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Algorithm to compute corrections to the Sudakov form factor

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A general method is explained for computing high-energy processes in gauge theories when "Sudakov double logarithms" are present. An expansion for the electron form factor in quantum electrodynamics is constructed; it gives the leading-logarithmic result of Sudakov, together with corrections to it to arbitrary logarithmic accuracy. The method can be applied to other processes, such as low-transverse-momentum parton phenomena, and form factors and elastic scattering of composite particles.

I. INTRODUCTION

There are many processes in gauge theories whose high-energy behavior has a "double logarithmic" form like that found by Sudakov.[1] Sudakov computed the leading-logarithm series for the electromagnetic form factor of the electron in QED with a result $\exp[-\alpha \ln^2 Q^2/(4\pi)]$. It is not at all obvious that nonleading logarithms do not completely change this result; normal renormalization-group methods only permit a single logarithm per loop, so they cannot be directly relevant. The purpose of this paper is to explain a general method of computing these processes including the corrections.

Although the new method applies to any process involving double logarithms, I explain it in its simplest form by investigating the form factor of an electron in QED with a massive photon. Now the method can only be applied to gauge-invariant quantities, in this case an on-shell matrix element of a gauge-invariant operator. Thus it is necessary to avoid infrared divergences by giving the photon a mass; if the photon were massless it would be necessary to work with some kind of inclusive cross section rather than a form factor. The final result is an expansion given in Sec. IV for the *logarithm* of the form factor. This looks like a generalized operator-product expansion with the operator having an anomalous dimension $\gamma_S(e)\ln(Q^2/\mu^2)$ where Q is the momentum transfer and μ is the renormalization point. Corrections are seen to be of the order $1/Q$, so that for the form factor itself they amount to a *factor* $1+O(1/Q)$. Until now there has been the possibility of an additive power-law correction in the form factor to the leading Sudakov result, which falls faster than any power of Q.

The new method extends immediately to other processes, on which work is in progress.

Mueller[2] has also computed corrections to the Sudakov form factor, but by a less direct method. However, his methods do not immediately extend to such processes of practical importance as parton processes at low transverse momentum, form factors and elastic scattering of composite particles, and the $x \to 1$ behavior of structure functions. However, Mueller does emphasize the usefulness of such an extension.

Although the present paper exclusively treats the Abelian case, the methods of Ref. 3 are available to make the extension to non-Abelian theories.

The basic tools of this paper are those used to prove parton-model-type results in quantum chromodynamics (QCD).[4-8] Let us consider a form factor $V(Q,m,M,\mu,e(\mu))$, where Q is the momentum transfer (which may be timelike or spacelike), m and M are the photon and electron masses, μ is the renormalization mass, and $e(\mu)$ is the electric charge renormalized at μ. We will consider the dominant regions of momentum space for Feynman graphs for V, as $Q\to\infty$. There are as usual[4-7] three types of virtual momenta:

(1) ultraviolet, i.e., off-shell by order Q;

(2) collinear to one or other of the electron momenta p_1, p_2;

(3) soft, i.e., small compared with Q.

The crucial steps will involve application of a Grammer-Yennie approximation[9] for the soft momenta; this has the effect of a gauge transformation, but a different one for each of the two electrons entering V.

The main result is Eq. (4.17) below, where $\ln V$ is given as a sum of three terms. This is a kind of generalization of the operator-product expansion[10] in which there is a momentum-dependent anomalous dimension of the form $\gamma(e)\ln(Q^2/\mu^2)$ (compare Ref. 11). Now the double logarithms arise when a single virtual-photon line is able to be both soft and collinear as its momentum is integrated over. In graphs with many loops, the various soft and collinear contributions overlap in a very complicated way. The crux of the proof of our result is to show that in $\ln V$ the overlap is no worse than in the one-loop case. A substantial cancellation between different graphs is involved. Without this cancellation, the anomalous dimension would have more than just the single logarithm.

In Sec. II I show a simple factorization for the one-particle-irreducible vertex in the Coulomb gauge:

$$V \sim z_2(Q/2)\Gamma_{\rm IR}\Gamma_{\rm UV}\,, \tag{1.1}$$

where z_2 is the (noncovariant) residue of the propagator pole, $\Gamma_{\rm IR}$ is independent of Q, and $\Gamma_{\rm UV}$ is finite when masses are set to zero. Then in Sec. III I obtain a factorization of z_2 (which contains the Sudakov logarithms). Finally, the results are put together in Sec. IV to give a form suitable for calculations in a covariant gauge.

Section V sketches the application of the methods to super-renormalizable theories.

II. FACTORIZATION OF EXCHANGES

A. Definitions

Let us consider the theory of a photon of mass m coupled to one or more charged fields. For convenience of exposition, we choose the case of a single charged scalar field of mass M, with Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}{}^2 + \tfrac{1}{2}m_0{}^2 A_\mu{}^2 + D_\mu\phi^\dagger D^\mu\phi - M_0{}^2\phi^\dagger\phi - \frac{\lambda_0}{4}(\phi^\dagger\phi)^2 + \text{gauge terms}\,, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu - ieA_\mu$. The extension to fermions or to several species of charged particle will be immediate. We call the charged particle an electron.

Let $S_F(p)$ be the renormalized electron propagator and let $G_\mu(p_1, p_2)$ be the renormalized Green's function of an electron and a photon (Fig. 1), with the photon propagator amputated and divided by e_R. We let $q^\mu = p_1^\mu - p_2^\mu$; q^μ may be timelike or spacelike. Then the form factor is

$$V_\mu = z_2(p_1)^{1/2} z_2(p_2)^{1/2}\Gamma_\mu(p_1, p_2)\,, \tag{2.2}$$

where Γ_μ is the one-particle-irreducible (1PI) Green's function and $z_2(p)$ is the residue of $S_F(p)$ at the electron pole. (Note that we will be working in the Coulomb gauge much of the time, so z_2 may depend on $\vec{p}$.) For any particle, V can be decomposed in terms of scalar form factors.

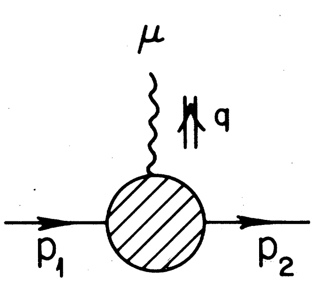


FIG. 1. Electron form factor.

Now V is gauge independent. However, it is convenient to work in the Coulomb gauge in order to construct a method to compute the large-Q behavior ($Q = |q^2|^{1/2}$). The free photon propagator is

$$D^c_{\mu\nu}(k) = \frac{i}{k^2 - m^2 + i\epsilon}\left[-g_{\mu\nu} + \frac{(k_\mu n_\nu + n_\mu k_\nu)n\cdot k}{(m^2 - \tilde{k}^2)n^2} + \frac{k_\mu k_\nu}{\tilde{k}^2 - m^2}\right]\,, \tag{2.3}$$

where n^μ is the timelike gauge-fixing vector and $\tilde{k}^2 \equiv k^2 - (k\cdot n)^2/n^2$, so that $\tilde{k}^2 = -\vec{k}^2$ in the rest frame of n^μ. Note that $D^c_{\mu\nu}$ is invariant under scaling of n^μ to λn^μ.

Renormalization is most conveniently done by dimensional renormalization,[12] though this is not essential. It will be necessary in any practical calculation to relate e_R, m_R, M_R to their physical values e, m, M.

