Hadronic transitions between heavy-quark states in quantum chromodynamics

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A general theory is formulated in quantum chromodynamics for hadronic transitions between heavy-quark states. Many predictions follow simply from the tensor properties of the multipoles that dominate in a transition. When a transition involves only soft-pion emission, partial conservation of the axial-vector current provides information on the mass and angular distributions of the pions. Applications to hadronic transitions within the Υ family are presented.

I. INTRODUCTION

The new particles, the ψ's and the Υ's, have proved to be ideal systems for studying the interactions between heavy quarks. As the quark mass increases, so does the number of heavy-quark-antiquark ($Q\bar{Q}$) bound states below the Okubo Zweig-Iizuka threshold. Consequently, many hadronic transitions between them will be accessible experimentally. The important transitions in the Υ family were anticipated by Eichten and Gottfried[1]; they are reproduced in Fig. 1. Clearly, a theoretical understanding of these processes will offer additional insight into the nature of heavy-quark dynamics.

In quantum chromodynamics (QCD) a transition between two heavy-quark states proceeds in two steps[2]: the emission of gluons by the heavy quarks and the subsequent conversion of gluons into light hadrons. This is depicted in Fig. 2. The energy available to the light hadrons is small, and the emitted gluons are soft. Therefore, perturbative QCD is not applicable to these soft processes. Nevertheless, Gottfried[3,4] has pointed out that the heavy-quark system moves slowly and has dimensions small compared to those of the emitted light-quark system. Therefore, the heavy quarks can be treated nonrelativistically, and a multipole expansion of the changing gauge field converges rapidly for sufficiently large heavy-quark mass. Gottfried showed that a multipole expansion of the color gauge field leads to selection rules and rate estimates for various hadronic transitions between $Q\bar{Q}$ states.

An important issue in the multipole expansion of QCD is gauge invariance. This question as well as others has been considered by several authors.[5-8] They have made a detailed study in perturbative QCD of very heavy quarks that interact by a Coulomb potential. In particular, Peskin[7] concludes that the transition amplitude for two-gluon emission is indeed described by a gauge-invariant second-order color-electric dipole transition.

Since the potentials for the ψ and Υ families

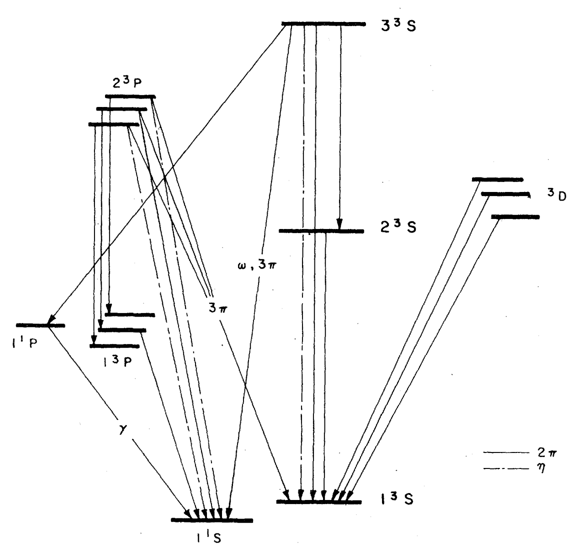

FIG. 1. Important hadronic transitions in the Υ family. Not every transition is shown. This is taken from Ref. 1.

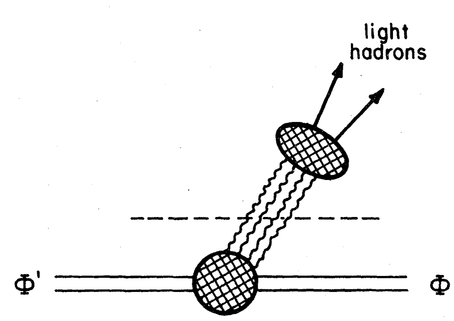

FIG. 2. A hadronic transition as a two-step process; emission of gluons from heavy quarks (lower half) and the conversion of gluons into light hadrons (upper half).

are still far from being Coulombic,[9] we have reexamined the problem of hadronic transitions in QCD. Our goal is to go beyond the usual perturbation theory in the coupling constant. Instead, we rely on the small size of the heavy-quark system. The multipole expansion of QCD is derived in Sec. II. It begins with a brief review of Peskin's[7] perturbative analysis. We find that his result can be represented by a gauge-invariant effective action for the heavy-quark system; it also suggests a natural variable for a constituent quark [see Eq. (2.10)]. Generalizing this observation, we use a gauge-invariant effective action to summarize the complicated interactions between heavy quarks and the couplings of heavy quarks to gluons.[10] In terms of the constituent-quark field, a gauge-invariant multipole expansion of QCD follows.

The *relative* importance of the multipoles is estimated by dimensional analysis. When a multipole dominates a transition, application of the Wigner-Eckart theorem gives many predictions. For example, there are six relations for the nine transitions $2\,^3P_{J'} \to 1\,^3P_J + 2\pi$ in the Υ family. Details of these predictions are presented in Sec. III.

In Sec. IV we combine the soft-pion technique with the QCD multipole expansion to predict the mass and angular distributions for two-pion emissions. In some cases these distributions are uniquely determined, only the overall normalization is unknown. For example, the 2π system in a transition $^3D_1 \to {}^3S_1 + 2\pi$ has unique mass and angular distributions, distinct from those in a 2π transition between two spin-triplet S states. If the 2π emission is observable, this property can be used to decide whether $\psi(3.772)$ is indeed a D state. It can also help to distinguish a vibrational state[11] expected in the Υ spectroscopy from a predominantly D state.

In Sec. V we discuss our results and make some concluding remarks.

Some technical details are relegated to two appendices. In Appendix A we show that high-order leading corrections to the Coulomb propagator in an external field form an exponential series. In Appendix B we give an elementary derivation of the multipole expansion of QED; the method is generalized to obtain a similar expansion of QCD.

II. MULTIPOLE EXPANSION OF QCD

In this section we will derive the multipole expansion of QCD with the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\,\mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[\gamma^\mu(i\partial_\mu - gA_\mu) - m]\psi + \cdots . \tag{2.1}$$

Here ψ is the field operator for a heavy quark; and we have used the matrix notation for the gauge fields,

$$\begin{aligned} A_\mu &= \sum_a \tfrac{1}{2}\lambda_a A_{\mu a}\,, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]\,. \end{aligned} \tag{2.2}$$

In (2.1) the contribution from light quarks is understood to be included, but it is not explicitly displayed. The Lagrangian (2.1) is invariant under the local gauge transformation

$$\begin{aligned} A_\mu(x) &\to V(x)A_\mu(x)V^{-1}(x) - \frac{i}{g}V(x)\partial_\mu V^{-1}(x)\,, \\ \psi(x) &\to V(x)\psi(x)\,. \end{aligned} \tag{2.3}$$

At present it is still not possible to derive a realistic interquark potential from QCD. We therefore must find an approach which will not rely on the details of this potential. To motivate the approach that we shall adopt, we will begin with a brief review of the perturbative analysis by Peskin.[7] We will then proceed to the general case.

Consider the amplitude for two-gluon emission by a bound state of a heavy quark and its antiquark. In perturbation theory, this amplitude is given by the diagrams[12] in Fig. 3. Peskin[7] has calculated the contribution of each diagram, and the sum is shown to be a gauge-invariant second-

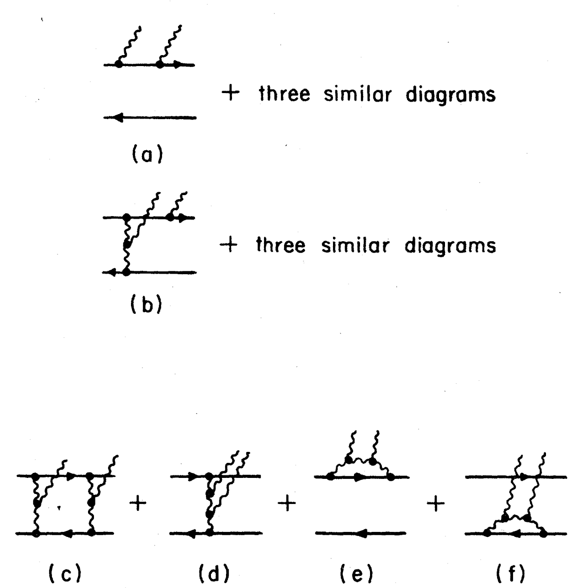

FIG. 3. Lowest-order Feynman diagrams contributing to two-gluon emission. Coulomb interactions between the heavy quark and its antiquark are present but not shown.

order color-electric dipole transition. Peskin's analysis[7] is very difficult to generalize since it requires the calculation of all diagrams to a given order in the perturbation. To reveal its general features we will analyze the problem from a different viewpoint.

First of all, the diagrams in Fig. 3 may be obtained by iterations of the two-particle irreducible diagrams shown in Fig. 4. The non-Abelian nature of QCD is reflected in Figs. 4(d)—4(g). It is convenient to combine these diagrams with the Coulomb-potential term of Fig. 4(a). The results obtained by Peskin[7] give the leading contributions

Sum of Figs. 4(a) and 4(d)—4(g)

$$= \sum_{a,b} \frac{g^2}{4\pi}\frac{1}{r}\frac{1}{2}\lambda_{1a}\left[1 - g\vec{r}\cdot\sum_c F_c\vec{A}_c(0) + \tfrac{1}{2}\left(g\vec{r}\cdot\sum_c F_c\vec{A}_c(0)\right)^2\right]_{ab}\tfrac{1}{2}\lambda_{2b}, \tag{2.4}$$

where

$$\vec{r} = \vec{x}_1 - \vec{x}_2, \tag{2.5}$$

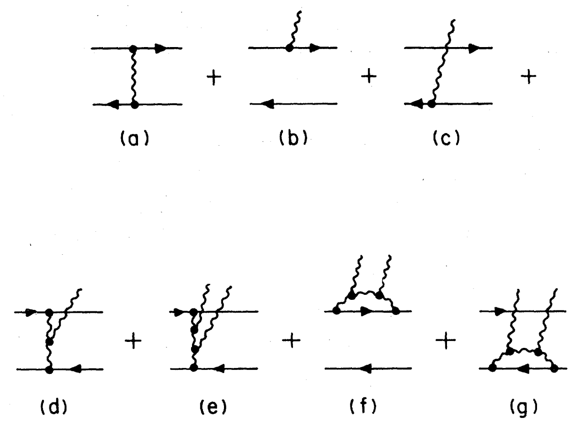

FIG. 4. Two-particle irreducible diagrams from which diagrams in Fig. 3 can be generated by iterations.

$$(F_c)_{ab} = -f_{abc}. \tag{2.6}$$

Obviously, Eq. (2.4) exhibits the first three terms of an exponential series. The whole series is readily derived in the Coulomb gauge; this is shown in Appendix A. Therefore, when higher-order terms are included, we have

$$\frac{g^2}{4\pi}\frac{1}{r}\sum_a \tfrac{1}{2}\lambda_{1a}\tfrac{1}{2}\lambda_{2a} - \frac{g^2}{4\pi}\frac{1}{r}\sum_{a,b}\tfrac{1}{2}\lambda_{1a}\left[\exp\left(-g\vec{r}\sum_c F_c\vec{A}_c(0)\right)\right]_{ab}\tfrac{1}{2}\lambda_{2b}. \tag{2.7}$$

This is an interesting result: Contributions from nonlinear couplings of the gauge fields modify the Coulomb potential by an exponential factor. The significance of this result will become clear shortly.

