y \to -iY$ and, similarly, in (3.13) and (3.14). Therefore, the Euclidean Robinson identity (3.14) has a sum of two positive-definite and two negative-definite terms on the right-hand side. Hence when one integrates the Euclidean Robinson identity over the manifold it is no longer possible to conclude that each term on the right-hand side must separately equal zero. Therefore, one cannot conclude from this analysis that the Euclidean Kerr solution is unique.

One can introduce a new set of variables for which there exists a Robinson identity with the right-hand side being positive definite.[21] We start from the Lorentzian field equations in terms of the metric quantities W and X, as given, e.g., by Carter,[19]

$$\nabla\left(\frac{X\nabla W - W\nabla X}{\rho}\right) = 0\,, \qquad \nabla\left(\frac{\rho\nabla X}{X}\right) + \frac{|X\nabla W - W\nabla X|^2}{\rho X^2} = 0\,. \tag{4.3}$$

The Euclidean equations ($W \to -iW$) are therefore

$$\nabla\left(\frac{X\nabla W - W\nabla X}{\rho}\right) = 0\,, \qquad \nabla\left(\frac{\rho\nabla X}{X}\right) - \frac{|X\nabla W - W\nabla X|^2}{\rho X^2} = 0\,. \tag{4.4}$$

Introduction of the quantities $\tilde{X} = \rho/X$ and $\tilde{Y} = W/X$ leads to

$$\nabla\left(\frac{\rho\nabla\tilde{X}}{\tilde{X}}\right) + \frac{\rho|\nabla\tilde{Y}|^2}{(\tilde{X})^2} = 0\,, \qquad \nabla\left(\frac{\rho\nabla\tilde{Y}}{\tilde{X}^2}\right) = 0\,. \tag{4.5}$$

These equations for the Euclidean variables $\tilde{X}$, $\tilde{Y}$ are identical to Eqs. (3.13) for the Lorentzian variables X and Y. Therefore, the Robinson identity (3.14) exists on the Euclidean section in terms of the Euclidean variables $\tilde{X}$, $\tilde{Y}$. Integration of the tilded identity over the manifold leads to a sum of four positive terms on the right-hand side as desired. However, the tilded divergence on the left-hand side does not integrate up to a boundary term that vanishes, in fact, it diverges on the "horizon", i.e., the two-dimensional fixed-point locus (bolt) of the Killing vector ∂_t. Once again it is impossible to prove the uniqueness of the Euclidean Kerr black hole using a Euclidean Robinson theorem. In Sec. V we try (and fail) to disprove uniqueness by searching for possible counterexamples.

V. ATTEMPTS AT COUNTEREXAMPLES

The failure of the Euclidean Robinson uniqueness theorem, discussed in Sec. IV, suggests that perhaps another Euclidean solution exists satisfying the boundary conditions. One manner in which stationary, axisymmetric Euclidean solutions may be found is by analytically continuing stationary, axisymmetric Lorentzian solutions to the Euclidean section. Clearly all Lorentzian solutions, apart from Kerr, will be pathological in some sense since the Lorentzian Robinson uniqueness theorem works. The idea would be that the pathologies would not be present on the Euclidean section. Some Euclidean solutions cannot be obtained by analytic continuation of Lorentzian ones. A sufficient, but not necessary, condition for this is that the curvature be (anti) self-dual. In this section we explore examples from both categories.

Apart from the Lorentzian Kerr solution, the only other stationary, axisymmetric, asymptotically flat solution for which the metric is explicitly known is the Lorentzian Tomimatsu-Sato solution.[22,23] There is actually a family of such solutions, characterized by a parameter δ taking integer values with $\delta = 1$ being the Kerr solution. The complexity of the metric grows rapidly with δ and therefore we will only display the metric for $\delta = 2$. The metric can be written locally as

$$ds^2 = m^2[e^{2\mu_2}dx^2 + e^{2\mu_3}dy^2 + e^{2\psi}(d\phi - \Omega dt)^2] - e^{2\nu}dt^2\,, \tag{5.1}$$

where

$$\Omega = -\frac{8qC(x,y)}{mD(x,y)}\,, \quad e^{2\nu} = p^2\frac{(x^2-1)B(x,y)}{D(x,y)}\,, \quad e^{2\psi} = \frac{(1-y^2)D(x,y)}{4B(x,y)}\,, \tag{5.2}$$