It is convenient to choose $\vec{p}_1$ and $\vec{p}_2$, the electron momenta, to be along the z axis and at various stages we will use light-cone coordinates where $V^\pm = (V^0 \pm V^3)/\sqrt{2}$, $V_T = (V^1, V^2)$ for any vector V^μ. In the overall center-of-mass frame $n^\mu \propto (1, 1, O_T)$ in $(+, -, T)$ coordinates, while for the timelike form factor $q^\mu = (Q/\sqrt{2}, Q/\sqrt{2}, O_T)$ and for the spacelike case $q^\mu = (Q/\sqrt{2}, -Q/\sqrt{2}, O_T)$.

B. Leading graphs for Γ

In the Coulomb gauge we can use the methods of Refs. 4–8 to find the dominant regions of momentum space in graphs for the 1PI form factor Γ at large Q^2. These have the structure of Fig. 2. There J_1 and J_2 are two "jets,"[4-8] i.e., the momenta of their lines are collinear to p_1 and p_2, respectively. All internal lines of the "hard vertex" H are off-shell by an amount of order Q. There is an arbitrary number of connected graphs S_i which exchange soft momenta [i.e., $o(Q)$] between J_1 and J_2. All external lines of the S_i are photons.

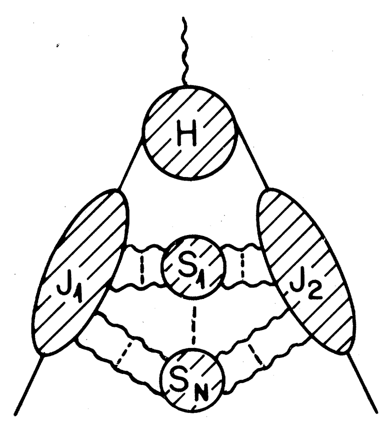


FIG. 2. Leading graphs for Γ^μ have this structure.

It is important that the statements in the previous paragraph are true for the leading power Q^0 with all its logarithmic corrections and not merely for the leading logarithms. Now any graph for Γ can be written in the topology of Fig. 2, and this decomposition is unique provided we observe the following rules:

(1) Each S_i is connected and only has photons for external legs.

(2) No external photon of S_i attaches to an internal electron loop; that is, every external line of S_i attaches to J_1 or J_2 at a vertex that is connected to the external electrons by a path involving electron lines only.

(3) H has no nontrivial decomposition into a graph of the form of Fig. 2.

The purpose of rule (2) is to avoid ambiguities from graphs such as Fig. 3 with internal loops in an S_i. Note that at this stage we have made a topological decomposition and we have made no restriction on the external momenta of S_i.

Let us go back to the power-counting arguments of Refs. 4–8. We find the following:

(a) All momenta in H are off-shell by order Q; thus H is a reduced vertex in the sense of Ref. 4 since all collinear or soft lines are in a J_i or S_i.

(b) Momenta in J_i or S_i may be soft, collinear or off-shell. We can incorporate off-shell lines into reduced vertices of the form of interaction vertices.

But:

(c) Although it is possible for some external lines of an S_i to be collinear, there must overall be only soft-photon exchange between J_1 and J_2, as in Fig. 4. The collinear lines are connected to the soft photons by collinear electron *loops*. Thus we can apply the Grammer-Yennie approximation of the next subsection to the soft lines attaching to these electron loops. Summing over all graphs and applying Ward identities gives zero. Hence after summing over all graphs, the external lines of S_i can only be soft or off-shell.

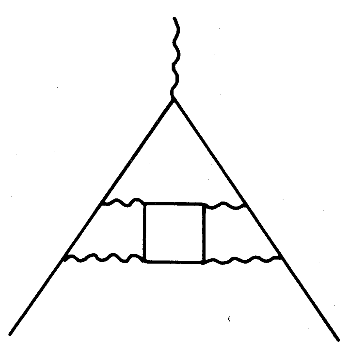


FIG. 3. Graph with potentially ambiguous decomposition into form of Fig. 2.

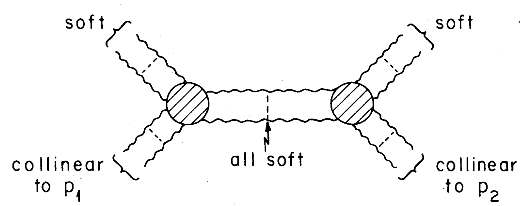


FIG. 4. Momentum-space structure of an S_i.

C. Grammer-Yennie approximation

We will now define the Grammer-Yennie approximation,[9] K_i, of S_i. This will equal S_i in the limit that all its external lines are soft, and in general it will obey the same power counting as S_i. Let us write

$$G_i \equiv S_i - K_i \tag{2.4}$$

for the remainder of S_i. Then the 1PI Green's function Γ is a sum over all versions of Fig. 2 when each S_i is replaced by either a K_i or a G_i. Since K_i is a good approximation to S_i when its external lines are soft, Eq. (2.4) says that exchange of a G_i is dominated by off-shell lines. To get a leading power we must have a structure of the form of Fig. 5. There, all parts of a graph forced by the presence of a G_i to be off-shell have been absorbed into a new hard vertex Γ_{UV}. Finally we will be able to apply Ward identities to the K_i's to obtain a simple result.

The Grammer-Yennie approximation is obtained by noticing that if a soft photon is attached to a collinear scalar electron (Fig. 6) then the summation over μ in

$$(2p^\mu + k^\mu) D^c_{\mu\nu}(k)$$

is dominated by $\mu = +$ $(-)$ if p is collinear to p_1 (p_2). The same result holds for any coupling of a soft photon to a collinear electron, and follows

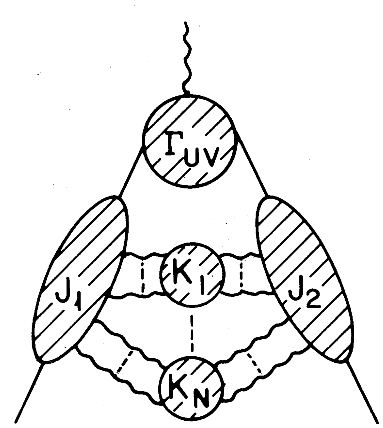


FIG. 5. Structure obtained after writing $S_i = G_i + K_i$.

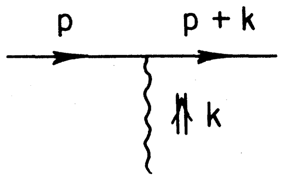


FIG. 6. Coupling of a soft photon k^μ to a collinear electron.

from Refs. 4–8. Thus when k is soft and p collinear to p_1, we may replace $D^c_{\mu\nu}$ by its Grammer-Yennie approximation[9]

$$K^1_{\mu\nu} = k_\mu \omega_1^\kappa D^c_{\kappa\nu}/(\omega_1 \cdot k + i\epsilon), \tag{2.5}$$

where ω_1 is any light-like vector collinear to p_1, e.g., $\omega_1^\mu = (1, 0, 0_T)$ in $(+, -, T)$ coordinates. The $i\epsilon$ in Eq. (2.5) follows from the arguments of Ref. 3; it is to reproduce the $i\epsilon$ of the propagators of the collinear electron lines in Fig. 2. Similarly, when p is collinear to p_2, we use

$$K^2_{\mu\nu} = k_\mu \omega_2^\kappa D^c_{\kappa\nu}/(\omega_2 \cdot k \pm i\epsilon), \tag{2.6}$$

where $\omega_2^\mu = (0, 1, 0_T)$ is collinear to p_2. The $+i\epsilon$ or $-i\epsilon$ in Eq. (2.6) is to be taken according as p_2 is incoming (q^μ timelike) or outgoing (q^μ spacelike). Notice that both (2.5) and (2.6) are invariant under scaling of ω_1 and ω_2.