We represent the diagrams of Fig. 4 by an effective Lagrangian for the heavy quarks:

$$L_Q = \int d^3x\,\bar{\psi}[\gamma^\mu(i\partial_\mu - gA_\mu) - m]\psi - \tfrac{1}{2}\int d^3x\,d^3y\sum_{a,b}\bar{\psi}(\vec{x},t)\gamma^0\tfrac{1}{2}\lambda_a\psi(\vec{x},t)\,\frac{g^2}{4\pi}\left[\frac{P\exp[-g\int_{\vec{y}}^{\vec{x}} d\vec{r}\cdot\sum_c \vec{A}_c(\vec{r},t)]}{|\vec{x}-\vec{y}|}\right]_{ab}\bar{\psi}(\vec{y},t)\gamma^0\tfrac{1}{2}\lambda_b\psi(\vec{y},t). \tag{2.8}$$

Here we have replaced the simple exponential in (2.7) by a more accurate version of a path-ordered line integral.[13] After rearrangements we may write L_Q as

$$L_Q = \int d^3x\,\bar{\psi}[\gamma^\mu(i\partial_\mu - gA_\mu) - m]\psi - \frac{1}{2}\frac{g^2}{4\pi}\sum_a\int d^3x\,d^3y\,\rho_a(\vec{x},t)\frac{1}{|\vec{x}-\vec{y}|}\rho_a(\vec{y},t), \tag{2.9}$$

$$\rho_a(\vec{x},t) \equiv \Psi^\dagger(\vec{x},t)\tfrac{1}{2}\lambda_a\Psi(\vec{x},t),$$

where $\rho_a(\vec{x},t)$ is a color charge density, and

$$\Psi(x) = U^{-1}(\vec{x},t)\psi(x), \qquad U(\vec{x},t) = P\exp\left(ig\int_0^{\vec{x}} d\vec{x}'\cdot\vec{A}(\vec{x}',t)\right). \tag{2.10}$$

In (2.10) P stands for path ordering, and the line integral is along the straight-line segment connecting the two end points. The origin $\vec{x}=0$ will be identified with the center of the $Q\bar{Q}$ system. Under the local gauge transformation (2.3), the operators U and Ψ respond as follows:

$$U(\vec{x},t) \to V(\vec{x},t)U(\vec{x},t)V^{-1}(\vec{0},t), \qquad \Psi(\vec{x},t) \to V(\vec{0},t)\Psi(\vec{x},t). \tag{2.11}$$

The operator $U^{-1}(\vec{x},t)$ transports color from the point $\vec{x}$ to the origin, and $\Psi(x)$ behaves as if its color resides at $\vec{x}=0$. The significance of the exponential factor in (2.7) is now clear. It leads to the introduction of the variable Ψ; as a consequence of (2.11) L_Q is gauge invariant. Also, it is the quarks and antiquarks represented by Ψ, not ψ, that interact via a Coulomb potential. We therefore identify Ψ with the constituent quark. This identification conforms with another observation. In the quark model a meson is interpreted as a $Q\overline{Q}$ bound state. It also must be a gauge-invariant color singlet. Such a state can only be constructed with Ψ, not ψ—for example,

$$|Q\overline{Q}\rangle = \int d^3x f(\vec{x})\overline{\Psi}(\vec{x},t)\Gamma\Psi(0,t)|0\rangle\,, \qquad (2.12)$$

with a matrix Γ in spin and flavor. Therefore, for discussions of hadronic transitions the constituent-quark field Ψ is the natural variable. In terms of Ψ we get

$$L_Q = \int d^3x \overline{\Psi}\,[\gamma^\mu(i\partial_\mu - gA'_\mu) - m]\Psi - \frac{1}{2}\frac{g^2}{4\pi}\sum_a \int d^3x\, d^3y\, \rho_a(\vec{x},t)\frac{1}{|\vec{x}-\vec{y}|}\rho_a(\vec{y},t)\,, \qquad (2.13)$$

where

$$A'_\mu = U^{-1}A_\mu U - \frac{i}{g}U^{-1}\partial_\mu U\,. \qquad (2.14)$$

It is shown in Appendix B that A'_μ takes the more useful form

$$A'_0(\vec{x},t) = A_0(0,t) - \int_0^1 ds\, U^{-1}(s\vec{x},t)\,\vec{x}\cdot\vec{E}(s\vec{x},t)U(s\vec{x},t)\,, \qquad (2.15)$$
$$\vec{A}'(\vec{x},t) = -\int_0^1 ds\, s\, U^{-1}(s\vec{x},t)\vec{x}\times\vec{B}(s\vec{x},t)U(s\vec{x},t)\,.$$

The color-electric field $\vec{E}$ and color-magnetic field $\vec{B}$ are defined by

$$E^k = F^{k0}\,, \quad B^i = -\tfrac{1}{2}\epsilon^{ijk}F_{jk} \quad (\epsilon^{123}=1)\,. \qquad (2.16)$$

Expanding A'_μ in powers of $\vec{x}$, we obtain

$$L_Q = L_O + L_I\,, \qquad (2.17)$$

$$L_O = \int d^3x\, \overline{\Psi}(\gamma^\mu i\partial_\mu - m)\Psi - \frac{1}{2}\frac{g^2}{4\pi}\sum_a\int d^3x\, d^3y\, \rho_a(\vec{x},t)\frac{1}{|\vec{x}-\vec{y}|}\rho_a(\vec{y},t)\,, \qquad (2.18)$$

$$L_I = \sum_a [-Q_a A_{0a}(0,t) + \vec{d}_a\cdot\vec{E}_a(0,t) + \vec{m}_a\cdot\vec{B}_a(0,t) + \cdots]\,, \qquad (2.19)$$

where

$$Q_a = g\int d^3r\, \rho_a(\vec{r},t)\,,$$
$$\vec{d}_a = g\int d^3r\, \vec{r}\rho_a(\vec{r},t)\,,$$
$$\vec{m}_a = g\int d^3r\, \tfrac{1}{2}\vec{r}\times\vec{j}_a(\vec{r},t)\,, \qquad (2.20)$$
$$\vec{j}_a \equiv \overline{\Psi}\vec{\gamma}\tfrac{1}{2}\lambda_a\Psi\,.$$

Equation (2.19) is the multipole expansion of QCD. We have only displayed the first three terms in the expansion: the monopole, the electric dipole, and the magnetic dipole. (Since no confusion will occur in this paper, we will refer to these multipoles by the terminology of their electromagnetic analogs. We will often omit the modifier "color.") Apart from the monopole, all the other multipole interactions are gauge invariant. Again, this is an important consequence of using the variable Ψ.

Equations (2.19) and (2.20) are identical to their counterparts in QED except for the appearance of color indices. It is important to recognize that L_Q is an effective action that represents the result of partial summation of the perturbation series. It only includes what Peskin[7] called the connected pieces of the diagrams. The gauge fields in L_Q are those generated by light quarks of ordinary hadrons produced in the hadronic transitions. Therefore, they may appropriately be regarded as external fields as far as the heavy-quark system is concerned. The amplitude for a hadronic transition is obtained by applying the perturbation theory to L_I. For example, Peskin's result corresponds to using the electric dipole twice.

We now turn to the general case. In order to go beyond perturbation theory, we will make hypotheses concerning the nature of $Q\overline{Q}$ interactions and the couplings of the heavy quarks to the gauge fields. The basic idea is to assume that the lower half of Fig. 2 can be generated by an effective action. More specifically, we assume the following:

(1) The $Q\overline{Q}$ interaction can be described by a nonrelativistic (confining) potential, and it can be represented by a gauge-invariant addition to the effective action for the $Q\overline{Q}$ system. The latter is true in the case of the Coulomb potential, as we have explicitly verified; it must be true in general if the potential has a physical meaning.

(2) The couplings of the heavy quarks to the

gauge fields can be represented by effective interactions that are consistent with gauge invariance. Moreover, when the gauge fields carry sufficiently small momenta, they can be expanded in a Taylor series around the center of the heavy-quark system. For a nonrelativistic $Q\bar{Q}$ system, its effective Hamiltonian will have a well defined limit as the quark mass approaches infinity. Consequently, no positive powers of the heavy-quark mass should appear as coefficients in the expansion.

The use of an effective action is a common practice in many branches of physics. These hypotheses are certainly very natural. The requirement of local gauge invariance severely restricts the form of the effective action. The simplest possible gauge-invariant effective Lagrangian for the heavy-quark system is

$$L_Q = \int d^3x\, \bar{\psi}[\gamma^\mu(i\partial_\mu - gA_\mu) - m]\psi$$
$$- \frac{1}{2}\int d^3x\, d^3y \sum_{a=0}^{8} \bar{\Psi}(\vec{x},t)\gamma^0 \tfrac{1}{2}\lambda_a \Psi(\vec{x},t)[\delta_{a0}V_1(|\vec{x}-\vec{y}|) + (1-\delta_{a0})V_2(|\vec{x}-\vec{y}|)]\bar{\Psi}(\vec{y},t)\gamma^0\tfrac{1}{2}\lambda_a\Psi(\vec{y},t) + \cdots , \tag{2.21}$$

where Ψ is the constituent-quark field defined by (2.10). Equation (2.21) and Ψ are defined in terms of a renormalized coupling constant g appropriate for the heavy-quark system. The SU(3) color matrices λ_a in the last term of (2.21) have been extended to include the identity

$$\tfrac{1}{2}\lambda_0 = 1 . \tag{2.22}$$

Compared with (2.9), (2.21) differs only in the general form assumed for the $Q\bar{Q}$ potential. The minimum coupling to the gauge field is dictated by gauge invariance. But other types of gauge-invariant couplings may also appear; they are indicated by the dots in (2.21). Later we will discuss the effects of these possible interactions. Also neglected in (2.21) are possible three-body and other many-body potentials. In terms of Ψ, (2.21) has the same multipole expansion as that given in (2.19). A transcription into a nonrelativistic quantum-mechanical Hamiltonian gives

$$H_{Q\bar{Q}} = H + H_I , \tag{2.23a}$$

$$H = \frac{\vec{p}^{\,2}}{m} + V_1(|\vec{x}|) + \sum_a \tfrac{1}{2}\lambda_a \tfrac{1}{2}\bar{\lambda}_a V_2(|\vec{x}|) , \tag{2.23b}$$

$$H_I = Q_a A_{0a}(0,t) - \vec{d}_a \cdot \vec{E}_a(0,t) - \vec{m}_a \cdot \vec{B}_a(0,t) + \cdots , \tag{2.23c}$$

where $\bar{\lambda}_a$'s are the color matrices for an antiquark. In this notation we have[14]

$$Q_a = g(\tfrac{1}{2}\lambda_a + \tfrac{1}{2}\bar{\lambda}_a) ,$$
$$\vec{d}_a = \tfrac{1}{4} g \vec{x}(\lambda_a - \bar{\lambda}_a) , \tag{2.24}$$
$$\vec{m}_a = \frac{g}{4m}\tfrac{1}{2}(\lambda_a - \bar{\lambda}_a)(\vec{\sigma} - \vec{\sigma}') ,$$

where $\vec{x}$ is the relative $Q\bar{Q}$ separation and $\vec{\sigma}$ and $\vec{\sigma}'$ are the spin matrices for Q and $\bar{Q}$, respectively. We should emphasize again that the gauge fields in (2.23) are external fields due to the light quarks produced in hadronic transitions.

