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Confinement and chiral-symmetry breakdown: Estimates of F_π and of effective quark masses

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We illustrate how a field theory of confinement automatically leads to spontaneous breakdown of (flavor nonsinglet) chiral symmetry, with accompanying massless pions. This breakdown is described as a tunneling process involving a quark and virtual $q\bar{q}$ pairs (as in superconductivity), and is driven by the infrared singularities of the confinement mechanism. Effective quark masses (not to be confused with either current or constituent quark masses) are defined in terms of $\{\gamma_5, S^{-1}(p)\}$ at zero momentum, where $S(p)$ is the effective quark propagator (an entire function in most gauges), and these masses yield values for F_π. Aside from messy numbers of $O(1)$ discussed in the text and leaving out short-distance corrections, F_π is essentially $2\sqrt{3}\, a(2\pi)^{-3/2}$, where $a \simeq 400$ MeV appears in the linearly rising potential $V(r) = a^2 r$. We calculate chiral-breaking effects in four models, beginning with $d = 2 + 1$ QED for massless electrons, to introduce techniques [using Ward identities and the full Dyson equation for $S(p)$] for dealing with confining field theories. Two other models are $d = 3 + 1$ propagator models of confinement and the fourth explicitly exhibits the tunneling to a $q\bar{q}$ pair mentioned above, in a theory with area-law confinement.

I. INTRODUCTION

There is a sense in which we understand the pion less well in today's quark models than we did twenty years ago as a Goldstone boson.[1] The pion coexists successfully in an SU(6) multiplet with heavy particles like the ρ and the properties of this multiplet are for the most part well-described in terms of $q\bar{q}$ bound states.[2] Yet this language leads to no immediate understanding of the Goldstone nature of the pion (in particular, its nearly zero mass) as we see, for example, from early bag-model calculations of the pion's properties.[3] And this language does not make it clear what the nature of the chiral symmetry-breaking mechanism is. Inevitably, the mechanism involves formation of a $q\bar{q}$ condensate and it is problematical how to integrate this with a simple bound-state picture. The difficulty is that in a quasipotential view of a bound state, there is no machinery for a quark to tunnel through to a virtual pair and couple to the condensate. Such a possibility is inherent in any relativistic field theory, but knowledge of the (continuum) techniques necessary to treat even a phenomenological confining field theory is not yet widespread.

In fact, such techniques have already been partially developed,[4,5] and used in various confinement contexts. We have noted that in $d = 2 + 1$ QED for zero-bare-mass electrons, the logarithmically confining potential generates in perturbation theory an infrared-singular electron "mass"; the singularity is removed in the full Dyson equation and replaced by an electron propagator $S(p)$ which is entire in momentum space. In this paper we extend these techniques to show that $\{\gamma_5, S(p)\} \neq 0$ in several confining models. We also develop another set of methods based on path-integral representations of quark propagators to show the explicit connection between chiral breakdown and tunneling to a virtual $q\bar{q}$ pair.

Quite independent of the above considerations, other authors have given reasons to believe that confinement and breakdown of chiral symmetry are related. It has been appreciated for some time that instantons can break $SU(N) \times SU(N)$ chiral symmetries[6] [as well, of course, as axial U(1) symmetry which involves anomalies[7]]. Callan *et al.* also argue[8] that instantons confine, but the connection between these two roles played by instantons has not yet been made quantitative. Recently, Casher[9] has argued in a general way that vectorlike confining forces automatically break chiral symmetry, and Johnson and Donoghue[10] have calculated F_π in the bag model, where chiral symmetry is violated by bag boundary conditions. This interesting result is obscured by a lack of precise understanding of how the pion becomes massless in a bag model.

In preconfinement times, it was easy to exploit the connection between superconductivity and spontaneous generation of a fermion mass.[1] One wrote down a gap equation which could have two solutions: zero-mass fermions and Wigner-Weyl realization of chiral symmetry, or massive fermions and massless pions. The gap equation was a statement about the pole of a propagator. Confinement poses a serious challenge to this picture because the quark propagator in quantum chromodynamics (QCD) does not even exist, strictly speaking. We can—and will—usefully define an effective propagator $S(p)$, but it has no poles (in most gauges); in fact, $S(p)$ is entire in $\not{p}$.[4]

These features show up as a very specific problem in the naive gap equation: The fermion mass is infrared divergent. It is no accident that exactly the same divergence appears in the static potential of the confining theory: The mass divergence and the potential divergence conspire to cancel each other in (and only in) color-singlet channels. In a certain sense, the mass is subsumed in the gluon-exchange force, but it leaves its mark on the theory in the form of a nonvanishing value for $\{\gamma_5, S^{-1}(p)\} \equiv -2\gamma_5 M(p^2)$. This momentum-dependent "mass" breaks chiral symmetry and leads to zero-mass pions for which we can roughly calculate F_π.