Now we can define K_i to be S_i with each external photon propagator replaced by $K^1_{\mu\nu}$ or $K^2_{\mu\nu}$ according as it is attached to J_1 or J_2. Of course, exchange of a single photon (e.g., Fig. 7) requires

$$K^{12}_{\mu\nu} = \frac{k_\mu \omega_1^\kappa D^c_{\kappa\lambda} \omega_2^\lambda k_\nu}{(\omega_1 \cdot k + i\epsilon)(\omega_2 \cdot k \mp i\epsilon)}, \tag{2.7}$$

where the reversed $i\epsilon$ for $\omega_2 \cdot k$ compared with (2.6) happens because k flows out of J_2. Then G_i is defined by Eq. (2.4).

Because of the factor k_μ in Eqs. (2.5) and (2.6) we will be able to apply Ward identities to simplify the exchanges of K_i's. Notice that we make no restriction on the momenta of the K_i graphs, even though we defined K_i in order to have a useful approximation to S_i in the soft region.

Now when we set $S_i = G_i + K_i$ in Fig. 2, many K_i's are forced to be off-shell because of the presence of a G. An example is shown in Fig. 8. Thus it is useful to use Fig. 5 to define Γ_{UV}, which is to have no nontrivial decomposition of the form of Fig. 5. All soft and collinear contributions are isolated into the K_i's and the J_i's, respectively.

D. Proof of factorization

Application of Ward identities to the external lines of every K_i in Fig. 5 where they attach to J_1 or J_2 gives Fig. 9. The external electron lines p_1 and p_2 are, of course, amputated and on-shell. The structure E is as follows.

It is a sum over all graphs with external photon lines. Each photon line is attached to the vertex where p_1 or p_2 enters Γ_{UV}. The rule for a line of E attached to p_1 is that its propagator is replaced by

$$\frac{-e_R \omega_1^\kappa D^c_{\kappa\nu}(k)}{\omega_1 \cdot k + i\epsilon} = \frac{-e_R D^{c-}{}_\nu}{k^- + i\epsilon}, \tag{2.8}$$

when k is outgoing from E. When the line is attached to p_2 we have

$$\frac{e_R \omega_2^\mu D^c_{\mu\nu}}{\omega_2 \cdot k \pm i\epsilon} \tag{2.9}$$

with the same $i\epsilon$ as in Eq. (2.6). Finally, for a line going direct from p_1 to p_2 we have

$$\frac{e_R{}^2 \omega_1^\mu D^c_{\mu\nu} \omega_2^\nu}{(\omega_1 \cdot k + i\epsilon)(\omega_2 \cdot k \mp i\epsilon)} = \frac{e_R{}^2 D^c_{+-}}{(k^- + i\epsilon)(k^+ \mp i\epsilon)}. \tag{2.10}$$

For each set of N identical connected graphs in E there is a factor $1/N!$.

It is interesting to note that individual graphs for Fig. 5 have collinear contributions. Thus

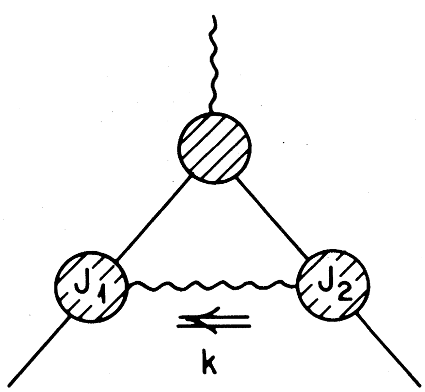


FIG. 7. Illustrating Eq. (2.7).

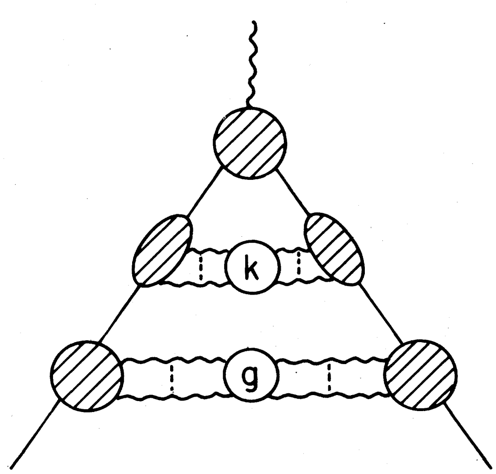


FIG. 8. The momenta in K cannot be soft in this graph.

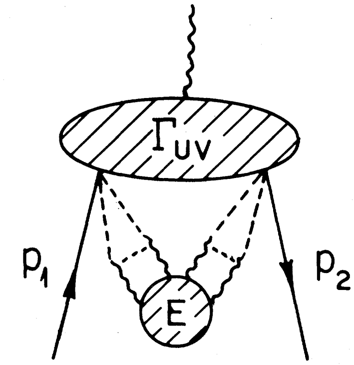


FIG. 9. Result of applying Ward identities to Fig. 5.

graphs (a) and (c) of Fig. 10 behave as $\ln^3 Q^2$ as $Q\rightarrow\infty$. After replacing the exchanged photon by its Grammer-Yennie approximation (2.7) we get zero for the sum of Fig. 10. The remainder, with the photon replaced by its G version, has no soft and no collinear contribution. Hence it is dominated by ultraviolet momenta, and its leading behavior is $\ln^2 Q$ with both logarithms renormalization-group controlled.

Returning to Fig. 9, we note that there are ultraviolet divergences involving E. In addition we would like to display the factorized contribution coming from soft momenta in E. Now exchanges of G_i's and K_i's have the same ultraviolet power-counting as S_i's, so there is an UV divergence in Γ_{UV}. This can be subtracted graph-by-graph by use of the forest formula.[10,13] There will be two sorts of extra counterterms:

(a) Those for a graph or subgraph of the topology of a contribution to Γ; such divergences are multiplicatively renormalized, by a factor Z_4, say.

(b) Other subgraphs within J_1, J_2, and the S_i's with some K photons in the interior of the subgraphs. Examples are the boxed subgraphs in Fig. 11. The counterterms cancel after summing over

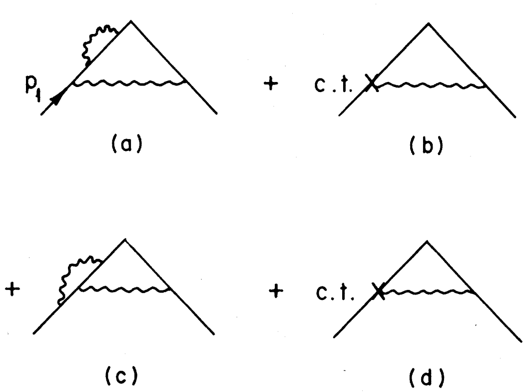


FIG. 10. A triply logarithmic contribution to Fig. 2 or Fig. 5. Graphs (b) and (d) contain counterterms for subdivergences of (a) and (c).