This is a good place to comment on the effects of possible additional gauge-invariant couplings. Since each multipole coupling except the monopole is gauge invariant by itself, higher-order QCD effects may modify the numerical coefficients. This will not effect our later discussion because we will not be able to compute absolute rates anyway. However, we will assume that high-order QCD effects do not change the *relative* importance of the multipole moments from the natural scaling estimates.[15] We will call this the "scaling hypotheses." For example, according to (2.23) and (2.24), the electric dipole is more important than the magnetic dipole, since the former scales like x and the latter like m^{-1}. Similar simple dimensional analysis will be used to estimate the magnitude of other gauge-invariant couplings. Consider a possible coupling of the form

$$H_I' = C\left(\sum_a \vec{d}_a \cdot \vec{E}_a\right)^2 . \tag{2.25}$$

The constant C has the dimension of a length. There are two dimensional parameters for a heavy-quark system, the quark mass m and the size of the system a. Dimensional analysis gives $C \sim m^{-1}$ or $C \sim a$. Thus, H_I' is of order a^3 or a^2/m. It is therefore at least smaller by a^2 compared with the electric dipole or magnetic-dipole terms. Indeed, gauge-invariant couplings other than those displayed in (2.23) have higher dimensions; therefore they are suppressed by powers of m^{-1} and/or a.

Our scaling hypothesis enables us to determine which multipole dominates in a particular transition. As we will see in the next two sections, the symmetry and tensor property of this multipole will lead to many interesting experimental consequences.

Hadronic transitions are caused by the perturbation H_I. Since Q_a is of order unity we must treat the monopole to all orders. However, it annihilates a color singlet; therefore a transition is produced by other terms in H_I. As a concrete example, consider a transition $\Phi' \to \Phi + h$ which utilizes the color-electric dipole twice. Standard perturbation theory gives the transition amplitude

$$M_{ee} = \langle \Phi h | T | \Phi' \rangle$$

$$= \sum_{a,b} \int_0^\infty dt \left\langle \Phi h \middle| T\, \vec{d}_a(0) \cdot \vec{E}_a(0) \exp\left(-i \int_{-t}^0 dt' \sum_c Q_c A_{0c}(0,t')\right) \vec{d}_b(-t) \cdot \vec{E}_b(0,-t) \middle| \Phi' \right\rangle \tag{2.26}$$

Equation (2.26) can be simplified by using the relations

$$Q_a |\Phi'\rangle = 0\,, \tag{2.27}$$

$$[Q_a, d_b] = ig \sum_c f_{abc} d_c\,. \tag{2.28}$$

We rewrite

$$\prod_{i=1}^{n} \left(\sum_a Q_a A_{oa}(t_i') \right) \vec{d}_b(-t) = (ig)^n \sum_c \vec{d}_c(-t) [\bar{A}_o(t_i') \cdots \bar{A}_o(t_n')]_{cb}\,, \tag{2.29}$$

where the matrix $\bar{A}_0$ is defined by

$$[\bar{A}_o]_{cb} \equiv \sum_a f_{abc} A_{oa}\,. \tag{2.30}$$

Since both Φ and Φ' are color singlets, we may make the replacement

$$d_a^i(0) d_c^j(-t) \to \frac{1}{N^2-1} \sum_b d_b^i(0) d_b^j(-t) \delta_{ac}\,, \tag{2.31}$$

where $N=3$ is the number of colors. The transition amplitude becomes

$$M_{ee} = \frac{1}{N^2-1} \sum_{ij} \int_0^\infty dt \left\langle \Phi \,|\, T \sum_a d_a^i(0) d_a^j(-t) \,|\, \Phi' \right\rangle \left\langle h \middle| T E^i(0) \exp\left(g \int_{-t}^0 dt' \bar{A}_o(t')\right) E^j(0,-t) \middle| 0 \right\rangle, \tag{2.32}$$

where a short-hand notation is employed:

$$E^i(0) \exp\left(g \int_{-t}^0 dt' \bar{A}_o(t')\right) E^j(0,-t) = \sum_{b,c} E_b^i(0) \exp\left(\int_{-t}^0 dt' \bar{A}_o(t')\right)_{bc} E_c^j(0,-t)\,. \tag{2.33}$$

We have written (2.32) in the factorized form to emphasize the point that the gauge fields which appear explicitly in (2.32) couple only to the light hadrons. To simplify (2.32) we will use the symbolic notation

$$E^i(0) \exp\left(g \int_{-t}^0 dt' \bar{A}_0(t')\right) E^j(0,-t) = E^i e^{-tD_o} E^j\,, \tag{2.34}$$

where $E^i \equiv E^i(\vec{x}=0, t=0)$ and D_0 is the gauge-covariant time derivative

$$D_o = \partial_o - g\bar{A}_o \tag{2.35a}$$

and

$$D_o^n E^j = [\partial_o - g\bar{A}_o(0,t)]^n E^j(0,t)\big|_{t=0}\,. \tag{2.35b}$$

The integration over t then gives

$$M_{ee} = \frac{i}{N^2-1} \left\langle \Phi h \middle| \vec{d}_a \cdot \vec{E} \frac{1}{E' - H + iD_o} \vec{d}_a \cdot \vec{E} \middle| \Phi' \right\rangle, \tag{2.36}$$

where E' is the energy of Φ' and H is the effective $Q\bar{Q}$ Hamiltonian (2.23b). We emphasize again that D_o acts only on the color-electric field $\vec{E}$, and both D_o and $\vec{E}$ are independent of heavy-quark variables. The color matrices can be eliminated with the aid of the projection operators P_1 and P_8 into the $Q\bar{Q}$ singlet sector and octet sector, respectively:

$$P_1(\lambda_a - \bar{\lambda}_a) P_8 (\lambda_a - \bar{\lambda}_a) P_1 = 8\, \frac{N^2-1}{N}\,. \tag{2.37}$$

Finally, we have

$$M_{ee}=i\frac{g^2}{2N}\left\langle\Phi h\left|\,\vec{x}\cdot\vec{E}\,\frac{1}{E'-H_8+iD_0}\,\vec{x}\cdot\vec{E}\,\right|\Phi'\right\rangle, \tag{2.38}$$

where H_8 is the octet component of H:

$$H_8=P_8HP_8\,. \tag{2.39}$$

The resolvent

$$G(E')=\frac{1}{E'-H_8+iD_0} \tag{2.40}$$

describes the propagation of the intermediate states between the two color-electric dipole vertices: These are color-singlet states consisting of the $Q\overline{Q}$ octet, gluons, and light quarks. The interaction between the $Q\overline{Q}$ octet and gluons as well as light quarks is represented by $\vec{A}_0$ in the iD_0 term. Equation (2.38) is manifestly gauge invariant.

In the next section we will present the detailed predictions that follow from Eq. (2.38). For completeness, we give the transition amplitudes that involve two magnetic-dipole transitions (M_{mm}) and that involve one electric-dipole and one magnetic-dipole transition (M_{em}):

$$M_{mm}=i\frac{g^2}{2N}\frac{1}{4m^2}\times\left\langle\Phi h\left|(\vec{\sigma}-\vec{\sigma}')\cdot\vec{B}\,\frac{1}{E'-H_8+iD_0}(\vec{\sigma}-\vec{\sigma}')\cdot\vec{B}\right|\Phi'\right\rangle, \tag{2.41}$$

$$M_{em}=i\frac{g^2}{2N}\frac{1}{2m}\times\left\langle\Phi h\left|(\vec{\sigma}-\vec{\sigma}')\cdot\vec{B}\,\frac{1}{E'-H_8+iD_0}\,\vec{x}\cdot\vec{E}+\vec{x}\cdot\vec{E}\,\frac{1}{E'-H_8+iD_0}\,(\vec{\sigma}-\vec{\sigma}')\cdot\vec{B}\right|\Phi'\right\rangle. \tag{2.42}$$

We conclude this section with two remarks:

(1) We have assumed that the sum of diagrams with vertices connected (as defined by Peskin[7]) to the $Q\overline{Q}$ system can be represented by a gauge-invariant effective action. Although the $c\bar{c}$ and $b\bar{b}$ systems are not small enough to permit a perturbative analysis of their interactions, it should be a good approximation to treat the small $Q\overline{Q}$ states as pointlike in hadronic transitions. The heavy-quark dynamics is summarized by the multipole moments. The disparity in the two sizes of $Q\overline{Q}$ states and light hadrons makes the multipole expansion converge. In this respect, the multipole expansion is reminiscent of the short-distance expansion of operator products: The multipole moments play the roles of coefficient functions.

(2) The validity of the multipole expansion depends on the magnitude of the expansion parameter ka, where k is the typical momentum of the emitted gluons and a is the size of the $Q\overline{Q}$ system. If the gluons were to emerge as real particles, we would have

$$k\sim\tfrac{1}{2}(m_{\Phi'}-m_{\Phi}) \tag{2.43}$$

for a two-gluon emission. When Eq. (2.43) is combined with potential model estimates[9] of the sizes of $c\bar{c}$ and $b\bar{b}$ systems, we get for the transitions $\psi'\to\psi+2\pi$ and $\Upsilon'\to\Upsilon+2\pi$,

$$(ka)_\psi\sim0.7\,, \tag{2.44}$$

$$(ka)_\Upsilon\sim0.3\,. \tag{2.45}$$

It is not clear how confinement affects these estimates. However, similar results are obtained if we choose $1/k\sim1$ fm, the typical size of light hadrons. We therefore expect the multipole expansion to work reasonably well for Υ particles. It may also work for ψ particles. It is known in nuclear physics that the classification of multipole orders is valid even for $ka\sim1$.