It would be nice to show these features in QCD by making an *ab initio* calculation of the confinement machinery. Unfortunately, we cannot do this, so we turn to models. Four models will be taken up: $d=2+1$ QED with massless electrons, two propagator models of four-dimensional confinement, and an area-law model for four-dimensional confinement. (By a propagator model we mean one in which the forces between quarks come from the exchange of an effectively Abelian gluon whose propagator is chosen to have specific confining properties.[5,11]

The first model, $d=2+1$ QED, is included for two reasons: First, it really is Abelian, and an undressed single-photon exchange correctly gives the confinement law (corresponding to a logarithmic static potential). The second reason is that it is a useful proving ground for the necessarily nonperturbative technique needed to deal with confining forces. Furthermore, as far as we can tell the electron propagator $S(p)$ actually exists and there is no reason to invoke an effective propagator. We will explicitly show that $\{\gamma_5, S^{-1}(p)\} \neq 0$ by calculating $S(p)$ from the full Dyson equation using a Ward-identity-preserving solution for the longitudinal part of the vertex function [conserved transverse parts do not contribute to the infrared-singular behavior responsible for the entirety of $S(p)$]. This idea, which actually linearizes the Dyson equation, was independently exploited in Refs. 4 and 12. However, in Ref. 4 it was only carried out for eikonal vertices (i.e., $\gamma_\mu \to v_\mu$, a forward timelike velocity) which does not allow for a proper consideration of antiparticle effects; Ref. 12 uses Dirac matrix vertices, and we will use this development of Delbourgo and West. The upshot is that $S(p)$ no longer has any logarithmic infrared divergences associated with the electron mass; it solves that problem by rejecting the notion of a mass shell and becoming entire.

Once it has been shown that $\{\gamma_5, S^{-1}(p)\} \neq 0$, an argument due to Goldstone[13] shows that the homogeneous Bethe-Salpeter equation in the pion channel has a solution at zero-momentum transfer. We take this to mean the existence of a zero-mass pion, both in three and in four dimensions [an assumption related to the rapid decrease of $M(p^2)$ at large p^2, discussed below]. It is then more or less straightforward to use chiral Ward identities to calculate F_π, a calculation which amounts to normalizing the pion Bethe-Salpeter wave function.

This work can be trivially extended to logarithmic confinement in four dimensions, which requires the gluon propagator to have the behavior $(-k^2)^{-3/2}$ for small k. Logarithmic potentials have been seriously advocated[14] for charmonium and $b\bar{b}$ states and the strength of the potential for heavy-quark systems is known. The resulting values for quark masses (~1 GeV) and F_π (~300 MeV) are disappointing.

One feature of four-dimensional theories which is absent from $d=2+1$ QED is the presence of renormalizable short-distance singularities. We propose to omit these from the present treatment, although their inclusion is not difficult in principle. There are two reasons: First, the models we use below for four-dimensional QCD may be valid in the extreme ultraviolet and extreme infrared regions, but there is no reason to believe that they model the transitions between these extremes very well. Second, we have a feeling based on some experience that short-distance corrections are not likely to be more than about ±25%, and we cannot even claim corresponding accuracy for the part of the theory that we save. Work is in progress to look at short-distance corrections and will be reported later. With the proviso that we consider only confining forces, let us take up more realistic models of QCD than logarithmic confinement.