$$e^{2\mu_2} = B(x,y)/[4p^2(x^2-1)(x^2-y^2)^3]\,, \tag{5.3}$$

$$e^{2\mu_3} = \frac{x^2-1}{1-y^2}\,e^{2\mu_2}\,, \tag{5.4}$$

$$B(x,y)=[p^2(x^2+1)(x^2-1)-q^2(y^2+1)(1-y^2)+2px(x^2-1)]^2 + 4q^2y^2[px(x^2-1)+(px+1)(1-y^2)]^2, \tag{5.5}$$

$$C(x,y)=-p^3x(x^2-1)[2(x^2+1)(x^2-1)+(x^2+3)(1-y^2)] - p^2(x^2-1)[4x^2(x^2-1)+(3x^2+1)(1-y^2)]+q^2(px+1)(1-y^2)^3, \tag{5.6}$$

$$\begin{aligned}D(x,y)=&\,p^6(x^2-1)(x^8+28x^6+70x^4+28x^2+1)-16q^6(1-y^2)^3\\&+p^4q^2\{(x^2-1)[32x^2(x^4+4x^2+1)-4(1-y^2)(x^2-1)^3\\&\qquad+(-6x^4+12x^3+10)(1-y^2)^3]-4(1-y^2)^3(x^4+6x^2+1)\}\\&+p^2q^4\{(x^2-1)[64x^4+(1-y^2)^2(y^4+14y^2+1)]-16(1-y^2)^3(x^2+2)\}\\&+8p^5x(x^4-1)(x^4+6x^2+1)-32pq^4x(1-y^2)^3\\&+8p^3q^2x\{(x^2-1)[8x^2(x^2+1)+(1-y^2)^2(2y^2-x^2+1)]-4(1-y^2)^3\}.\end{aligned} \tag{5.7}$$

The coordinates x, y are ellipsoidal polar-type coordinates with $x\in[1,\infty)$ a radial variable and $y\in[-1,1]$ a polar variable. The constants p, q are constrained by $p^2+q^2=1$. Analysis of the metric in the asymptotically flat region shows that m is the mass and m^2q is the angular momentum. The surface $x=1$ is a Killing horizon, and also an event horizon for odd δ. However, Gibbons[24] has shown that the $x=1$ surface is not totally null for even δ and hence not an event horizon for even δ. There are ring-curvature singularities in the equatorial plane, $y=0$, defined by $B(x,y=0)=0$, which for all $\delta>1$ have at least one ring inside the $x=1$ surface and at least one ring outside. The x, y chart breaks down at the poles of the Killing horizon, i.e., $x=1$, $y=\pm 1$. Tomimatsu and Sato[23] computed one component of the Riemann tensor,

$$R_{(t)(\phi)(t)(\phi)}=\frac{\delta^4q^2}{m^2}(1-y^2)(x^2-y^2)^{\delta^2-1}\frac{D^2}{B^3}\left(\frac{\partial}{\partial x}\frac{C}{D}\right)^2, \tag{5.8}$$

and claimed that $x=1$, $y=\pm1$ are points of curvature singularity since (5.8) diverges as $y^2\to 1$ in the $x^2=1$ surface. However, Economou and Ernst[25] computed the complete Weyl tensor invariants and showed that curvature singularities occur whenever the following complex scalar is zero:

$$Z=[p^2(x^4-1)+q^2(y^4-1)+2px(x^2-1) - 2ipqxy(x^2-y^2) - 2iqy(1-y^2)]/(x^2-y^2). \tag{5.9}$$

Z vanishes nowhere off the equatorial plane $y=0$, while on the plane it vanishes at the two ring singularities mentioned previously. The resolution is that when the signature is Lorentzian the curvature invariant contains terms of positive and negative sign, and thus there must be another divergent term in the invariant, identical to the square of (5.8), which occurs with the opposite sign. The poles are, however, still somewhat strange because the limit of Z at $x=1$, $y=\pm1$ depends upon the manner in which they are approached.

The Euclidean section of the Tomimatsu-Sato solution may be defined by $t\to it$, $q\to -iq$ in analogy with the continuation used to find a Euclidean section of Kerr (1.3). The naked ring singularity in the equatorial plane $y=0$ at $x>1$ is no longer present since $B(x,y=0)$, or equivalently $Z(x,y=0)$, no longer vanishes for $x>1$. This is quite encouraging. However, it may be seen that the poles at $x=1$, $y=\pm1$ become curvature singularities on the Euclidean section by noticing that when the metric signature is positive definite the unboundedness of the analytically continued Riemann tensor (5.8) at the poles implies that the complete invariant diverges. This may be verified from the analytically continued expression (5.9) and also the directional nature of the singularity verified. Therefore, although the $\delta=2$ Euclidean Tomimatsu-Sato solution no longer suffers from the naked ring singularities, it acquires curvature singularities at the poles of the Killing horizon $x=1$, $y=\pm1$, and hence cannot be a counterexample to the conjectured uniqueness of the Euclidean Kerr solution. Expression (5.8) (with $q\to iq$) shows that the poles are curvature singularities for the Euclidean Tomimatsu-Sato solution for all $\delta\geq 2$ and therefore these solutions are not counterexamples either.