III. PREDICTIONS–APPLICATION OF THE WIGNER-ECKART THEOREM

In this section we present predictions that follow from the Wigner-Eckart theorem[16] and scaling properties of the multipole moments. We will concentrate our discussion on second-order $E1$ transitions except on one occasion where second-order $M1$ transitions are also considered. In the next section further predictions for soft-pion emissions will be given.

A. Wigner-Eckart theorem

As mentioned in the previous section, the gauge-field operators in the transition amplitude (2.38) are independent of heavy-quark variables. Therefore, the transition amplitude is a reducible second-rank tensor. It is decomposed into a sum of irreducible tensors according to

$$T_{ij}=x_iG(E')x_j=\sum_{k=0}^{2}T_{ij}^{(k)}\,, \tag{3.1}$$

where

$$T_{ij}^{(0)}=\delta_{ij}T^{(0)}=\delta_{ij}\sum_l x_lG(E')x_l\,, \tag{3.2}$$

$$T_{ij}^{(1)}=\tfrac{1}{2}[x_iG(E')x_j-x_jG(E')x_i]\,, \tag{3.3}$$

$$T_{ij}^{(2)}=\tfrac{1}{2}\left(x_iG(E')x_j+x_jG(E')x_i-\tfrac{2}{3}\delta_{ij}\sum_l x_lG(E')x_l\right). \tag{3.4}$$

These Cartesian components are related to the spherical components by

$$T_{ij}^{(1)}=\sum_{q=-1}^{q=+1}\epsilon_{ijk}\epsilon_k^{(q)}T_{-q}^{(1)}(-1)^q\,, \tag{3.5a}$$

$$T_{ij}^{(2)}=\sum_{q=-2}^{q=+2}\epsilon_{ij}^{(q)}T_{-q}^{(2)}(-1)^q\,, \tag{3.5b}$$

where $\epsilon_k^{(q)}$ and $\epsilon_{ij}^{(q)}$ are components of the irreducible spherical tensor operators of rank 1 and 2, respectively, constructed from the unit vector $\hat{x}$. Equations (3.2)–(3.5) can be written in the general form

$$T_{ij}^{(k)}=\sum_q C_{ij,q}^{(k)}T_q^{(k)}(-1)^q\,. \tag{3.6}$$

Consider a transition $\Phi'\to\Phi+h$, where h denotes n light hadrons. Let the initial $Q\overline{Q}$ state be characterized by its total angular momentum J', its z component M', its orbital angular momentum l', its spin s, and other quantum numbers summarized by α'. The corresponding quantum numbers of the final $Q\overline{Q}$ state are denoted by unprimed symbols. Since the transition operator is spin independent, the initial and final spin are the same. The Wigner-Eckart theorem gives

$$\langle\Phi h|E_iT_q^{(k)}E_j|\Phi'\rangle=(-1)^{J'+J-M+l+k+s}[(2J'+1)(2J+1)]^{1/2}\begin{pmatrix}J & k & J'\\ -M & q & M'\end{pmatrix}\begin{Bmatrix}l & k & l'\\ J' & s & J\end{Bmatrix}\langle l\alpha h\|E_iT^{(k)}E_j\|l'\alpha'\rangle\,, \tag{3.7}$$

where ( ) and { } denote the 3-j and 6-j symbols, respectively. We now introduce the tensors

$$\frac{1}{[(2\omega_1)(2\omega_2)\cdots(2\omega_n)]^{1/2}}t_q^{(k)}=\sum_{ij}C_{ij,q}^{(k)}\langle l\alpha h\|E_iT^{(k)}E_j\|l'\alpha'\rangle\,. \tag{3.8}$$

The differential transition rate becomes

$$d\Gamma(\Phi'\to\Phi+h)=(2\pi)\delta(E_i-E_f)(2J+1)(2J'+1)\sum_{k,k'}\sum_{q,q'}\sum_M(-1)^{k+k'}(-1)^{q+q'}\begin{pmatrix}J & k & J'\\ -M & q & M'\end{pmatrix}\begin{pmatrix}J & k' & J'\\ -M & q' & M'\end{pmatrix}$$
$$\times\begin{Bmatrix}l & k & l'\\ J' & s & J\end{Bmatrix}\begin{Bmatrix}l & k' & l'\\ J' & s & J\end{Bmatrix}t_q^{(k)}t_{q'}^{(k')*}\prod_{i=1}^n\frac{d^3k_i}{2\omega_i}\,. \tag{3.9}$$

The final heavy particle Φ moves slowly in the rest frame of Φ'. So its mass-to-energy ratio is approximately unity: $M/E\approx1$. After inserting this factor, the phase-space volume element takes a Lorentz-invariant form

$$\delta(E_i-E_f)\prod_{l=1}^n\frac{d^3k_l}{2\omega_l}=2M\,d\mathfrak{M}^2\left[\frac{d^3K}{2K^0}\,\frac{d^3P}{2E}\,\delta^4(P'-P-K)\right]\left[\prod_{l=1}^n\frac{d^3k_l}{2\omega_l}\,\delta^4\Big(K-\sum_l k_l\Big)\right]\,, \tag{3.10}$$

where P', P, and K are the four-momenta of Φ', Φ, and the light hadron system, respectively, and $\mathfrak{M}^2=K^2$ is the invariant mass squared of the light hadrons.

The result simplifies greatly if we integrate over all momenta. Rotational invariance implies

$$4\pi M\int\frac{d^3K}{2K_0}\,\frac{d^3P}{2E}\prod_{l=1}\frac{d^3k_l}{2\omega_l}\delta^4(P'-P-K)\delta^4\Big(K-\sum_l k_l\Big)t_q^{(k)}t_{q'}^{(k')*}=\delta_{kk'}\,\delta_{qq'}A_k \tag{3.11}$$

and therefore[17]

$$\frac{d\Gamma(\Phi'\to\Phi+h)}{d\mathfrak{M}^2}=(2J+1)\sum_k\begin{Bmatrix}k & l' & l\\ s & J & J'\end{Bmatrix}^2A_k\,. \tag{3.12}$$

This is the central result of this section. It follows from spin independence of H in (2.23b) and angular momentum conservation. The significance of spin-dependent forces may be measured by the spin splittings.[18] In the $c\overline{c}$ system, the splitting of the first P states are of order 100–150 MeV. The corresponding splittings in the $b\overline{b}$ system are expected to be smaller by a factor of $(m_c/m_b)^2\simeq0.16$, or about 20–30 MeV. These splittings are much smaller than the level spacings between multiplets. Consequently, spin independence should be a good approximation for the $b\overline{b}$ system.

For emission of a light-hadronic system, such as 2π, Eq. (3.12) relates mass distributions among transitions between different states of two multiplets. In this section we will consider the application of (3.12) to several important transitions.

For these transitions, (3.12) is the most general distribution allowed by spin independence; angular momentum conservation requires that the amplitude, at most, is a comination of an irreducible second-rank tensor, a vector, and a scalar. Our predictions are direct tests of spin independence.

B. Transitions between two $l=0$ multiplets

When both l and l' are zero, we have

$$\begin{Bmatrix} k & 0 & 0 \\ s & J' & J \end{Bmatrix} = \frac{1}{(2J+1)^{1/2}}\delta_{k0}\delta_{sJ'}\,. \tag{3.13}$$

Therefore, only the tensor with $k=0$ contributes. It follows from (3.9) that

$$d\Gamma(\Phi'(J'=0)\to\Phi(J=0)+h)$$
$$=d\Gamma(\Phi'(J'=1)\to\Phi(J=1)+h)\,, \tag{3.14}$$

i.e., the differential rate is the same for a transition between two spin-triplet S states and that between the two spin-singlet hyperfine partners. The most important examples of this kind are

$$d\Gamma(\psi'\to\psi+2\pi)=d\Gamma(\eta_c'\to\eta_c+2\pi)\,, \tag{3.15}$$

$$d\Gamma(\Upsilon'\to\Upsilon+2\pi)=d\Gamma(\eta_b'\to\eta_b+2\pi)\,, \tag{3.16}$$

$$d\Gamma(\Upsilon''\to\Upsilon+2\pi)=d\Gamma(\eta_b''\to\eta_b+2\pi)\,, \tag{3.17}$$

where η_c, η_c', η_b, η_b', and η_b'' are the spin-singlet partners of ψ, ψ', Υ, Υ', and Υ'', respectively. From the known rate for $\psi'\to\psi+2\pi$, we predict

$$\Gamma(\eta_c'\to\eta_c+2\pi)\cong 110\text{ keV}\,. \tag{3.18}$$

Furthermore, the angular and mass distributions of the pions should be identical in both cases. It is hoped that these predictions will be tested at SPEAR and CESR in the near future.

C. Transitions between two $l=1$ multiplets

For transitions between two multiplets with nonzero orbital angular momenta, the transition amplitude in general contains more than one reduced matrix element. The most important example in this category is the transitions between the first two P multiplets in the $b\bar{b}$ system. According to popular potential models,[9,19] these two multiplets will lie below the Zweig-rule-allowed threshold and their energy differences are large enough for two-pion emission. The spin-triplet $2P$ states can be populated by photon transitions from the spin-triplet $3S$ produced in e^+e^- annihilation.

There are three reduced matrix elements for the nine transitions between the two spin-triplet multiplets. There exist, therefore, six constraints. First of all, we observe that Eq. (3.12) implies the reciprocity relation

$$(2J'+1)\frac{d\Gamma(\Phi'_{J'}\to\Phi_J+2\pi)}{dM_{\pi\pi}}=(2J+1)\frac{d\Gamma(\Phi'_J\to\Phi_{J'}+2\pi)}{dM_{\pi\pi}}\,, \tag{3.19}$$

which gives

$$\begin{aligned} &d\Gamma(0\to1)=3d\Gamma(1\to0)\,,\\ &d\Gamma(0\to2)=5d\Gamma(2\to0)\,,\\ &3d\Gamma(1\to2)=5d\Gamma(2\to1)\,. \end{aligned} \tag{3.20}$$

Substituting numerical values for the 6-j symbols,[20] we find

$$\begin{aligned} &d\Gamma(1\to1)=d\Gamma(0\to0)+\tfrac{1}{4}d\Gamma(0\to1)+\tfrac{1}{4}d\Gamma(0\to2)\,,\\ &d\Gamma(1\to2)=\tfrac{5}{12}d\Gamma(0\to1)+\tfrac{3}{4}d\Gamma(0\to2)\,,\\ &d\Gamma(2\to2)=d\Gamma(0\to0)+\tfrac{3}{4}d\Gamma(0\to1)+\tfrac{7}{20}d\Gamma(0\to2)\,. \end{aligned} \tag{3.21}$$

In both Eqs. (3.20) and (3.21), we have used the notation $d\Gamma(J'\to J)$ for $(d\Gamma/dM_{\pi\pi})(\Phi'_{J'}\to\Phi_J+2\pi)$.