The first of these is a propagator theory in which the $q\bar{q}$ force law is like QED, except that e^2k^{-2} is replaced by $C_F\bar{g}^2(k)k^{-2}$ ($C_F=\frac{4}{3}$ is the quark Casimir eigenvalue). A linearly rising static potential follows from $C_F\bar{g}^2(k)=-m^2k^{-2}$:

$$V(r)=\frac{m^2}{8\pi}r\equiv a^2 r\,. \tag{1.1}$$

Heavy-quark physics[15] tells us that $a\simeq 400$ MeV, $m\simeq 1.9$ GeV. [For details of how to extend this theory to ultraviolet momenta, see Ref. 16. The effective Abelian propagator $\bar{g}^2(k)k^{-2}$ correctly reproduces all *leading* asymptotic-freedom results.] The same effective gluon propagator is used for a gluon emitted and absorbed on the same quark line, but in baryons the qq force is attractive and one-half the strength of the $q\bar{q}$ force.

We wish to imitate the successful techniques used in $d=2+1$ QED, and we begin by looking at the naive gap equation. The result is a *linearly* infrared-divergent fermion mass, which exactly cancels the linear divergence of the static potential (both for baryons and for mesons). One might hope that this divergence goes away in the full Dyson equation, as it did in $d=2+1$ QED, but it does not; the kernel of the Dyson equation appears to have a *logarithmic* infrared divergence. This is very much to be expected in QCD because the conventional propagator of quarks (or other colored fields) vanishes (by virtue of gauge invariance of the vacuum, which is not the case for QED). We can define an effective propagator, but only with reference to the particular *color-singlet* process in which this propagator is to be used. Because of the way the model is constructed to look like QED, any color-singlet object like a hadronic Green's function has an Abelian gauge invariance which lets us shuffle various nonphysical phenomena between the gauge-variant constructs (such as the effective propagator) into which we artificially resolve the color-singlet Green's function.

Our essential point is that the logarithmic infrared singularity of the effective propagator is purely a gauge artifact totally absent from the color-singlet Green's function. This means that $\ln\mu^2$ (μ is the infinitesimal infrared-regulator mass) will be replaced by $\ln\omega^2$ in all physical quantities, where ω is a finite mass characteristic of hadrons. Since F_π is only logarithmically sensitive to what ω is, we do not, in this first attempt to estimate F_π, worry about its exact value. Instead, we simply replace logarithmic factors by unity, which could be inaccurate by 50% (we hope not by much more).

To calculate F_π it is necessary to go through the intermediate device of an effective quark mass M, defined as

$$M=M(p^2=0), \quad \{\gamma_5, S^{-1}(p)\}\equiv-2\gamma_5 M(p^2). \qquad (1.2)$$

$M(p^2)$ is not in itself a gauge-invariant quantity, but it is useful to think of it as physically real. To make its definition precise we specify that $M(p^2)$ is to be calculated in the Feynman gauge, the reasons for which will be given in Sec. II. F_π depends not just on M, but on $M(p^2)$ for all momenta; if we took $M(p^2)=M$, F_π would be logarithmically divergent. There are good reasons[17,18] for thinking that $M(p^2)\sim p^{-2}$ for large p, so we use the simple parametrization

$$M(p^2)=\frac{M\Lambda^2}{\Lambda^2-p^2}\ . \qquad (1.3)$$

As might be expected, F_π is only logarithmically sensitive to Λ, which we argue below should be of the order of the ρ mass. There is a precise formula (see the Appendix) for F_π in terms of M and Λ, but we can orient ourselves to the main features of the problem by once more setting logarithms equal to unity because some simple formulas emerge:

$$M\simeq\frac{m}{2\pi}\simeq 300\ \text{MeV}\ , \qquad (1.4)$$

$$F_\pi\simeq\frac{3^{1/2}m}{(2\pi)^2}=\frac{2a(3^{1/2})}{(2\pi)^{3/2}}\simeq 87\ \text{MeV}\ . \qquad (1.5)$$

Of course, no attention should be paid to the accidental and embarrassing closeness of F_π to the experimental value of 94 MeV; we have made so many approximations and simplifications that any value from 50 to 150 MeV would be plausibly the exact result for this model. But it is amusing to see a number of order 100 MeV emerge from another, namely m, of order 2 GeV.

Before going on to the last model, area-law confinement, we should say a few words about the meaning of M. It is certainly not the current mass $\hat{M}$ of a few MeV for up and down quarks induced by the weak interactions and responsible for the nonzero pion mass. Nor should it be confused with the so-called constituent mass which is really an energy of the form $\hat{M}\gamma$, where γ is the usual relativistic factor; this energy is of course finite even when $\hat{M}=0$ (as illustrated concretely in Ref. 5). At the moment, M merely stands for a parameter characterizing the strength of chiral-symmetry breakdown; other dynamical aspects await clarification.