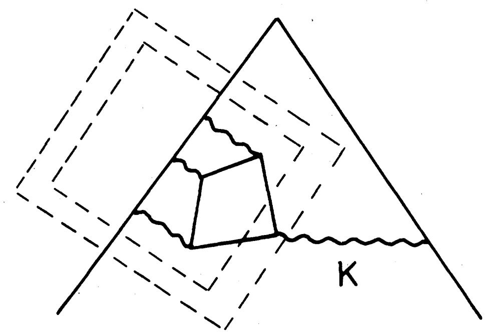


FIG. 11. The boxed subgraphs are divergent; the line K has (2.7) as its propagator.

a gauge-invariant set of graphs.

To get the cancellation to work without paying special attention to the renormalization prescription, it is convenient to use dimensional renormalization,[12,13] where only the divergent part is subtracted.

The only divergences involving E give an overall multiplicative renormalization Z_4^{-1}; this has to cancel the divergence in Γ_{UV} since the total is just the 1PI vertex Γ, which has no knowledge of Grammer and Yennie.

Finally we must construct a factorized form for Γ. To do this we consider a connected contribution $E_c(k)$ to E, where k is the total momentum transfer. We write

$$E_c(k)=\delta^{(4)}(k)\int d^4\rho\, E_c(\rho)+E_c(k)-\delta^{(4)}(k)\int d^4\rho\, E_c(\rho)$$
$$=\delta^{(4)}(k)\int d^4\rho\, E_c(\rho)+\tilde{E}_c(k)\,. \tag{2.11}$$

When $\tilde{E}_c(k)$ is convoluted with Γ_{UV} the soft contribution cancels. Hence we have

$$\Gamma=\tilde{\Gamma}_{RUV}\exp\left[\int d^d k E_c(k)-\ln Z_5\right]$$
$$\equiv\tilde{\Gamma}_{RUV}\exp(\mathcal{E}_R)\,. \tag{2.12}$$

Here $\tilde{\Gamma}_{RUV}$ is Fig. 9 with every connected subgraph of E replaced by its $\tilde{E}_c$. The result is multiplicatively renormalized by a factor of Z_5. All the soft contributions are contained in the exponential, where E_c is now the sum over all connected graphs for E, and the term $-\ln Z_5$ exactly cancels the UV divergence at $d=4$.

Since $\tilde{\Gamma}_{RUV}$ has no soft or collinear contributions we can take the limit that all masses are zero and obtain

$$\Gamma\sim e^{\mathcal{E}_R}\tilde{\Gamma}_{RUV}(m=M=p_1^2=p_2^2=0) \tag{2.13}$$

as $Q\rightarrow\infty$.

Rather than going through its long sequence of definitions we can compute $\tilde{\Gamma}_{RUV}$ as follows. First compute $\mathcal{E}_R$ from its definition, as illustrated in

Fig. 12. This calculation gives Z_5. Next we set all masses to zero in $\mathcal{E}_R$ and in Γ to obtain $\tilde{\Gamma}_{RUV}$ from (2.13). Dimensional continuation is used to regulate infrared divergences.[14] We directly compute Γ. Next note that after integration over k^0 and k^3 in Fig. 12 each graph for $\mathcal{E}_R$ is the integral over k_T of a power of k_T; there are no masses available to give a scale. Such integrals are zero.[15] Thus $\tilde{\Gamma}_{RUV}=\Gamma$ (zero mass) Z_5.

The integral over k^0 and k^3 in $\mathcal{E}_R$ is convergent even though the denominators in (2.8)–(2.10) are $1/k^+$ or $1/k^-$ rather than $1/(p_1\cdot k+k^2)$ or $1/(p_2\cdot k+k^2)$. This is because there is no leading collinear contribution. We will find the opposite situation when we investigate the self-energy.

Finally let us note the renormalization-group equations for $\mathcal{E}_R$ and $\tilde{\Gamma}$:

$$\mathfrak{D}\mathcal{E}_R=-\gamma_5\,, \tag{2.14}$$

$$\mathfrak{D}\tilde{\Gamma}_{RUV}=(\gamma_2+\gamma_5)\tilde{\Gamma}_{RUV}\,, \tag{2.15}$$

where

$$\mathfrak{D}=\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial e_R}-\gamma_M M_R{}^2\frac{\partial}{\partial M_R{}^2}+\gamma_3 M_R{}^2\frac{\partial}{\partial M_R{}^2}\right)\,, \tag{2.16}$$

$$\gamma_5=\beta\frac{\partial}{\partial e_R}\ln Z_5\,, \tag{2.17}$$

and β, γ_2, γ_3, and γ_M have their usual definitions.[16]

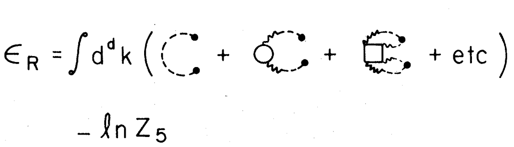


FIG. 12. Definition of $\mathcal{E}_R$.

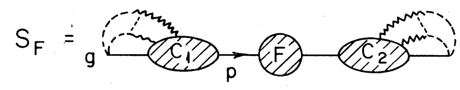
III. FACTORIZATION OF $z_2(p)$

A. Grammer-Yennie

Now the renormalized electron propagator S_F is a function of p^2 and $n\cdot p/\sqrt{n^2}\simeq Q/2$. We wish to find its residue $z_2(p\cdot n/\sqrt{n^2})$ at $p^2=M^2$. First we write the free photon propagator as

$$D^c_{\mu\nu}=\frac{i}{k^2-m^2+i\epsilon}\times\left\{-g_{\mu\nu}+\left[\frac{k_\mu(n_\nu n\cdot k/n^2-\frac{1}{2}k_\nu)}{m^2-\vec{k}^2}+\mu\longleftrightarrow\nu\right]\right\}\,. \tag{3.1}$$

Then we apply Ward identities to the k_μ factor in (3.1), whenever $D^c_{\mu\nu}$ occurs in S_F. This results in Fig. 13, where the internal photon propagators are in the Feynman gauge (i.e., $-g_{\mu\nu}/(k^2-m^2+i\epsilon)$), while the extra vertices with their attached photons are defined by Fig. 14. They have a label "g" (for "gauge") to distinguish then from the Grammer-Yennie vertices which we will define shortly. Also, $F(p)$ is a complete renormalized electron propagator in the Feynman gauge, and C_1 and C_2 are 1PI in the electron lines attached to F. Moreover $C_1=C_2$ ($=C$, say) by charge-conjugation invariance, and C^2 is multiplicatively renormalized by the ratio of the Coulomb to the Feynman gauge wave-function renormalizations. Note that in lowest order $C=1$, with no g vertices.



+ terms with no electron pole

FIG. 13. The Coulomb-gauge electron propagator written in terms of the Feynman-gauge quantities.

Now since the electron pole is totally contained in F, we have

$$z_2\left(\frac{Q}{2}\right)=z_{2F}C_R{}^2(Q)\,, \tag{3.2}$$

where z_{2F} is the residue of the pole of F_R, and C_R is evaluated on the electron mass shell $p^2=M^2$.

If it were not for the gauge-fixing vector n^μ, the momenta contributing to C_R would be collinear to p^μ. But we also have important contributions from soft momenta—these are small and have rapidity in the rest frame of n^μ that is much less than $\ln(Q/M)$. We wish to take $Q\to\infty$ and it is necessary to identify the regions that give a leading contribution to C_R.