It will be argued in the next section that $d\Gamma(1\to0)$ and $d\Gamma(0\to1)$ vanish in the soft-pion limit. So we expect these rates to be much smaller than others. In that case, Eq. (3.21) can be further simplified.

We note in passing that the rate for the spin-singlet transition $2\,^1P_1\to1\,^1P_1+2\pi$ is given by

$$d\Gamma(1\to1)\big|_{\text{spin singlet}}=[d\Gamma(0\to0)+d\Gamma(0\to1)+d\Gamma(0\to2)]_{\text{spin triplet}}\,. \tag{3.22}$$

However, it would be difficult to observe this transition since the spin-singlet state $2\,^1P_1$ is hard to reach in e^+e^- annihilation experiments.

D. Transitions between $l=2$ and $l=0$ multiplets

These transitions provide another example where only one irreducible tensor contributes to the matrix elements. In the $b\bar{b}$ system the first D multiplet is expected to lie below the Zweig-rule-allowed threshold. The transition to $1\,^3S_1$ by emission of 2π can be a significant decay made for these states. The $1\,^3D_J$ multiplet can be reached by cascade photon transitions $3\,^3S_1\to2\,^3P\to1\,^3D_J$. The transitions $1D\to1S+2\pi$ are controlled by a second-rank tensor. Equation (3.12) gives

$$\begin{aligned} d\Gamma(1\,^3D_3\to1\,^3S_1+2\pi)&=d\Gamma(1\,^3D_2\to1\,^3S_1+2\pi)\\ &=d\Gamma(1\,^3D_1\to1\,^3S_1+2\pi)\\ &=d\Gamma(1\,^1D_2\to1\,^1S_0+2\pi)\,. \end{aligned} \tag{3.23}$$

For completeness we have also included the prediction for the spin-singlet transition.

E. Scaling laws

The scaling properties of the multipole moments allow us to estimate the ratios of transition rates. Unlike the predictions discussed earlier that follow from spin independence and angular momentum conservation, these results are model dependent. They require the dominance of a particular multipole and need a model for numerical estimate. Furthermore, the naive scaling may be modified by high-order QCD effects. Nevertheless, these results are interesting and can be tested experimentally. Two examples will be given.

As the first example we estimate the ratio of the two transition rates, $\psi' \to \psi + 2\pi$ and $\Upsilon' \to \Upsilon + 2\pi$. This has been considered by Gottfried.[3,4] These are second-order $E1$ transitions. The amplitude (3.1) depends quadratically on the $Q\overline{Q}$ separation, so we have[9]

$$\frac{\Gamma(\psi' \to \psi + 2\pi)}{\Gamma(\Upsilon' \to \Upsilon + 2\pi)} \simeq \frac{\langle r^2\rangle_{\psi'}{}^2}{\langle r^2\rangle_{\Upsilon'}{}^2} \simeq 10\,, \tag{3.24}$$

which agrees with Gottfried's result. Equation (3.24) neglects the variation of the resolvent. This is not unreasonable since the level spacings of low-lying states of the $c\overline{c}$ and $b\overline{b}$ system are quite similar. As will be discussed in Sec. V, this ratio is sensitive to the spin of the gluon. For a scalar gluon, it would have been of order unity instead of 10. It is therefore interesting to have a direct experimental determination of the ratio.

We now consider the second example: the transitions $\psi' \to \psi + \eta$ and $\Upsilon' \to \Upsilon + \eta$. These transitions have a different scaling law from that for the 2π emission. Because η is a pseudoscalar, this decay is induced by $M1$-$M1$ and $E1$-$M2$ transitions. Both have roughly the same scaling properties. The matrix element is

$$M = \frac{g^2}{m_Q{}^2} \vec{\epsilon}\,' \times \vec{\epsilon}\,^* \cdot \vec{P}_\eta A\,, \tag{3.25}$$

where $\vec{\epsilon}\,'$, $\vec{\epsilon}$, and $\vec{P}_\eta$ are the polarization vectors of ψ' and ψ (or Υ' and Υ), and the momentum of η, respectively. Neglecting the variation of the reduced matrix element A, we find

$$\frac{\Gamma(\psi' \to \psi + \eta)}{\Gamma(\Upsilon' \to \Upsilon + \eta)} \cong \left(\frac{m_b}{m_c}\right)^4 \left[\frac{P_\eta(\psi)}{P_\eta(\Upsilon)}\right]^3 \cong 400\,. \tag{3.26}$$

For our estimate we have used[9] $m_b = 5.2$ GeV, $m_c = 1.8$ GeV. We see a dramatic difference between (3.24) and (3.26). Part of the difference is due to the phase-space suppression

$$\left[\frac{P_\eta(\psi)}{P_\eta(\Upsilon)}\right]^3 \cong 6\,. \tag{3.27}$$

Future experiments will test the validity of these naive scaling predictions.

IV. SOFT-PION EMISSION

Multipole expanding the gauge fields gives the explicit dependence of the transition amplitude on heavy-quark variables. Its dependence on the light-hadron momenta is in general unknown. However, when the transition involves only soft-pion emission, its variation with the pion momenta is constrained by PCAC (partial conservation of axial-vector current) and current algebra. In this section we exploit this additional information in conjunction with the results obtained in the last section. In some cases the multipole expansion and PCAC together determine uniquely the shape of angular and mass distributions; only the overall normalization is undetermined.

The PCAC and current-algebra approach to pion emission was first studied by Brown and Cahn[21] in their work on $\psi' \to \psi + 2\pi$. If the σ term is neglected, the transition amplitude has the form

$$\begin{aligned} &M(\Phi' \to \Phi + \pi_a(q_1) + \pi_b(q_2)) \\ &\quad = F_\pi^{-2} \frac{q_1^\mu q_2^\nu}{(2\omega_1 2\omega_2)^{1/2}} \langle \Phi' | T J_{5\mu}^a(q_1) T J_{5\nu}^b(q_2) | \Phi\rangle \\ &\quad = \delta_{ab} \frac{q_1^\mu q_2^\nu}{(2\omega_1 2\omega_2)^{1/2}} A_{\mu\nu}\,, \end{aligned} \tag{4.1}$$

where $J_{5\mu}^a$ is the axial-vector current. According to the PCAC hypothesis, we may neglect the variation of $A_{\mu\nu}$ with the pion momenta. Therefore, the transition amplitude M is homogeneous and bilinear in q_1 and q_2. To make use of this result, we will assume that the multipole expansion and PCAC are compatible.[22] Then, the transition amplitudes in the multipole analysis take the form

$$\begin{aligned} &\langle l\alpha\pi_a(q_1)\pi_b(q_2) \| E_i T^{(0)} E_j \| l'\alpha'\rangle \\ &\quad = \delta_{ab}\,\delta_{ij}(A q_{1\mu} q_2^\mu + B q_{10} q_{20}) \frac{1}{(2\omega_1 2\omega_2)^{1/2}}\,, \end{aligned} \tag{4.2}$$

$$\langle l\alpha\pi_a(q_1)\pi_b(q_2) \| E_i T^{(1)} E_j \| l'\alpha'\rangle = 0\,, \tag{4.3}$$

$$\begin{aligned} &\langle l\alpha\pi_a(q_1)\pi_b(q_2) \| E_i T^{(2)} E_j \| l'\alpha'\rangle \\ &\quad = \delta_{ab}[q_{1i}q_{2j} + q_{1j}q_{2i} - \tfrac{2}{3}\delta_{ij}(\vec{q}_1 \cdot \vec{q}_2)] \frac{C}{(2\omega_1 2\omega_2)^{1/2}}\,. \end{aligned} \tag{4.4}$$

Equation (4.3) is a consequence of Bose statistics of pions: The amplitude must be symmetric in q_1 and q_2, but the only vector available, $\vec{q}_1 \times \vec{q}_2$, is antisymmetric. The transition amplitude is now characterized by three constants, A, B, and C.

An immediate consequence of Eq. (4.3) is that in the transitions $2\,^3P_{J'} \to 1\,^3P_J + 2\pi$ we have

$$d\Gamma(0\to 1) = \tfrac{1}{3} d\Gamma(1\to 0) \approx 0\,. \tag{4.5}$$

Equation (3.21) then simplifies:

$$\begin{aligned} d\Gamma(1\to 1) &= d\Gamma(0\to 0) + \tfrac{1}{4} d\Gamma(0\to 2)\,,\\ d\Gamma(1\to 2) &= \tfrac{3}{4} d\Gamma(0\to 2)\,,\\ d\Gamma(2\to 2) &= d\Gamma(0\to 0) + \tfrac{7}{20} d\Gamma(0\to 2)\,. \end{aligned} \tag{4.6}$$

In the following we consider a few further applications of Eqs. (4.2)–(4.4).

A. Transitions with $l' = l = 0$ or $J' = J = 0$

These transitions are determined by two constants A and B. A third parameter that appears in the analysis of Brown and Cahn[21] is excluded by the multipole expansion. As noted by Brown and Cahn, the transition $\psi' \to \psi + 2\pi$ must be dominated by the parameter A in order to conform with the observed isotropic angular distributions.[23] We will assume that $B \approx 0$ in both transitions $\psi' \to \psi + 2\pi$ and $\Upsilon' \to \Upsilon + 2\pi$. The mass distribution is obtained by substituting Eq. (4.2) into (3.8) and (3.9). It reproduces the result of Brown and Cahn,[21]

$$\frac{d\Gamma_0}{dM_{\pi\pi}} = K(M_{\pi\pi}^2 - 4m_\pi^2)^{1/2}(M_{\pi\pi}^2 - 2m_\pi^2)^2\,, \tag{4.7}$$

where K is the recoil momentum of Φ in the rest frame of Φ'. It is given by

$$K = \frac{1}{2M'}\,[(M'+M)^2 - M_{\pi\pi}^2]^{1/2}[(M'-M)^2 - M_{\pi\pi}^2]^{1/2}\,, \tag{4.8}$$

where M' and M are masses of Φ' and Φ, respectively. Equation (4.7) is plotted in Fig. 5 for $\psi' \to \psi + 2\pi$ and $\Upsilon' \to \Upsilon + 2\pi$. The shape for $\psi' \to \psi + 2\pi$ agrees reasonably well with data.[23]

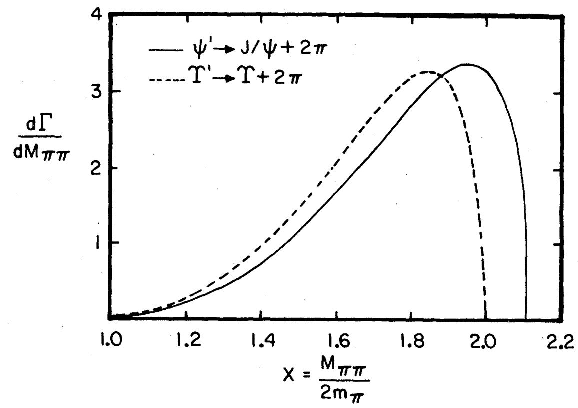

FIG. 5. Mass distribution of the 2π system in the transitions $\psi' \to J/\psi + 2\pi$ and $\Upsilon' \to \Upsilon + 2\pi$. The absolute normalization and relative normalization between the two transitions are arbitrary. These distributions assume $B=0$ in Eq. (4.2).