So far we have worked in momentum space. The last model[19] has to be worked out in coordinate space using another set of nonperturbative tools useful in confining theories. In some ways this makes it the most interesting model of all because it explicitly exhibits the connection between chiral breakdown and tunneling to a virtual $q\bar{q}$ condensate. The essential element, invoked by Feynman and Schwinger in works some thirty years old, is a representation of a charged-particle propagator as a four-dimensional path integral followed by a proper-time integral. Feynman graphs correspond to approximating the paths by straight lines, clearly inappropriate for confinement. In fact, the dominant classical paths must be periodic and confined to a finite region of three-space. Such paths automatically remove[5,16,19] all infrared singularities from infrared-singular theories like the propagator theory above, or the area-law theory of present concern. But Casher[9] has argued that these paths inevitably imply chiral-symmetry breakdown, as we will show.

Earlier works[4,5] using the bound-state path techniques restricted themselves to paths $x_\mu(s)$ (s is the proper time) which had a direct classical interpretation: The proper velocity $\dot{x}_\mu(s)$ was always forward timelike (FTL). This is quite appropriate for heavy-quark systems where the quarks do behave almost classically, but not so for light-or zero-mass quarks. These can give important contributions to the path integral from paths that fluctuate so that $\dot{x}_\mu$ is spacelike or backward timelike, corresponding respectively to a tunneling process or to an antiparticle. An otherwise FTL path which has a fold (a backward timelike segment joined by two spacelike segments, or a Z shape) has the physical interpretation of a virtual $q\bar{q}$ pair coupled to the original particle by tunneling—just the ingredient which characterizes the gap equation in superconductivity.

It turns out (see Sec. IV) that it is these folds, and only these folds, which give chiral-symmetry breakdown, either in the propagator model above or in the area-law model. Without them, quarks stay massless. So a static potential model, corresponding to pure FTL paths, cannot give a nonzero F_π (unless one wants to take seriously hybrid relativistic-potential models such as the Dirac equation with a static potential which exhibits a sort of tunneling in the form of the Klein paradox).

The area-law model is motivated by considerations of vortex condensate formation in massive gauge-invariant QCD,[19] but that need not concern us here. The main point is that the (Euclidean) action associated with a fixed Wilson loop is given by an appropriately scaled minimum area (independent of the spanning surface). The Wilson-loop expectation values are then subjected to proper-time and path integrals to give hadronic Green's functions. For FTL paths, the propagator model discussed above and the area-law model are virtually (in $d=2$, exactly) identical, but they differ somewhat for non-FTL paths (the difference is related to the existence of long-range van der Waals forces in the propagator theory).

We have tried to estimate F_π by summing over these non-FTL folds, a procedure very much the same as summing over barrier-penetration "instantons" in ordinary quantum mechanics.[20] The calculations are technically messy, and depend somewhat sensitively on hadronic parameters (e.g., radii and bound-state periods) which have been estimated[5] with path-integral techniques, but are not really accurately known. In consequence, values for the effective quark mass M could range from 50 to 500 MeV, with corresponding variability (15–150 MeV) in F_π. The importance of this calculation lies not in this wide range of values for F_π, but in the insight it yields as to how infrared singularities can be tamed to give some nonzero value for F_π independent of spurious cutoff effects.

A final note on the practical significance of F_π. It is very natural that the first predictions from QCD (or models of it) will be properties of isolated hadrons: masses, static moments, and F_π. It will be much harder to calculate, from QCD itself, multihadron properties such as coupling constants. Fortunately, we do not need to do this: Knowledge of the mass spectrum and F_π, plus unitarity, completely determines *all* phenomenological hadron couplings of hadrons with spin ≤ 1, as discussed in Sec. V. So F_π is not just another hadron parameter; it helps to unblock almost every difficult calculation in QCD.

Section II of this paper describes general properties of propagators (or effective propagators) of confined particles, with emphasis on questions of entirety and gauge covariance. Section III illustrates the momentum-space Dyson equation in the paradigm of $d=2+1$ QED. Section IV takes up three models of confinement in four dimensions, and Sec. V contains conclusions and possible directions for the future. An appendix gives well-known formulas for getting F_π from $\{\gamma_5, S^{-1}(p)\}$.