Following Refs. 4–6 we define soft momenta to be loop momenta k^μ with $k^\mu\sim\lambda Q$ in the rest frame of n^μ with $\lambda\to 0$. Collinear momenta have

$$k^+\simeq xp^+\,,\quad k^-\sim\lambda Q\,,\quad k_T\sim\lambda^{1/2}Q\,,$$

where $x\neq 0$ and $\lambda\to 0$. There are also important ultraviolet contributions to C with all components of k^μ of order Q. The Sudakov double logarithms come from the region interpolating between soft and collinear momenta.

The graphs for C_R can be decomposed as in Fig. 15, where the external lines of S_l are photons. The decomposition is unique provided we copy

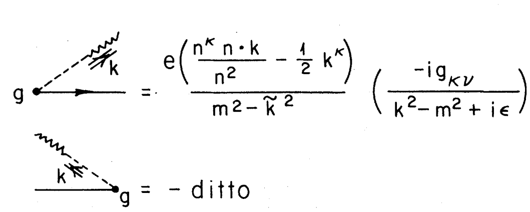


FIG. 14. Definitions of the g vertices in Fig. 13.

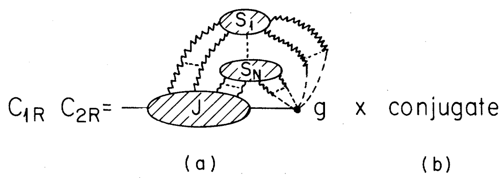


FIG. 15. Graphs for $C_R{}^2$.

the rules for the S_i's in Sec. II. (Every connected component of an S_i is attached to a g vertex. It may or may not have external photons coupling to the jet J; but it it does, then the photon must couple to the exhibited electron line, or to any electron attached to it by ϕ^4 vertices.)

As in Sec. II, we split each connected graph S_i into a "K graph" and a "G graph." Each line of a K graph flowing into the "jet" J is given its Grammer-Yennie approximation shown in Fig. 16, while G is the remainder. All soft contributions are in the K graphs and in Sec. III B we will apply the Ward identities. Notice that the $i\epsilon$'s for the K photons in Figs. 15(a) and 15(b) are different. This is why we did not apply the Grammer-Yennie approximation directly to the propagator for we would not have known which sign of $i\epsilon$ to use. The factor $\frac{1}{2}$ with the k^2 is arbitrary; any other nonzero positive number would do equally well except for some complications in lowest-order calculations. Similarly the $p_1^\kappa \pm \frac{1}{2}k^\kappa$ can be replaced by p_1^κ except for the same complications. However, we cannot replace $p_1 \cdot k \pm \frac{1}{2}k^2$ by $\omega_1 \cdot k$ for we would obtain spurious divergences. The $1/\omega_1 \cdot k$ would reproduce the soft region correctly but the k^+ integral would diverge logarithmically at large k^+ (if the k^- integral is performed first). This happens since there is both a soft and a collinear contribution. Whereas $p_1 \cdot k \gg k^2$ in the soft region, the two terms are comparable in the collinear region, and k^2 dominates when $k^+ > p^+$. It is necessary for the K photons to provide a quantitatively good approximation in the soft region and at least a qualitative approximation elsewhere. In fact the definitions of Fig. 16 imply that our Grammer-Yennie approximation is exact, but only at the one-loop level, of course.

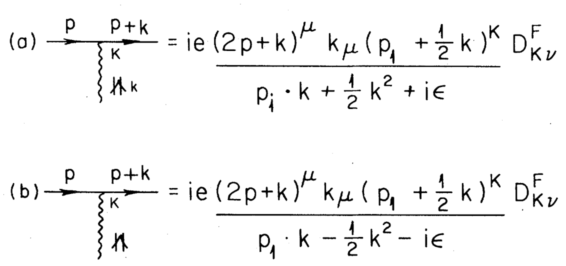


FIG. 16. Definitions of Grammer-Yennie approximations.

B. Factorization

We now apply Ward identities to the K photons to obtain Fig. 17. The rules for the K vertices in Fig. 17 are given in Fig. 18. Thus we have a factorization

$$z_2(\tfrac{1}{2}Q)=G_R{}^2(\tfrac{1}{2}Q)z_{2F}\exp\left(2\int Bd^dk-\ln Z_6\right). \qquad (3.3)$$

Here we have set $n\cdot p/\sqrt{n^2}=Q/2$, and we have defined $\int Bd^dk$ to be the set of connected graphs for the second factor of Fig. 17, as illustrated in Fig. 19. To exhibit the overall ultraviolet divergence in $\int Bd^dk$, which is renormalized by a term $\frac{1}{2}\ln Z_6$, we have displayed the momentum coming out of the k vertices and into the g vertices. G_R is the renormalized contribution of the graphs with the Grammer-Yennie contribution subtracted. It contains a renormalization factor $Z_6{}^{1/2}$. By Heckathorn's[17] argument Z_6 is independent of Q.

All the soft contributions are now in B, which we will discuss in Sec. III C. The Q dependence of G_R comes from large transverse momentum, and we will compute its large Q asymptote by an operator-product expansion in Sec. III D.

C. Computation of $\int B$

Now B is the set of graphs in Fig. 19, with k being the momentum leaving at the k vertices and entering at the g vertices; all other loop integrals are performed inside B. We will examine carefully the integral over k, since it is the source of the Sudakov double logarithms. In the case that B has no k vertices there is no k integral to perform, and we then define B to have a factor $\delta^{(4)}(k)$.

If we work in the rest frame of p^μ and if $n^+ \to 0$ (corresponding to $Q\to\infty$), then the g vertex of Fig. 14 goes to

$$\frac{-ien_\nu}{k\cdot n(k^2-m^2+i\epsilon)}. \qquad (3.4)$$

This has a singularity at $k^+=0$ and there is no $i\epsilon$ prescription to avoid it. Its neighborhood is the soft region and gives rise to the double logarithms. Note that the only knowledge that $\int B$ has of Q comes from the g and k vertices.

It is convenient to integrate first over k^- at fixed k^+ and k_T, and to write

$$\tilde{B}[(k_T, k^+(p_1^-/p_1^+)^{1/2}, k^+(n^-/n^+)^{1/2}] \equiv k^+\int dk^- B(k)\,, \qquad (3.5)$$

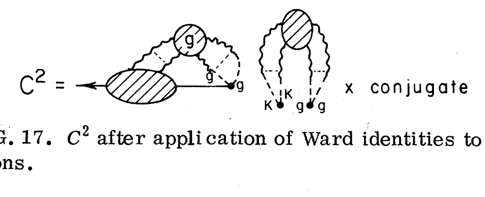
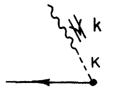


FIG. 17. C^2 after application of Ward identities to K photons.



$$= \frac{e\,(p_1 + \frac{1}{2}k)^{\kappa}\, D^F_{\kappa\nu}}{p_1 \cdot k + \frac{1}{2}k^2 + i\epsilon}$$

FIG. 18. Definition of K vertices in Fig. 17.

where the explicit factor k^+ gives invariance under boosts in the z direction. The only (z-boost-invariant) quantities on which $\tilde{B}$ can depend are k_T, $k^+(p_1^-/p_1^+)^{1/2}$, and $k^+(n^-/n^+)^{1/2}$, since p^+p^- is fixed at $M^2/2$ and $\tilde{B}$ is invariant under scaling of n^μ.