B. Transitions with $l' \neq l$ or $J' \neq J$

Only the second-rank tensor contributes to these transitions, since

$$\begin{Bmatrix} 0 & l' & l \\ s & J & J' \end{Bmatrix} = \frac{(-1)^{l+J+s}}{[(2l+1)(2J+1)]^{1/2}}\,\delta_{l',l}\delta_{J',J} = 0 \quad (l' \neq l \text{ or } J' \neq J)\,. \tag{4.9}$$

For these transitions Eq. (4.4) determines completely the shapes of the mass and angular distributions.

The mass distribution is the same for all transitions of this type. It is given by

$$\frac{d\Gamma_2}{dM_{\pi\pi}} = K(M_{\pi\pi}^2 - 4m_\pi^2)^{1/2}\left[(M_{\pi\pi}^2 - 4m_\pi^2)^2\left(1 + \frac{2}{3}\frac{K^2}{M_{\pi\pi}^2}\right) + \frac{8}{15}\frac{K^4}{M_{\pi\pi}^4}(M_{\pi\pi}^4 + 2M_{\pi\pi}^2 m_\pi^2 + 6m_\pi^4)\right]. \tag{4.10}$$

Equation (4.10) is shown in Fig. 6 for several choices of the masses M' and M. We observe that Eqs. (4.7) and (4.10) differ markedly in shapes. Equation (4.7) is sharply peaked at high masses, but Eq. (4.10) is spread over the whole available mass range. This distinction can be tested at CESR by comparing the 2π mass distributions in the transitions of the $b\bar{b}$ system, $\Upsilon' \to \Upsilon + 2\pi$, $2\,^3P_{J'} \to 1\,^3P_J + 2\pi$ $(J' \neq J)$, and $1\,^3D_J \to \Upsilon + 2\pi$.

On the other hand, angular distributions depend on the initial and final values of J and l, and initial polarizations. We give two simple examples.

Consider first the transition $2\,^3P_0 \to 1\,^3P_2 + 2\pi$. Since the initial state has zero total angular momentum, there is only angular correlation. Let θ be the angle between the relative pion momentum in the $\pi\pi$ center-of-mass c.m. system and the recoil momentum $\vec{K}$ of Φ. The angular correlation is then given by

$$\begin{aligned} \frac{d\Gamma(2\,^3P_0 \to 1\,^3P_2 + 2\pi)}{dM_{\pi\pi}\,d\cos\theta} = K(M_{\pi\pi}^2 - 4m_\pi^2)^{1/2}[&(M_{\pi\pi}^2 - 4m_\pi^2)^2 + (M_{\pi\pi}^2 - 4m_\pi^2)K^2 + K^4 \\ &-(2K^2 + M_{\pi\pi}^2 + 8m_\pi^2)(M_{\pi\pi}^2 - 4m_\pi^2)(K^2/M_{\pi\pi}^2)\cos^2\theta \\ &+(M_{\pi\pi}^2 - 4m_\pi^2)^2(K^4/M_{\pi\pi}^4)\cos^4\theta]\,. \end{aligned} \tag{4.11}$$

This function is plotted in Fig. 7. The angular correlation disappears at both extremes: $M_{\pi\pi}=2m_\pi$ and $M_{\pi\pi}=M'-M$.

Next, consider the transition $1\,^3D_1\to 1\,^3S_1+2\pi$. If the $1\,^3D_1$ is produced directly in e^+e^- annihilation, it will be transversely polarized. The complete angular distributions are rather complicated. But the angular distribution of the total momentum vector of the 2π system (or the recoil momentum of the $1\,^3S_1$) with respect to the e^+e^- beam direction is quite simple. If we call this angle θ, we then have

$$\frac{d\Gamma(1\,^3D_1\to 1\,^3S_1+2\pi)}{dM_{\pi\pi}d\cos\theta}=K(M_{\pi\pi}^2-4m_\pi^2)^{1/2}\left\{(M_{\pi\pi}^2-4m_\pi^2)^2\left(1+\frac{47}{60}\frac{K^2}{M_{\pi\pi}^2}\right)+\frac{2}{3}\frac{K^4}{M_{\pi\pi}^2}(M_{\pi\pi}^4+2m_\pi^2M_{\pi\pi}^2+6m_\pi^4)\right.$$
$$\left.-\left[\frac{7}{20}\frac{K^2}{M_{\pi\pi}^2}(M_{\pi\pi}^2-4m_\pi^2)^2+\frac{2}{5}\frac{K^4}{M_{\pi\pi}^4}(M_{\pi\pi}^4+2m_\pi^2M_{\pi\pi}^2+6m_\pi^4)\right]\cos^2\theta\right\}. \tag{4.12}$$

This function is shown in Fig. 8. We find the special cases

$$M_{\pi\pi}=2m_\pi:\ \frac{d\Gamma}{d\cos\theta}=1-\tfrac{3}{5}\cos^2\theta\,, \tag{4.13}$$

$$M_{\pi\pi}=M'-M:\ \frac{d\Gamma}{d\cos\theta}=1\,. \tag{4.14}$$

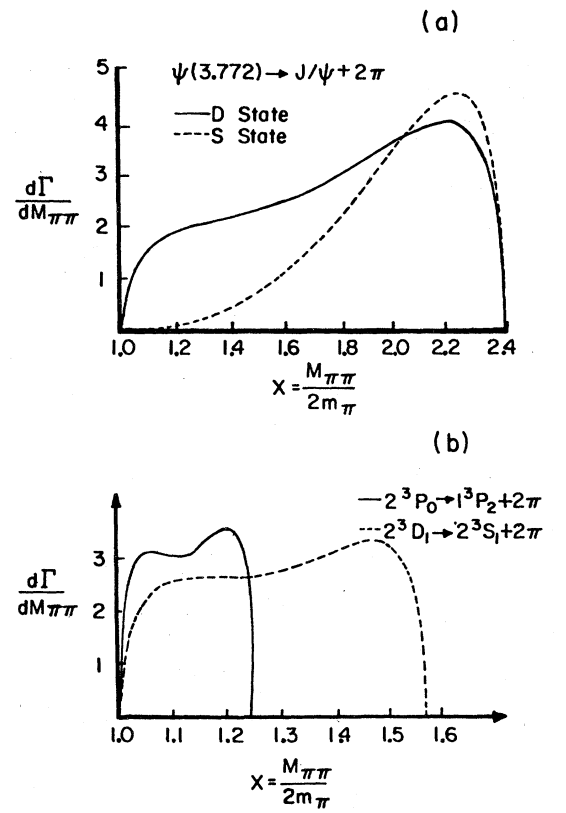

FIG. 6. (a) Mass distributions of the 2π system in the transition $\psi(3.772)\to J/\psi+2\pi$. The solid line and dashed line distinguish a pure D state from a pure S state. (b) Mass distributions of the 2π system in the transitions between $b\bar{b}$ states: $2\,^3P_0\to 1\,^3P_2+2\pi$ and $2\,^3D_1\to 2\,^3S_1+2\pi$. The masses are taken from the model of Ref. 9: $M(2P)=10.27$ GeV, $M(1P)=9.92$ GeV, $M(2D)=10.46$ GeV, and $M(2S)=10.02$ GeV. All normalizations are arbitrary.

It is easy to verify that after angular integrations, both Eqs. (4.11) and (4.12) reproduce the mass distribution given by Eq. (4.10).

As mentioned in the Introduction, we can use the unique mass and angular distributions of the 2π system in the transition $D\to S+2\pi$ to distinguish a predominantly D state from an S state. It has been suggested[11] that in the $b\bar{b}$ system an $l=0$ vibrational state is expected at a mass $M\sim 10.4$–10.5 GeV. But a conventional potential model[9] also predicts a D state nearby. If the 2π transition is observable, the mass and angular distributions of the 2π system may reveal the identity of the state.

C. Other transitions

Transitions with $l'=l\neq 0$ and $J'=J\neq 0$ will in general depend on all three constants. For these transitions the multipole expansion does not supply additional information concerning the mass and

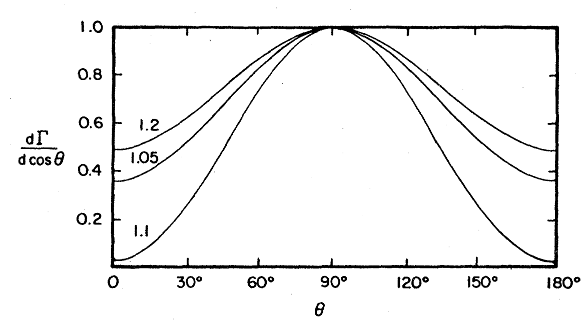

FIG. 7. Angular correlation in the transition between $b\bar{b}$ states, $2\,^3P_0\to 1\,^3P_2+2\pi$. θ is the angle between the momentum of a pion in the 2π center-of-mass system and the recoil momentum of $1P$ in the rest frame of $2P$. Masses of the $b\bar{b}$ states are the same as those in Fig. 6. Each curve is labeled by the value of the parameter $x=M_{\pi\pi}/2m_\pi$. The curves are normalized to unity at $\theta=90°$. The angular correlation disappears at both kinematic limits: $x=1$ and $x=x_{max}$. As x increases, the correlation gets stronger first and then it weakens.

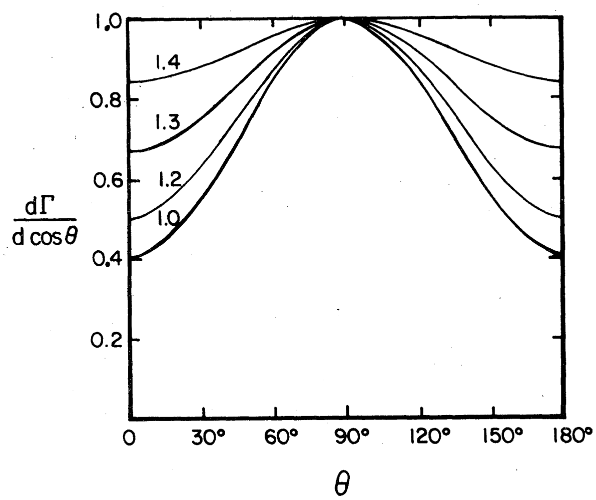

FIG. 8. Angular distribution of the momentum vector of the 2π system in the transition between $b\bar{b}$ states: $2\,^3D_1 \to 2\,^3S_1 + 2\pi$. The (pure) D state is assumed to be produced in e^+e^- annihilation. The angle θ is measured with respect to the colliding-beam direction. Masses are the same as those in Fig. 6. The number on each curve is the value of the parameter $x = M_{\pi\pi}/2m_\pi$. The curves are normalized to unity at $\theta = 90°$.

angular distributions. In all the other cases multipole analysis reduces the number of parameters needed for describing a transition. For example, our analysis shows that a transition between two $l=0$ states requires two parameters, and the transition $^3D_J \to {}^3S_1 + 2\pi$ requires only one. A general analysis in PCAC and current algebra would require three parameters for both transitions.