Note added. As I was writing this, I saw a work by Pagels and Stokar[21] which overlaps the present work. These authors relate the effective quark mass to F_π and give exactly the same formulas and approximations as in the Appendix of this paper. They also relate $M(p^2)$ to the pion electromagnetic form factor, emphasizing the importance of $M(p^2)\sim p^{-2}$ for large p. They do not address the question which is the crux of my work: How does confinement break chiral symmetry? My attention has also been brought to the work of Brout, Englert, and Frère[31] which discusses the issues raised here in the $d=2$ 't Hooft version of QCD.

II. CONFINEMENT AND ENTIRE PROPAGATORS

Presumably for QCD in four or fewer dimensions, the conventionally defined propagator of confined quarks (or of any gauge-variant quantity) is identically zero because the vacuum is unique and gauge invariant. Propagators exist in ordinary QED because the vacuum is one of a degenerate set related by gauge transformations. (Apparently this holds even for $d=2+1$ QED, a confining theory.) It is nonetheless useful to speak of quark propagators in QCD, because in the phenomenological models of confinement[5,19] developed in Sec. IV there are effective propagators with all the properties discussed in this section. In particular, they have an *Abelian* gauge-covariance property, just as in QED. So we need make no distinction between QCD and $d=2+1$ QED in what follows.

The question is how to extract physical, gauge-independent information from the propagator. If there were a mass shell this would be easy; the position of the propagator pole is gauge invariant. Unfortunately, the general case for confinement is that there is no mass shell. In this section we develop several standard representations for the propagator (in coordinate and in momentum space), which will be used in succeeding sections, and find their gauge-transformation properties. We will always be interested in infrared effects, that is, large distance and small momentum, which leads to certain simplifications. There is one trivial result which holds in any case: $\{\gamma_5, S(p)\}\neq 0$ is gauge invariant.

There are only two continuum techniques known to the author for treating propagators in confining theories (of course, perturbation theory fails). One[4,5,19] uses the proper-time techniques of Feynman and Schwinger and is formulated in coordinate space; the other[4] turns the full momentum-space Dyson equation into an equation linear in the propagator by approximating the proper vertex by its longitudinal part. These techniques are closely related, and are especially suited to treating infrared-divergence problems nonperturbatively (because transverse proper vertex parts vanish at small momentum, thus eliminating infrared singularities). In the eikonal approximation they are identical,[4] but this approximation is too drastic to use for chiral-symmetry breakdown.

In the propagator models of this paper, we are given an effective Abelian gluon propagator of the form $\bar{g}^2(k)k^{-2}$ in momentum space; whether or not a model is confining can be immediately deduced from this propagator. We assert: *A theory is confining if and only if the lowest-order graph for the fermion self-mass M is infrared divergent (with zero bare mass).* Similarly, the theory is confining if and only if the static potential deduced from the zero-frequency propagator is infrared divergent; these divergences are correlated in such a way that they cancel in color-singlet (or zero-charge, for QED) sectors. These assertions are elementary and will be illustrated by examples in succeeding sections.

Evidently, an infrared-singular mass signifies the nonexistence of a fermion mass shell. The nonperturbative expression of this is the entirety of the fermion propagator in momentum space (at least for a wide class of gauges). An entire propagator has the usual spectral representation

$$S(\not p)=\int_{-\infty}^{\infty}\frac{dW'\rho(W')}{\not p - W'}\,, \qquad (2.1)$$

with some special features. First, $\rho(W')$ is itself entire, so there are no thresholds and no gap on the real W' axis where $\rho=0$. Second, in the normal case with a gap $S(\not p=W)$ (W being an ordinary complex variable, not a Dirac matrix) is defined for real W as the boundary value on top of the right-hand cut but below the left-hand cut. This definition cannot be used for the case of no gap and we define $S(W)$ explicitly by (2.1) in the upper-half W plane, and by its analytic continuation in the lower-half plane. We will assume that $S(W) \sim W^{-1}$ for large W anywhere in the upper-half plane, which requires that $\int dW'\rho(W')=1$. [Of course, in the lower-half plane $S(W)$ diverges at large W, since there must be an essential singularity at $W=\infty$.] The behavior $S(W)\sim W^{-1}$ is violated by powers of logarithms in an asymptotically free theory because of short-distance effects, but we have already said that we will ignore these.