A class of Euclidean solutions which cannot be obtained from Lorentzian solutions are those with (anti) self-dual curvature. A reasonable physical requirement to impose on any Euclidean solution is that the manifold admit spin structure. Gibbons and Pope[7] have constructed an argument proving that self-dual, asymptotically Euclidean solutions (i.e., the curvature falls off to zero at infinity in the four-dimensional sense) with spin structure cannot exist. Their argument applies equally well to the asymptotically flat situation (curvature falls off to zero in the three-dimensional sense) under consideration here. The argument proceeds as follows. The index of the Dirac operator, $\gamma^a\nabla_a$ for a manifold with boundary is given by

$$\text{index}[\gamma^a\nabla_a] = \frac{1}{192\pi^2}\int R_a{}^b \wedge R^a{}_b d(\text{vol}) - \frac{1}{192\pi^2}\int_{\partial\mathfrak{M}} \theta^a{}_b \wedge R^b{}_a d(\text{surf}) - [\eta_D(0)] \,, \tag{5.10}$$

where R_a^b is the curvature two-form in an orthonormal basis, $\theta^a{}_b$ is the second fundamental form of the boundary, and $\eta_D(0)$ is the expression

$$\eta_{D(s)}\big|_{s=0} = \sum_n \text{sgn}(\lambda_n)\,|\lambda_n|^{-s}\Big|_{s=0} \,, \tag{5.11}$$

where the eigenvalues λ_n are eigenvalues of the Dirac operator restricted to the boundary. $\eta_D(0)$ measures the "handedness" of the manifold and vanishes if the boundary of the manifold admits an orientation reversing isometry as does the boundary $S^2\times S^1$ under consideration here, and also the S^3 boundary considered by Gibbons and Pope. The second term in the index (5.10) vanishes by virtue of asymptotic flatness while the first vanishes by the condition of (anti) self-duality. Hence an asymptotically flat, self-dual solution, if it exists, should admit at least one normalizable spinor. However, Lichnerowicz's theorem[8] proves that spinors on manifolds with $R \geq 0$ are covariantly constant and therefore not normalizable if the manifold is noncompact. Hence one must conclude that asymptotically flat, self-dual solutions do not exist.

VI. A EUCLIDEAN NO-HAIR THEOREM

The phrase "no-hair theorem" usually refers to the theorem of Carter[19]: Stationary, axisymmetric spacetimes satisfying the usual black-hole boundary conditions fall into discrete families depending on at most two parameters, the mass m and the angular momentum $J=ma$; and that continuous variations of these solutions are uniquely determined by continuous variations of m and J. Hence the only regular perturbations of the Kerr solution are the "trivial" perturbations in m and J. A corollary is that the Kerr solution is the unique family with a regular zero angular momentum ($J=0$) limit. The method of proof involves a linearized version of the Robinson identity (3.14), where "linearized" means X_1, Y_1 differs from X_2, Y_2 by quantities of the first order. Clearly this theorem will have the same difficulties on the Euclidean section as the Robinson uniqueness theorem (Sec. IV). Teukolsky[10,11] and Wald[12] have employed a different method to show that no bifurcations occur off the Kerr sequence. The idea behind their method is to explicitly solve the Teukolsky[10] master equation for perturbations off the Kerr background solution and thereby show that the only stationary, regular perturbations are the trivial perturbations $m\to m+\delta m$, $J\to J+\delta J$. We shall attempt this method on the Euclidean section.

In the Boyer-Lindquist coordinate system, (1.4), Kinnersley's null tetrad[26] has the following $[t,r,\theta,\phi]$ components:

$$\begin{aligned} l^a &= [(r^2+a^2)/\Delta, 1, 0, a/\Delta] \,, \\ n^a &= [r^2+a^2, -\Delta, 0, a]/2\rho^2 \,, \\ m^a &= [ia\sin\theta, 0, 1, i/\sin\theta]/2^{1/2}(r+ia\cos\theta). \end{aligned} \tag{6.1}$$