Next we integrate over k^+. A potential problem arises since the three momentum variables on which $\tilde{B}$ depends can have widely different values. Thus large logarithms which invalidate the use of perturbation theory can occur. *Showing how this problem does not in fact exist is the key step in this paper.*

Let us perform the k^+ integration to obtain

$$\overline{B}(k_T, p_1^+n^-/(p_1^-n^+)) \equiv \int dk^+dk^-B$$
$$= \int \frac{dk^+}{k^+}\tilde{B}\,, \qquad (3.6)$$

where $p_1^+n^-/(p_1^-n^+) = Q^2/M^2 + O(1)$ as $Q \to \infty$. When k_T is of order Q or larger we can safely say that all momenta are of order k_T, since B is connected, so that we have a purely ultraviolet problem. This is controlled by the renormalization group with the aid of an operator-product expansion, as we will show shortly. So we will now examine the region $k_T \ll Q$.

The crucial point is that there are three regions to distinguish, in each of which either p^- or n^+ or both is effectively zero so that one or both of the variables in Eq. (3.5) that depend on k^+ will drop out.

Soft region: $k^+ \sim |k_T|(n^+/n^-)^{1/2}$. Here we can treat p^- as zero: The only dependence on p^- comes from k vertices when the momentum through it has its $k^- \ll k^+$, i.e., the region of "Coulomb momenta" $p_1 \cdot k = 0$.[3] But the contour of integration can be deformed away.[3,18] In the rest frame of n^μ the dominant region of integration (by a power) is[4] k^+, $k^- \sim k_T$ (or masses) so effectively all momenta are order k_T.

Collinear region: $k^+ \sim -p^+$. We would like to set $n^+ = 0$ so that $n^2 = 0$ and $k^+(n^-/n^+)^{1/2} \to \infty$. The limit exists provided we do not meet any $k \cdot n$ singularities. These singularities are only important if some g vertices have, in fact, $|k^+| \ll p^+$, i.e., if there are soft lines. But some other lines are collinear, and as B is connected the soft lines must couple to a loop of collinear electron lines by power counting,[4] as in Fig. 20. Thus application of the Grammer-Yennie approximation and of Ward identities shows that the total is zero. Hence we can set $n^+ = 0$ for the leading power in Q, but only after summing over a gauge-invariant set of graphs.

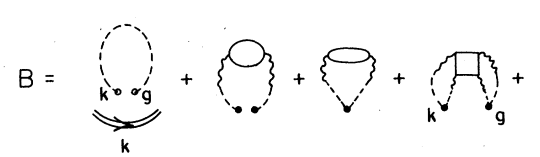


FIG. 19. Definition of B.

Intermediate region: $p^+ \gg -k^+ \gg [(k_T^2 + m^2)n^+/n^-]^{1/2}$. This region interpolates between the collinear and soft regions, and we can apply the results for both regions to set $p^- = n^+ = 0$ to give $\tilde{B}(k_T, 0, \infty)$ for the leading power. Consequently, $\overline{B}$ has a contribution of order $\tilde{B}(k_T, 0, \infty)\ln[Q/(k_T^2 + m^2)^{1/2}]$ from this region.

The limits $k^+ \to \infty$ or $-\infty$ can be considered as end points of the soft and collinear region, respectively, with the k^+-integral convergent by a power.

The result of the above is that in fact the only important momentum scale is k_T. Then if $Q \gg k_T$, m, M we have

$$2\overline{B}(k_T, Q, m, M, \mu, e) = f_1(k_T, m, M, \mu, e)\ln Q^2 + f_2(k_T, m, M, \mu, e) + O(1/Q)\,, \qquad (3.7)$$

where we deliberately choose not to fix the scale that comes with Q^2 in $\ln Q^2$. Since by power counting f_1 and f_2 are of order $1/k_T^2$ for large k_T, the Sudakov *double* logarithm comes from integrating the lowest-order term in f_1 up to $k_T \sim Q$ and multiplying by the explicit $\ln Q^2$ in (3.7).

To separate the behavior at small and large k_T we write

$$2\overline{B} = (f_1 \ln Q^2 + f_2)Q^2/(Q^2 + k_T^2) + 2\overline{B}_{\rm rem}(k_T, Q, m, M, \mu, e)\,. \qquad (3.8)$$

The factor $Q^2/(Q^2 + k_T^2)$ means that while

$$\int d^2k_T(f_1 \ln Q^2 + f_2)Q^2/(Q^2 + k_T^2)$$

is convergent at large k_T it also reproduces the complete $k_T \ll Q$ region. Since then $\overline{B}_{\rm rem} \to 0$ as

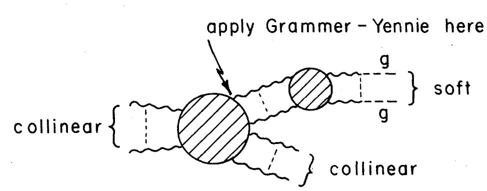


FIG. 20. Showing decoupling of soft and collinear contributions to B.

$k_T/Q\to 0$ we can set $m=M=0$ in $\overline{B}_{\rm rem}$ without encountering IR divergences.

Hence

$$2\int B-\ln Z_6=\int d^2k_T(f_1\ln Q^2+f_2)\frac{Q^2}{Q^2+k_T^2}+\int d^{d-2}k_T\overline{B}_{\rm rem}(k_T,Q,0,0,\mu,e)-\ln Z_6+O(1/Q)\,. \qquad (3.9)$$

We need to control the large-Q behavior of (3.9) by the renormalization group, for which purpose we notice that $\mathfrak{D}B=0$ so that $\mathfrak{D}f_1=\mathfrak{D}f_2=\mathfrak{D}\overline{B}_{\rm rem}=0$, where $\mathfrak{D}$ is defined by Eq. (2.16).

The large Q expansion of (3.9) is obtained by writing

$$S_1=-\int d^2k_T\left(\frac{Q^2}{Q^2+k_T^2}-\frac{\mu^2}{\mu^2+k_T^2}\right)\times[f_1(m=M=0)\ln Q^2+f_2(m=M=0)]-\int d^{d-2}k_TB_{\rm rem}(m=M=0)+\ln Z_6\,, \qquad (3.10)$$

$$I_1=-\int d^2k_T\left[f_1-\frac{k_T^2}{k_T^2+\mu^2}f_1(m=M=0)\right]\,, \qquad (3.11)$$

$$S_2=-\int d^2k_T\left\{f_2+f_1\ln\mu^2-\frac{k_T^2}{k_T^2+\mu^2}[f_2(m=M=0)+\ln\mu^2 f_1(m=M=0)]\right\}\,, \qquad (3.12)$$

so that

$$2\int B-\ln Z_6=-S_1-I_1\ln(Q^2/\mu^2)-S_2+O(1/Q)\,, \qquad (3.13)$$

and

$$\mathfrak{D}S_1=\gamma_s\ln(Q^2/\mu^2)+\gamma_{s'}+\gamma_6\,,$$
$$\mathfrak{D}I_1=-\gamma_s\,, \qquad (3.14)$$
$$\mathfrak{D}S_2=2I_1-\gamma_{s'}\,,$$

where

$$\gamma_s=2\int d^2k_T\frac{\mu^2k_T^2}{(\mu^2+k_T^2)^2}f_1(k_T,0,0,\mu,e)\,,$$

$$\gamma_{s'}=2\int d^2k_T\frac{\mu^2k_T^2}{(\mu^2+k_T^2)^2}[f_2(k_T,0,0,\mu,e)+\ln\mu^2f_1(k_T,0,0,\mu,e)]\,, \qquad (3.15)$$