The spectral function $\rho(W)$ is not unique even in a given gauge. One useful choice is $\rho(W)=\hat\rho(W)$, where $\hat\rho$ is real analytic; another is $\rho(W)=i(2\pi)^{-1} S(W)$, in which case (2.1) is Cauchy's theorem (with the contour closed at ∞ in the upper-half plane). Because $\hat\rho$ is ill behaved even in the upper-half plane (except on the real axis), it is not useful when one wants to close contours.

Next we want to look at the coordinate-space propagator $S(x)$, in part because it has simple gauge-transformation properties. It has the proper-time (Laplace transform) representation

$$S(x)=-i\int_0^{\infty} ds\, G(s;x)\,, \qquad (2.2)$$

where G is a path integral. For example, in scalar QED,

$$G(s;x)=N\int (d\text{ path})\exp\left[-i\left(\int_0^s d\tau\frac{\hat M}{2}(\dot x^2+1)+\frac{e^2}{2}\int_0^s d\tau\int_0^s d\tau'\,\dot x_\mu(\tau)\dot x_\nu(\tau')\Delta^{\mu\nu}(x(\tau)-x(\tau'))\right)\right] \qquad (2.3)$$

in the approximation of dropping closed charge-particle loops. Here N is a normalizing factor and $\hat M$ is the particle mass; the path $x_\mu(\tau)$ goes from 0 at $\tau=0$ to x_μ at $\tau=s$. Quite aside from any approximation such as (2.3), a change of gauge

$$\Delta_{\mu\nu}(x-y)\to\Delta_{\mu\nu}(x-y)+\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\Lambda(x-y) \qquad (2.4)$$

leads to the exact transformation law

$$S(x)\to S(x)\exp\{-ie^2[\Lambda(x)-\Lambda(0)]\}\,. \qquad (2.5)$$

We cannot conclude from (2.5) that it shows the s dependence of $G(s;x)$ to be gauge invariant, but this is true for the approximation (2.3) since the substitution (2.4) in (2.3) yields

$$G(s;x)\to G(s;x)\exp\{-ie^2[\Lambda(x)-\Lambda(0)]\}. \qquad (2.6)$$

Furthermore, it is well known that (2.3) correctly characterizes all the infrared singularities even of fermionic QED (and even for light fermions once the closed-loop contributions to $\Delta_{\mu\nu}$ are included). We therefore assume that the more restrictive gauge transformation law (2.6) holds, rather than the exact (2.5); of course, (2.5) is a consequence of (2.6).

Under these circumstances the s dependence of G is gauge invariant and physically meaningful consequences can be found in its behavior. Of particular interest is a "mass" term, which is characterized by a factor $e^{-iMs/2}$ in G for large s. (The other half of the mass comes from a term in G depending also on x.) In general, if $\ln G$ is $O(s)$ for large s there is a mass shell, but if $s^{-1}\ln G$ diverges for large s, the propagator is entire. [The propagator will also be entire if $(x^2)^{-1/2}\ln G$ diverges as $x^2\to\infty$, but this sort of entirety can be gauged away in most cases, including a case of practical interest to us.]

A convenient bridge between coordinate space and momentum space comes from a third representation of the propagator,

$$S(\not p)=-i\int_0^\infty ds\, e^{is\not p}F(s), \qquad (2.7)$$

where the inverse Laplace transform of (2.1) yields

$$F(s)=\frac{i}{2\pi}\int_{-\infty}^{\infty} dW\, e^{-isW}S(W). \qquad (2.8)$$

For $s<0$ the contour in (2.8) can be closed in the upper-half plane, giving $F(s)=0$ for $s<0$. [One can also define a function $\hat F(s)$, equal to $F(s)$ for $s>0$, but nonvanishing for $s<0$; this is

$$\hat F(s)=\int_{-\infty}^{\infty} dW\, e^{-isW}\hat\rho(W). \qquad (2.9)$$

$\hat F(s)$ is a real function of $-is$.] A concrete example of the connection between these functions is the choice $\hat\rho(W)=\pi^{-1/2}e^{-W^2}$, with $\hat F(s)=e^{-s^2/4}$, $F(s)=\theta(s)e^{-s^2/4}$; $S(W)$ is the complex error function. Note that even though $\hat\rho(W)$ is ill behaved at $W=\infty$, $S(W)\sim W^{-1}$ in the entire upper-half plane.