The scalars, $\psi_0 = -C_{abcd}l^am^bl^cm^d$, $\psi_4 = -C_{abcd}n^a\bar{m}^bn^c\bar{m}^d$, where C_{abcd} is a perturbation in the Weyl tensor, satisfy the Teukolsky equation [10],

$$[(r^2+a^2)^2/\Delta - a^2\sin^2\theta]\frac{\partial^2\psi}{\partial t^2} + \frac{4mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + [a^2/\Delta - \sin^{-2}\theta]\frac{\partial^2\psi}{\partial\phi^2} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left(a(r-m)/\Delta + \frac{i\cos\theta}{\sin^2\theta}\right)\frac{\partial\psi}{\partial\phi} - 2s[m(r^2-a^2)/\Delta - r - ia\cos\theta]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 0 \,, \tag{6.2}$$

where s takes the values 2 and -2 in association with ψ representing ψ_0 and $\psi_4/(r-ia\cos\theta)^4$, respectively. For stationary perturbations $\partial\psi/\partial t$, $\partial^2\psi/\partial t^2$ are zero, and Eq. (6.2) can be separated by writing $\psi = e^{i\mu\phi}S(\theta)R(r)$, where $S(\theta)$ is the spin weighted spherical harmonic[27] ${}_sS^{\mu}{}_l(\theta)$ and R satisfies

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \{[a^2\mu^2 + 2ia(r-m)s]\Delta^{-1} - (l-s)(l+s+1)\}R = 0 \,. \tag{6.3}$$

There are regular singular points at $\Delta=0$ and $r=\infty$ and hence the solution in the neighborhood of these points can be found in terms of power series. Near the singular points the dominant behavior of $R(r)$ is

$$\begin{aligned} R &\underset{r\to\infty}{\propto} r^{-1}, r^{-2s+1} \,, \\ R &\underset{r\to r_+}{\propto} (r-r_+)^{-ia\mu/2}, (r-r_+)^{-s+ia\mu/2} \,, \end{aligned} \tag{6.4}$$

where $r_+ = m + (m^2 - a^2)$.

If $\mu=0$, i.e., if the perturbation is axisymmetric, then the nature of the solution at the singular points is the same regardless of whether one chooses the Lorentzian Kerr solution (1.4), or the Euclidean Kerr solution (1.3). To proceed further it is necessary to determine the regular solution at r_+. Teukolsky[11] notes that the Boyer-Lindquist chart is singular at r_+ and therefore transforms to nonsingular Kerr coordinates $v, \tilde{\phi}$,

$$dv = dt + dr(r^2+a^2)/\Delta\ ,$$
$$d\tilde{\phi} = d\phi + a dr/\Delta \qquad (6.5)$$

to determine that the solution with behavior $R \sim (r-r_+)^{-s}$ is the regular solution in a chart regular at r_+. He goes on to show that regular behavior at $r=r_+$ is incompatible with regular behavior at $r=\infty$ and hence $\psi_4\equiv 0$. The Kerr metric (1.4) in Kerr coordinates (6.5) does not admit an obvious Euclidean section, however, the geometry at $r=r_+$ remains perfectly regular regardless of the metric signature (in contrast to the poles in the Tomimatsu-Sato solution c.f. Sec. V). We therefore choose the nondiverging behavior of R at $r=r_+$ on the Euclidean section and similiarly conclude that $\psi_4\equiv 0$.

Wald[12] has shown that the general perturbation of the Kerr solution with $\psi_4\equiv 0$ is completely specified by four perturbation parameters: δm, δa, δl, and $\delta\rho$. The first two perturbations correspond to the "trivial" perturbations in the mass m, and angular momentum parameter a. The last two perturbations are perturbations towards the Kerr-NUT (Newman-Unti-Tambourino) solution and the "rotating C" metric of Kinnersley,[28] i.e., those perturbations obtained by linearizing the exact Kerr-NUT and "rotating C" metrics about the Kerr metric. Both the latter perturbations induce unacceptable angular behavior in the metric, which is inherited from unacceptable angular behavior of the exact solutions indicating that they are not asymtotically flat in the usual sense. That this bad angular behavior is also present on the Euclidean section is a consequence of Lapedes and Perry's[29] investigation of the Euclidean section of the most general type-D solution. Hence the only regular, stationary, axisymmetric perturbations of the Kerr metric (1.3) are the trivial perturbations in m and α. There is no bifurcation off the Euclidean Kerr sequence.

VII. UNIQUENESS CONJECTURES

The Euclidean Schwarzschild and Euclidean Kerr solutions (1.1), (1.3) are nonsingular, non-Kähler, four-dimensional, positive-definite, Ricci flat metrics. In Sec. II the importance of the uniqueness of these solutions was outlined and a rough statement was formulated of the conditions under which the solutions are suspected to be unique. In this section we make these conjectures precise.