D. Phase-space corrections

Finally, we may use Eqs. (4.2)–(4.4) to correct for a major effect of spin splittings neglected so far. The phase space available to the 2π system is slightly different. As an example, consider again the transitions $2\,^3P_{J'} \to 1\,^3P_J + 2\pi$. Should the spin splittings turn out to be significant, Eq. (4.6) can be improved by including the phase-space effects. Define

$$I^{(0)}_{J'J} = \frac{M}{M'}\,\frac{d\Gamma_0}{dM_{\pi\pi}}\,,$$

$$I^{(2)}_{J'J} = \frac{M}{M'}\,\frac{d\Gamma_2}{dM_{\pi\pi}}\,, \tag{4.15}$$

where $d\Gamma_0/dM_{\pi\pi}$ and $d\Gamma_2/dM_{\pi\pi}$ are given by (4.7) and (4.10), and $M = M(J)$, $M' = M'(J')$. The improved version of (4.6) is

$$d\Gamma(1\to1) = \frac{I^{(0)}_{11}}{I^{(0)}_{00}}\,d\Gamma(0\to0) + \frac{1}{4}\,\frac{I^{(2)}_{11}}{I^{(2)}_{02}}\,d\Gamma(0\to2)\,,$$

$$d\Gamma(1\to2) = \frac{3}{4}\,\frac{I^{(2)}_{12}}{I^{(2)}_{02}}\,d\Gamma(0\to2)\,, \tag{4.16}$$

$$d\Gamma(2\to2) = \frac{I^{(0)}_{22}}{I^{(0)}_{00}}\,d\Gamma(0\to0) + \frac{7}{20}\,\frac{I^{(2)}_{22}}{I^{(2)}_{02}}\,d\Gamma(0\to2)\,.$$

V. DISCUSSION AND CONCLUSIONS

In this paper we have presented a general formalism for the multipole expansion of QCD and its application to hadronic transitions between heavy-quark states. A detailed analysis is made for second-order color $E1$ transitions. The Wigner-Eckart theorem leads to many interesting relations among transitions between two multiplets. Furthermore, when transitions involve only soft-pion emission we are able to determine, in some cases, the mass and angular distributions. The $b\bar{b}$ system should be an ideal place to test these ideas. There are several reasons for this. First of all, the system is much smaller in size and much more nonrelativistic than the $c\bar{c}$ system. The multipole expansion should work much better here. Secondly, there should be many more bound states below the Zweig-rule-allowed threshold, so a great number of transitions will be accessible. Among these the most interesting are the transitions between $2\,^3P_{J'}$ and $1\,^3P_J$.

The approach developed here is also applicable to many processes which have not been discussed in this paper. A particularly interesting and important process is $\Upsilon'' \to 1\,^1P_1 + 2\pi$. The singlet P state is difficult to reach by other means in e^+e^- experiments. According to conventional wisdom, the position of $^1P_1$ should coincide with the center of gravity of the spin-triplet states ^3P_J. As pointed out by Krammer and Kraseman,[24] the discovery of the spin-singlet P state, together with the spin-triplet states ^3P_J, will offer valuable information in the spin-dependent forces between heavy quarks.

We now turn to an important question: "Can the multipole-expansion predictions distinguish a vector gluon from a scalar gluon?" The answer is yes. Many features of hadronic transitions are sensitive to the spin of the gluon. Let us accept the color hypothesis that all hadrons are color-singlet states. The multipole expansion then has a different monopole term for a $Q\bar{Q}$ system:

$$\mathcal{L}_V = \tfrac{1}{2}g(\lambda_a + \bar{\lambda}_a)A_{0a} \quad \text{(vector gluon)}\,, \tag{5.1}$$

$$\mathcal{L}_S = \tfrac{1}{2}g(\lambda_a - \bar{\lambda}_a)\phi_a \quad \text{(scalar gluon)}\,. \tag{5.2}$$

In a vector-gluon theory the monopole term $\mathcal{L}_V$ cannot produce a transition since it annihilates a hadron state. So the leading operator for hadronic transitions is the color-electric dipole. On the other hand, the monopole term $\mathcal{L}_S$ in a scalar-gluon theory is the leading operator for $\Delta l = 0$

transitions. This distinction leads to several different predictions:

(1) As discussed in Sec. III, the ratio $\Gamma(\psi' \to \psi + 2\pi)/\Gamma(\Upsilon' \to \Upsilon + 2\pi)$ is about 10 in a vector-gluon theory. This ratio should be of order 1 in a scalar-gluon theory. That is,

$$\frac{\Gamma(\psi' \to \psi + 2\pi)}{\Gamma(\Upsilon' \to \Upsilon + 2\pi)} \approx 10 \quad \text{(vector gluon)}, \tag{5.3}$$

$$\approx 1 \quad \text{(scalar gluon)}. \tag{5.4}$$

Gottfried[3] has pointed out that the reported large ratio[25]

$$\left.\frac{B(d\sigma/dy)(\Upsilon' \to \mu^+\mu^-)}{B(d\sigma/dy)(\Upsilon \to \mu^+\mu^-)}\right|_{y=0} = 0.37 \pm 0.04 \tag{5.5}$$

implies that the rate $\Gamma(\Upsilon' \to \Upsilon + 2\pi)$ must be much smaller than $\Gamma(\psi' \to \psi + 2\pi)$. One then infers that Eq. (5.5) is inconsistent with Eq. (5.4) for a scalar-gluon theory. The rate $\Gamma(\Upsilon' \to \Upsilon + 2\pi)$ will be soon measured at CESR; its value will be of great importance to test (5.3) and (5.4).

(2) The two theories give different results for the transitions $2\,^2P_{J'} \to 1\,^3P_J + 2\pi$. In a scalar-gluon theory only transitions with $J' = J$ are allowed; those with $J' \neq J$ are smaller by $(ka)^4 \approx \frac{1}{10}$ by a rough estimate. On the other hand, all transitions are allowed in a vector-gluon theory; they should all be comparable except for the transitions $(J' = 0) \to (J = 1)$ and $(J' = 1) \to (J = 0)$ which are predicted to be much smaller by PCAC.

Since hadronic transitions are soft processes, they cannot be handled by the usual perturbation in the coupling constant. In order to make headway we have made several assumptions. They are explicitly stated in Sec. II. Success or failure of our predictions will be a test of these assumptions.

Note added. After completion of this work, I received a report from Dr. K. Shizuya [University of California Report No. LBL-10714, 1979 (unpublished)] in which he had studied the $Q\bar{Q}$ system in the Coulombic limit and derived a Hamiltonian similar to (2.19) by a Foldy-Wouthuysen transformation.

ACKNOWLEDGMENTS

This work grew out of a question asked by K. Gottfried. I should like to express my special thanks to him for valuable discussions and encouragement. I have also benefited from conversations with R. Cahn and P. Lepage. I should like to thank the Theory Group of Lawrence Berkeley Laboratory for their hospitality during a visit when part of this work was done. This work was supported in part by the National Science Foundation.

APPENDIX A: COULOMB PROPAGATOR IN AN EXTERNAL FIELD

In this appendix we derive Eq. (2.7). We will work in the Coulomb gauge. In this gauge, the third term in the exponential series (2.4) is entirely given by the diagram of Fig. 9(a). In fact, the exponential series (2.7) is generated by the diagrams in Fig. 9(b) for the Coulomb propagator in an external field. Let $D_n(\vec{x}, \vec{y})$ be the contribution to the Coulomb propagator from a diagram with n external gluons; then we have

$$D_n(\vec{x}, \vec{y}) = (-g)^n \int d^3x_1 \cdots d^3x_n \vec{\mathfrak{a}}(\vec{x}_1, t) \cdot \vec{\nabla}_1 \frac{1}{4\pi|\vec{x} - \vec{x}_1|} \vec{\mathfrak{a}}(\vec{x}_2, t) \cdot \vec{\nabla}_2 \frac{1}{4\pi|\vec{x}_1 - \vec{x}_2|} \cdots \vec{\mathfrak{a}}(\vec{x}_n, t) \cdot \vec{\nabla}_n \frac{1}{4\pi|\vec{x}_{n-1} - \vec{x}_n|} \frac{1}{4\pi|\vec{x}_n - \vec{y}|}, \tag{A1}$$

where

$$\vec{\mathfrak{a}}(\vec{x}_i, t) = \sum_b F_b \vec{A}_b(\vec{x}_i, t), \tag{A2}$$

$$(F_b)_{ac} = f_{abc}. \tag{A3}$$

We are interested in the Coulomb propagator for a small system of heavy quarks. Therefore,

$$\vec{r} = \vec{x} - \vec{y} \tag{A4}$$

is small. We will choose a coordinate system such that both $\vec{x}$ and $\vec{y}$ are also small. In Peskin's[7] terminology the leading contribution to $D_n(\vec{x}, \vec{y})$ is given by its connected piece: when all the vertices $\vec{x}_i$ are close to the heavy quarks. Thus, we set $\vec{x}_i \sim \vec{x} \sim \vec{y} \sim 0$. Using the notation

$$\vec{\mathfrak{a}} \equiv \sum_b F_b \vec{A}_b(0, t), \tag{A5}$$

we find

$$D_n(\vec{x}, \vec{y}) = (2g)^n (\vec{\mathfrak{a}} \cdot \vec{\nabla})^n I_n(r), \tag{A6}$$

where

$$I_n(r) = \int d^3x_1 \cdots d^3x_n \frac{1}{4\pi|\vec{r} - \vec{x}_1|} \times \frac{1}{4\pi|\vec{x}_1 - \vec{x}_2|} \cdots \frac{1}{4\pi|\vec{x}_n|}. \tag{A7}$$

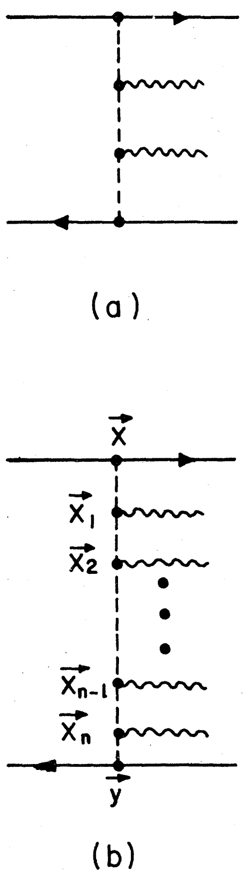

FIG. 9. (a) Second-order contribution to the Coulomb propagator in an external field. (b) nth-order contribution to the Coulomb propagator in an external field. Dashed lines in (a) and (b) are free Coulomb propagators.