$$\gamma_6=\beta\frac{\partial}{\partial e_R}\ln Z_6\,.$$

D. *OPE* for G_R

To the extent that its virtual lines have transverse momenta much less than Q, we see that G_R in Eq. (3.3) has a finite limit as $Q\to\infty$, which is obtained by setting $n^-=0$. This happens because in taking $n^-=0$, i.e., $n^2=0$, any divergences arise from $1/k\cdot n$ singularities at the g vertices. This divergence, at $k^-=0$, is in the soft region if $k_T\ll Q$, and, by the very definition of G_R, it has been subtracted off. Thus the problem of the $Q\to\infty$ limit of G_R is a problem in UV behavior. An immediate consequence is that the dominant contribution is of the form of Fig. 21. Here the momentum p^μ has small transverse momentum, while all lines in J', G' have transverse momentum of order Q. Transverse momenta in G and J are much less than Q. Thus we have an operator-product expansion

$$G_R\sim G_{R\rm IR}(m,M,n^-=0,\mu,e(\mu))G_{R\rm UV}(Q/\mu,e(\mu))\,. \qquad (3.16)$$

Here $G_{R\rm IR}$ is G_R with n^- set to zero. Since there is no longer any cutoff on k_T, $G_{R\rm IR}$ has an ultraviolet divergence which is multiplicatively renormalized by a factor Z_7^{-1} so that

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial e_R}\right)G_{R\rm IR}=\gamma_7(e_R)G_{R\rm IR}\,, \qquad (3.17)$$

where

$$\gamma_7=\beta\frac{\partial}{\partial e_R}\ln Z_7\,. \qquad (3.18)$$

IV. CALCULATIONS

Let us combine the results of Secs. II and III, viz., Eqs. (2.2), (2.13), (3.3), (3.13), and (3.16). It is convenient to take the logarithm of the form factor so that we have

$$\ln V=-U-I_1\ln(Q^2/\mu^2)-I_2+O(1/Q)\,, \qquad (4.1)$$

where I_1 is defined by Eq. (3.11), while

$$I_2=S_2-\mathcal{E}_R-2\ln G_{R\rm IR}-\ln z_{2F}\,,$$
$$U=S_1-2\ln G_{R\rm UV}-\ln\tilde{\Gamma}_{R\rm UV}\,. \qquad (4.2)$$

The quantities in Eq. (4.1) satisfy the following renormalization-group (RG) equations:

$$\mathfrak{D}\ln V=0\,, \qquad (4.3a)$$

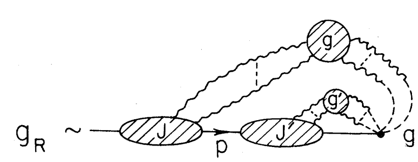


FIG. 21. Structure of dominant region for G_R.

$$\mathfrak{D}U=\gamma_s \ln(Q^2/\mu^2)+\gamma_R\,, \tag{4.3b}$$

$$\mathfrak{D}I_1=-\gamma_s\,, \tag{4.3c}$$

$$\mathfrak{D}I_2=-\gamma_R+2I_1\,, \tag{4.3d}$$

where γ_s is defined by Eq. (3.15) and

$$\gamma_R\equiv\gamma_{s'}+\gamma_6-2\gamma_7-\gamma_s-\gamma_{2C}\,. \tag{4.4}$$

The large-Q^2 behavior can be conveniently obtained by choosing μ of order m and M in I_1 and I_2. Then we have to write U in terms of its value when $\mu=Q$:

$$U(Q/\mu,\bar{e}(\mu))=-\int_\mu^Q \frac{d\mu'}{\mu'}\left[\gamma_R(\bar{e}(\mu'))+\gamma_s(\bar{e}(\mu'))\ln\frac{Q^2}{\mu'^2}\right] +U(1,\bar{e}(Q))\,, \tag{4.5}$$

where the effective coupling $\bar{e}(\mu)$ satisfies, as usual,

$$\mu\frac{d}{d\mu}\bar{e}(\mu)=\beta(\bar{e}(\mu))\,. \tag{4.6}$$

Hence

$$V=e^{-I_2(m,M,\mu)}\left(\frac{\mu^2}{Q^2}\right)^{I_1(m,M,\mu)}\exp\left\{-U(1,\bar{e}(Q))+\int_\mu^Q\frac{d\mu'}{\mu'}\left[\gamma_R(\bar{e}(\mu'))+\gamma_s(\bar{e}(\mu'))\ln\frac{Q^2}{\mu'^2}\right]\right\}[1+O(1/Q)]\,. \tag{4.7}$$

There is an overall normalization e^{-I_2} and a power $-I_1$ of Q^2 that are determined by infrared phenomena. In QED these are of course computable in perturbation theory. However, in view of extensions of the methods of this paper to QCD, it is useful to consider what would happen if QED were asymptotically free. Then I_1 and I_2 would have to be taken from experiment. But then it becomes important to see how they vary for the various form factors we might wish to consider. The variation is an ultraviolet problem.

It is also convenient at this stage to point out that there is a certain arbitrariness in our constructions of the terms in Eq. (4.2)—as can be seen, for example, in Fig. 16 where the $\frac{1}{2}$ can be replaced by any nonzero number. For calculational purposes it is unnecessary to know anything other than that Eqs. (4.1) and (4.3) are true and that U, γ_R, and γ_s have valid perturbation expansions. The purpose of our work in Secs. II and III is to assure us that this last sentence is in fact true and that the quantities considered can be defined outside of perturbation theory.

Suppose we compute $\ln V$ in perturbation theory:

$$\ln V=-\sum_{N,P}A_{NP}e^{2N}\ln^P(Q^2/\mu^2)+O(1/Q)\,, \tag{4.8}$$

where A_{N1} and A_{N0} may depend on m/μ and M/μ. The calculation can be done in the Feynman gauge, since V is gauge independent, and, given Eqs. (4.1) and (4.3), we would like to compute U, I_1, I_2, γ_S, and γ_R. However, the result is not unique, for if we make the transformation

$$I_1\to I_1+f(e)\,, \tag{4.9a}$$

$$I_2\to I_2+g(e)\,, \tag{4.9b}$$

then V and Eqs. (4.1) and (4.3) are invariant if we also let

$$U\to U-f\ln(Q^2/\mu^2)-g\,, \tag{4.9c}$$

$$\gamma_S\to\gamma_S-\beta\frac{\partial f}{\partial e}\,, \tag{4.9d}$$

$$\gamma_R\to\gamma_R-\beta\frac{\partial g}{\partial e}+2f. \tag{4.9e}$$

We can now impose some convenient conditions on these quantities to permit a unique calculation. Consider first the order e^2 terms in Eq. (4.8). The $e^2\ln^2 Q^2/\mu^2$ term in U has coefficient A_{12}, so that $\gamma_S=-4A_{12}e^2+O(e^4)$ by the RG Eq. (4.3b). We need two conditions to fix the rest of U; let us assume that $\gamma_R=0$ and that $U=0$ when $Q=\mu$. Then I_1 has to be $A_{11}e^2$, and the RG Eq. (4.3c) provides a consistency check. Finally, $I_2=A_{12}e^2$, with Eq. (4.3d) as a check.