Conjecture I

Let the pair $\{\mathfrak{M}, g_{ab}\}$ represent a noncompact four-dimensional manifold with an associated positive-definite metric. The Euclidean Schwarzschild solution $\{R^2\times S^2, g_{ab}\}$ with g_{ab} given by (1.1) is the unique solution that satisfies the following conditions.

(i) Ricci flat.

(ii) Geodesically complete.

(iii) Asymptotically flat, i.e., the induced metric $g_{\alpha\beta}$ on a regular noncompact embedded three-dimensional hypersurface satisfies

$$\lim_{r\to\infty} g_{\alpha\beta} = \delta_{\alpha\beta} + O(r^{-1}),\ \ \lim_{r\to\infty}\partial_\gamma g_{\alpha\beta} = O(r^{-2})\ ,$$

where $r^2=\delta_{\alpha\beta}X^\alpha X^\beta$ in suitable coordinates.

(iv) An $S^2\times S^1$ boundary at infinity such that in a suitable chart

$$ds^2 = d\tau^2 + dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2) + O(1/r)\ ,$$

where τ is identified with period $8\pi m$. r is a coordinate along a ray and θ, ϕ are the usual polar and azimuthal angles on S^2.

(v) Nontrivial topology.

Condition (v) excludes suitably identified flat space from being a counterexample.

Note that if one further requires that the metric admit a hypersurface orthogonal Killing vector then the Euclidean version of Israel's theorem (Sec. III) proves this more restricted conjecture.

Conjecture II

Let the pair $\{\mathfrak{M}, g_{ab}\}$ represent a noncompact, four-dimensional manifold with an associated positive definite metric as before. The Euclidean Kerr solution (1.3) is the unique solution satisfying conditions (i), (ii), (iii), and (v) (above) which has an $S^2\times S^1$ boundary at infinity such that in a suitable chart

$$ds^2 = d\tau^2 + dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2) + O(1/r)\ ,$$

where the pair $\{\tau, \phi\}$ is identified with $\{\tau+\beta, \phi+\beta\Omega\}$, r is a coordinate along a ray, θ and ϕ are the usual polar and azimuthal angles on S^2, and β and Ω are constants defined in Sec. I.

Note that if one further requires that the metric admit two commuting Killing vectors, one of which is nonhypersurface orthogonal, and the other is a generator of SO(2) (the Euclidean analog of stationarity and axisymmetry) then the Ernst, Carter, Robinson formalism of Sec. III does *not* prove this more restricted theorem. The formalism does provide a restatement of the more restrictive problem as follows.

Conjecture IIa

Subject to the following conditions, the unique solution X, Y, to the coupled set

$$\nabla\cdot(\rho X^{-2}\nabla X)+\rho X^{-3}(|\nabla X|^2-|\nabla Y|^2)=0,$$
$$\nabla\cdot(\rho X^{-2}\nabla Y)=0$$

in the background metric

$$ds^2=d\lambda^2/(\lambda^2-c^2)+d\mu^2/(1-\mu^2),$$

where

$$\rho^2=(\lambda^2-c^2)(1-\mu^2)$$

is

$$X=(1-\mu^2)\{(\lambda+m)^2-\alpha^2-\alpha^2(1-\mu^2)2mr/(r^2-\alpha^2\mu^2)\},$$
$$Y=2m\alpha\mu(3-\mu^2)+2\alpha^3\mu m(1-\mu^2)^3/[(\lambda+m)^2-\alpha^2\mu^2].$$

The conditions are the following:

(i) In the limit $\mu\to\pm1$, X and Y are well-behaved functions of λ and μ with

$$X=O(1-\mu^2),$$
$$X^{-1}X_{,\mu}=-2\mu(1-\mu^2)^{-1}+O(1),$$
$$Y_{,\lambda}=O((1-\mu^2)^2),\quad Y_{,\mu}=O(1-\mu^2).$$

(ii) In the limit $\lambda\to c$, X and Y are well behaved functions with

$$X=O(1),\quad X^{-1}=O(1),$$
$$Y_{,\lambda}=O(1),\quad Y_{,\mu}=O(1).$$

(iii) In the limit $\lambda^{-1}\to0$, Y and $\lambda^{-2}X$ are well-behaved functions of λ^{-1} and μ with

$$\lambda^{-2}X=(1-\mu^2)[1+O(\lambda^{-1})],$$
$$Y=2m\alpha\mu(3-\mu^2)+O(\lambda^{-1}),$$

m and α are constants.

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