The connected piece of $I_n(r)$ can be obtained by dimensional analysis:

$$I_n(r)=C_n r^{2n-1}\,. \tag{A8}$$

The equation

$$-\nabla^2 I_n = I_{n-1} \tag{A9}$$

gives the recursion relation

$$C_n=-\frac{1}{(2n)(2n-1)}C_{n-1}\,. \tag{A10}$$

Consequently,

$$C_n=\frac{1}{4\pi}\frac{(-1)^n}{(2n)!}\,, \tag{A11}$$

since

$$C_0=\frac{1}{4\pi}\,. \tag{A12}$$

Finally,

$$I_n(r)=\frac{1}{4\pi}\frac{(-1)^n}{(2n)!}r^{2n-1}\,. \tag{A13}$$

Carrying out the differentiation, we identify the connected piece of D_n with the contribution that is singular as $r\to 0$. Equation (A6) then gives

$$D_n(\vec{x},\vec{y})=\frac{1}{4\pi r}\frac{(-1)^n}{n!}(g\vec{r}\cdot\vec{a})^n\,. \tag{A14}$$

Summing over all n, we obtain the Coulomb propagator

$$D(\vec{x},\vec{y})=\sum_{n=0}^{\infty}D_n(\vec{x},\vec{y})=\frac{1}{4\pi r}e^{-g\vec{r}\cdot\vec{a}}\,. \tag{A15}$$

This is the result (2.7). A more complete calculation[13] will turn the simple exponential into a path-ordered line integral.

APPENDIX B: DERIVATION OF MULTIPOLE EXPANSION

In this appendix we derive Eq. (2.15) in the text. Our method applies both to Abelian and non-Abelian gauge theories. It is instructive to consider both cases: the comparison will show how the non-Abelian character enters the derivation of the multipole expansion.

We will imagine that the quantum field theory is formulated in Feynman's path integral. All the field operators are c-number variables. Since we do not have to deal with the self-couplings of the gauge fields, they may be treated as external fields.

The Lagrangian of a charged fermion field coupled to an electromagnetic potential is

$$\mathcal{L}_e=\bar{\psi}[\gamma^\mu(i\partial_\mu - eA_\mu)-m]\psi\,. \tag{B1}$$

We introduce the new variable

$$\psi(x)=U(\vec{x},t)\Psi(x)\,, \tag{B2}$$

where

$$U(\vec{x},t)=\exp\left(ie\int_0^{\vec{x}}d\vec{x}'\cdot\vec{A}(\vec{x}',t)\right) \tag{B3}$$

and the line integral is along a straight path connecting the end points; the origin $\vec{x}=0$ is chosen to be the center of the charged-particle system. In terms of Ψ, $\mathcal{L}_e$ becomes

$$\mathcal{L}_e=\bar{\Psi}[\gamma^\mu(i\partial_\mu - eA'_\mu)-m]\Psi\,, \tag{B4}$$

where A'_μ is given by

$$A'_\mu=U^{-1}A_\mu U-\frac{i}{e}U^{-1}\partial_\mu U\,. \tag{B5}$$

Using (B3), we have

$$A_0'(\vec{x},t)=A_0(\vec{x},t)+\int_0^{\vec{x}} d\vec{x}'\cdot\dot{\vec{A}}(\vec{x}',t)\,,$$
$$\vec{A}'(\vec{x},t)=\vec{A}(\vec{x},t)-\vec{\nabla}\int_0^{\vec{x}} d\vec{x}'\cdot\vec{A}(\vec{x}',t)\,, \tag{B6}$$

where $\dot{\vec{A}}$ is the time derivative of $\vec{A}$. Let us write

$$\vec{x}'=s\vec{x}\,,\quad 0<s<1\,; \tag{B7}$$

then

$$A_0'(\vec{x},t)=A_0(0,t)-\int_0^1 ds\,\vec{x}\cdot\vec{E}(s\vec{x},t)\,,$$
$$\vec{A}'(\vec{x},t)=-\int_0^1 ds\,s\,\vec{x}\times\vec{B}(s\vec{x},t)\,, \tag{B8}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, respectively. Expanding $\vec{E}$ and $\vec{B}$ in powers of $\vec{x}$ around the origin, we obtain the familiar multipole expansion in electrodynamics.

Next, consider a non-Abelian gauge theory. The Lagrangian of a heavy quark coupled to a non-Abelian gauge potential is

$$\mathcal{L}_Q=\bar{\psi}[\gamma^\mu(i\partial_\mu-gA_\mu)-m]\psi\,, \tag{B9}$$

where A_μ is now a matrix,

$$A_\mu=\sum_a \tfrac{1}{2}\lambda_a A_{\mu a}\,. \tag{B10}$$

We repeat the steps leading to (B5). The only difference is that the operator U now is defined in terms of a path-ordered exponential. So we have

$$\mathcal{L}_Q=\bar{\Psi}[\gamma^\mu(i\partial_\mu-gA_\mu')-m]\Psi\,, \tag{B11}$$

$$A_\mu'=U^{-1}A_\mu U-\frac{i}{g}U^{-1}\partial_\mu U\,, \tag{B12}$$

$$U(\vec{x},t)=P\exp\left(ig\int_0^{\vec{x}} d\vec{x}'\cdot\vec{A}(\vec{x}',t)\right)\,. \tag{B13}$$

To simplify Eq. (B12) we establish several identities satisfied by U. We note first that[26]

$$\partial_0 U=igU(\vec{x},t)\int_0^1 ds\,U^{-1}(s\vec{x},t)\vec{x}\cdot\dot{\vec{A}}(s\vec{x},t)U(s\vec{x},t)\,, \tag{B14}$$

$$\vec{\nabla}U=igU(\vec{x},t)\int_0^1 ds\,U^{-1}(s\vec{x},t)\times\left[s\sum_k x^k\vec{\nabla}A^k(s\vec{x},t)+\vec{A}(s\vec{x},t)\right]\times U(s\vec{x},t)\,. \tag{B15}$$

Introducing

$$a_\mu(s)=U^{-1}(s\vec{x},t)A_\mu(s\vec{x},t)U(s\vec{x},t)\,, \tag{B16}$$

we find

$$\frac{da_\mu}{ds}=U^{-1}(s\vec{x},t)\{(\vec{x}\cdot\vec{\nabla})A_\mu(s\vec{x},t)+ig[A_\mu(s\vec{x},t),\vec{x}\cdot\vec{A}(s\vec{x},t)]\}U(s\vec{x},t)\,. \tag{B17}$$

Integrating Eq. (B17) we get

$$U^{-1}(\vec{x},t)A_0(\vec{x},t)U(\vec{x},t)=a_0(1)=A_0(0,t)+\int_0^1 ds\,U^{-1}(s\vec{x},t)\vec{x}\cdot\{\vec{\nabla}A_0(s\vec{x},t)+ig[A_0(s\vec{x},t),\vec{A}(s\vec{x},t)]\}U(s\vec{x},t)\,, \tag{B18}$$

$$U^{-1}(\vec{x},t)\vec{A}(\vec{x},t)U(\vec{x},t)=\vec{a}(1)=\int_0^1 ds\,U^{-1}(s\vec{x},t)\vec{A}(s\vec{x},t)U(s\vec{x},t)+\int_0^1 ds\,sU^{-1}(s\vec{x},t)\{(\vec{x}\cdot\vec{\nabla})\vec{A}(s\vec{x},t)+ig[\vec{A}(s\vec{x},t),\vec{x}\cdot\vec{A}(s\vec{x},t)]\}U(s\vec{x},t)\,. \tag{B19}$$

When we combine Eqs. (B14), (B15), (B18), and (B19), Eq. (B12) finally becomes

$$A_0'(x)=A_0(0,t)-\int_0^1 ds\,U^{-1}(s\vec{x},t)\vec{x}\cdot\vec{E}(s\vec{x},t)U(s\vec{x},t)\,,$$
$$\vec{A}'(x)=-\int_0^1 ds\,sU^{-1}(s\vec{x},t)\vec{x}\times\vec{B}(s\vec{x},t)U(s\vec{x},t)\,, \tag{B20}$$

which is Eq. (2.15). Equation (B20) is the non-Abelian analog of Eq. (B8). It differs from Eq. (B8) in two aspects: the appearance of the operators U that sandwich the field strength $\vec{E}$ and $\vec{B}$; when expressed in terms of the gauge potential, $\vec{E}$ and $\vec{B}$ contain nonlinear terms of A_μ.

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[9]See, for example, E. Eichten, K. Gottfried, T. Kinoshita, K. Lane, and T.-M. Yan, Phys. Rev. D 21, 203 (1980).

[10]An important ingredient in our analysis is the concept of connected and disconnected vertices introduced by Peskin (Ref. 7). We refer the reader to his paper for a detailed discussion of the concept. It is most easily understood by expressing the amplitude in coordinate space. A vertex is connected if its spatial distances from the heavy quarks shrink to zero with decreasing $Q\bar{Q}$ separation. A vertex is disconnected if its distances from the heavy quarks remain finite as the $Q\bar{Q}$ separation decreases. This concept permits a clear division of a transition amplitude into two parts as shown in Fig. 2. The lower half includes all the connected vertices and the upper half includes all the disconnected vertices. The gauge-invariant effective action to be constructed can then be used to compute the lower half of the amplitude represented by Fig. 2.

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[12]Diagrams involving four-gluon vertices were shown to be nonleading by Peskin (Ref. 7).

[13]The calculation can be done by a method used by D. Gross and S. Treiman, Phys. Rev. D 4, 1059 (1971).

[14]The color-magnetic dipole given here neglects a term that is proportional to the total color charge.

[15]These effects may introduce field-dependent coefficients. We assume that they do not change the tensor character of a multipole moment nor its importance relative to other multipoles. A possibility of this kind may be found in the calculation of J. Sapirstein [Phys. Rev. D 20, 3246 (1979)], who showed that the magnitude of the color-magnetic moment of a quark has a logarithmic dependence on the color-magnetic field.

[16]The use of the Wigner-Eckart theorem in hadronic transitions was anticipated by K. Gottfried, Ref. 4.

[17]If an interaction of the form (2.25) is included, it can be absorbed in the definition of the reduced matrix elements (3.7). Therefore the mass distribution is still given by (3.12).

[18]For a recent discussion of spin-dependent forces in QCD, see E. Eichten and F. Feinberg, Phys. Rev. Lett. 43, 1205 (1979).

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[22]It is not straightforward to implement the soft-pion theorems and the multipole expansion simultaneously. Because of the nonlocality (in time) of the resolvent in the reduced matrix elements (4.2)–(4.4), the usual PCAC reduction formula for soft pions does not lead to a useful result. Nevertheless, there appears no physical reason why PCAC and the multipole expansion should not be compatible.

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[26]I am indebted to Peter Lepage for his help in obtaining these identities.