In general, let us write

$$U=\sum_{P=1}^\infty A_P(e)\ln^P(Q^2/\mu^2)\,, \tag{4.10}$$

where $U=0$ when $Q=\mu$. Then

$$\gamma_S\ln(Q^2/\mu^2)+\gamma_R=\mathfrak{D}U =\sum_{P=0}^\infty[\beta A_P'-2(P+1)A_{P+1}]\ln^P(Q^2/\mu^2)\,. \tag{4.11}$$

This equation determines A_P in terms of β and A_{P-1}:

$$A_P=\beta A_{P-1}'/(2P) \tag{4.12}$$

for $P\geq 3$, while if $\gamma_R=0$ we have $A_1=0$ and

$$\gamma_S=-4A_2\,. \tag{4.13}$$

Now consider the $O(e^{2N})$ terms in $\ln V$. The only contributions to A_{NP} for $P\geq 2$ are from U. Thus A_{NP} for $P\geq 3$ is given by Eq. (4.12) in terms of lower-order values, and in particular $A_{NP}=0$ if $P>N+1$. Then the $\ln^2 Q^2/\mu^2$ comes from A_{N2}:

$$A_2=\sum_{N=1}^\infty e^{2N}A_{N2}\,. \tag{4.14}$$

Finally, the $\ln Q^2/\mu^2$ and Q-independent terms are fixed since we choose $A_0 = A_1 = 0$. Hence

$$I_1 = \sum_N e^{2N} A_{N1}\,, \tag{4.15}$$

$$I_2 = \sum_N e^{2N} A_{N0}\,. \tag{4.16}$$

In terms of the definitions with $\gamma_R = 0$ and $U(Q/Q) = 0$, Eq. (4.7) becomes

$$V \sim \exp - \left\{ I_2\left(\frac{m}{\mu}, \frac{M}{\mu}, \bar{e}(\mu)\right) + \ln\frac{Q^2}{\mu^2} I_1\left(\frac{m}{\mu}, \frac{M}{\mu}, \bar{e}(\mu)\right) + \int_\mu^Q \frac{d\mu'}{\mu'} \ln\frac{Q^2}{\mu'^2} \gamma_S(\bar{e}(\mu')) + O(1/Q) \right\}. \tag{4.17}$$

This reproduces Mueller's results.[2]

Now let us consider changing the particles or the operator whose form factor we are discussing. The only dependence on the operator is in $\tilde{\Gamma}_{R\mathrm{UV}}$. The only dependence of $\ln V$ on particle type that is not a sum of a term for each particle is in $\mathcal{S}_R$. But this term depends only on the particles' charges.

In Eq. (4.17) the only quantity that is independent of particle type and of the operator is the order e^2 value of γ_S, and this is what determines the leading, Sudakov behavior.

While Eq. (4.17) gives a useful summary of the large effects at high Q^2, it cannot be used when actually $Q^2 \to \infty$ for then the coupling $\bar{e}(Q)$ becomes large and perturbation theory inapplicable.[19] In view of the applications of the methods of this paper to non-Abelian gauge theories it is of interest to see what would happen if Eq. (4.17) were being used in an asymptotically free theory. In the first place Mueller's remarks[2] would apply, so that the leading behavior would be determined by $\beta = ae^3, \gamma = be^2$.

If we only wish to examine the electromagnetic form factor, we may choose $\gamma_R = 0$ and $U(Q/Q) = 0$. This is a convenient condition to apply in perturbation theory. To extend calculations to other operators we can require I_1, I_2, and γ_S to be the same as for the electromagnetic case and write

$$U = U_{\mathrm{em}} + \Delta U \tag{4.18}$$

with

$$\mathfrak{D}\Delta U = \gamma_0(e)\,, \tag{4.19}$$

where γ_0 is the anomalous dimension of $\ln V$:

$$\mathfrak{D} \ln V = -\gamma_0$$

since the operator is not now necessarily conserved. Thus although I_1 and I_2 in Eq. (4.7) are not computable in an asymptotically free theory, the changes due to change of operator are computable. These changes are in $U(1, \bar{e}(Q))$ and, if the operator has nonzero γ_0, in γ_S.

V. SUPERRENORMALIZABLE CASE

In space-time dimension $d < 4$, QED is superrenormalizable. It is easy to read off the alterations to the results of Sec. II, III, and IV. First of all, the anomalous dimensions vanish. Then the UV behavior of higher loops in the 1PI vertex is reduced, so that Eq. (2.13) becomes

$$\Gamma \sim e^{\mathcal{S}}(1 + O(1/Q^t))\,, \tag{5.1}$$

where $\mathcal{S}$ now needs no renormalization, and t is number greater than zero.

Then in Sec. III all the divergences at large k_T disappear so that Eq. (3.9) is replaced by

$$2\int B = \int d^{d-2}k_T(f_1 \ln Q^2 + f_2) + O(1/Q^t)\,, \tag{5.2}$$

while (3.16) becomes

$$G_R \sim G_{R\mathrm{IR}}\,. \tag{5.3}$$

Hence

$$V = G_R \exp\left(\ln Q^2 \int d^{d-2}k_T f_1 + \int d^{d-2}k_T f_2 + \mathcal{S}\right) = \left(\frac{m^2 + M^2}{Q^2}\right)^{a(m,M,e,d)} b(m,M,e,d)(1 + O(1/Q^t))\,, \tag{5.4}$$

where

$$a = -\int d^{d-2}k_T f_1 \tag{5.5}$$

and

$$b = G_{R\mathrm{IR}} \exp\left(\mathcal{S} - a\ln(m^2 + M^2) + \int d^{d-2}k_T f_2\right). \tag{5.6}$$

Hence there is a power law falloff of V which is *not* renormalization-group controlled. The purpose of the $m^2 + M^2$ factors in Eqs. (5.4) and (5.6) is to ensure that b is dimensionless.

VI. CONCLUSIONS

We have seen how to factorize the large-Q behavior of the electron's form factor, and that the UV part is the exponential of a quantity computable by renormalization-group methods. The main contribution came from the overlap of the soft and collinear regions, and was factored out by the Grammer-Yennie method.

There are many processes to which these techniques should be applied in QCD. The precise form of the factorization and exponentiation will depend very much on the process. There seem to be two main difficulties, viz.,

(1) The factorization and exponentiation are in coordinate space whereas the required answer is in momentum space. See Refs. 8 and 20 for calculational methods.

(2) QCD is non-Abelian, with two problems ensuing: (a) We proved that the K_t's in Fig. 5 cannot have collinear external lines. The proof relied on using Ward identities to show cancellation of soft contributions internal to a K_t. Because non-Abelian gluons are colored, this proof does not apply. Rather we must construct a definition of K_t in which internal soft contributions are subtracted off when the external lines are collinear. Exactly analogus remarks apply to our work in Sec. III. (b) When we apply Ward identities to a K gluon (as when we derived Fig. 9) we pick up a term for each colored external line of a jet. Such lines include soft gluons in a non-Abelian theory, when one obtains commutators of the vertices emitting the soft glue. In Ref. 3 it was shown that then one simply applies the Grammer-Yennie method to the commutator vertices.

None of these problems seems insurmountable, but they do prevent a trivial extension of the results of the present paper.

Note added: The extension to low-transverse-momentum dilepton production in hadron collisions (the Drell-Yan process) has been accomplished. See D. E. Soper and J. C. Collins, lectures at the XXI International Conference on High Energy Physics, Madison, Wisconsin (unpublished); report (unpublished).

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