A few steps of algebra show that the coordinate-space propagator in terms of $F(s)$ is

$$S(x)=\frac{1}{2\pi}\int_0^\infty ds\, F(s)\int_{-\infty}^{\infty} dW\, e^{iWs}S_W(x), \qquad (2.10)$$

where

$$S_W(x)=(i\not\partial-W)\frac{W}{8\pi\zeta}H_1^{(2)}(W\zeta), \quad \zeta=(x^2-i\epsilon)^{1/2} \qquad (2.11)$$

is the Dirac propagator for mass W. When the representation

$$H_1^{(2)}(W\zeta)=\frac{2}{\pi\zeta}\int_\zeta^\infty ds'\, e^{-iWs'}s'(s'^2-\zeta^2)^{-1/2} \qquad (2.12)$$

is used in (2.10), the integral over W yields derivatives of the function $\delta(s-s')$, and the lower limit $s'\geq\zeta$ in (2.12) shows that the large-x behavior of $S(x)$ can be read off from $F(\zeta)$ and conversely. In particular, we can see the effect of a change of gauge (2.5) on $F(s)$.

This fact, plus (2.6), are the main results of this section; they show how to untangle the gauge-independent and the gauge-dependent parts of the propagator both in coordinate space [through $G(s;x)$] and in momentum space [by comparing (2.5) and (2.10)–(2.12)]. For example, if we could find $G(s;x)$ independent of x (in a certain gauge), we know that this $G(s)$ is gauge invariant.

Actually, this is not a far-fetched hope. In Sec. IV we discuss a propagator model of QCD with $e^2\Delta_{\mu\nu}(x)$ in (2.3) given by $(m^2/16\pi)g_{\mu\nu}\theta(x^2)$. It is easy to see that, if the paths $x_\mu(\tau)$ in (2.3) are all forward timelike, the θ function is always unity. The double integral in the exponent of (2.3) is then path independent and produces the phase $\exp(-i\,m^2x^2/32\pi)$, which of course can be gauged away. The contribution of the near-forward-timelike paths gives a masslike term which is gauge invariant. It is clear for this model that there is a distinguished gauge, which is in fact a simple generalization of the Feynman gauge [since $\Lambda\sim(x-y)^2$ in (2.4) yields only a $g_{\mu\nu}$ term]. Such gauges would never be used in conventional QED because they completely distort the physics, but there is nothing wrong with them in principle. It turns out that there is no essential difference between using the ordinary Feynman gauge and this generalized Feynman gauge in the uses we make of them in Sec. IV, so we will refer to both as just the Feynman gauge.

III. NONPERTURBATIVE TREATMENT OF DYSON EQUATIONS AND WARD IDENTITIES IN $d=2+1$ QED

The main features of confinement in $d=2+1$ QED have already been outlined using eikonal vertices.[4] In the eikonal approximation γ_μ is replaced by v_μ, a forward timelike unit vector ($v^2=1$), and the free fermion propagator $S(p)=(v\cdot p-M)^{-1}$ has only a particle pole, no antiparticle pole. $G(s;x)$ in (2.3) is approximated by saving only straight-line paths with $\dot x_\mu(\tau)=v_\mu$. The result satisfies the full nonlinear Dyson equation with a special choice for the proper vertex Γ_μ, which satisfies the Ward identity. The theory is confining because $\ln G \sim s \ln s$.

Independent of Ref. 4, Delbourgo and West[12] have invented a better scheme which allows use of Dirac vertices γ_μ. The starting point is a representation for the vertex Γ_μ in terms of the spectral function $\rho(W)$ in (2.1) which determines the propagator $S(p)$:

$$S(p-k)\Gamma_\mu(p,k)S(p) = \int_{-\infty}^{\infty} dW\,\rho(W)\frac{1}{\not p-\not k-W}\gamma_\mu\frac{1}{\not p-W}\,. \qquad (3.1)$$

It is simple to check that

$$k_\mu\Gamma^\mu(p,k)=S^{-1}(p)-S^{-1}(p-k)\,. \qquad (3.2)$$

This solution of the Ward identity is, of course, not unique, but any transverse parts added to Γ_μ of (3.1), such as $i\sigma_{\mu\nu}k^\nu$, vanish at least linearly in k for small k, so (3.1) is especially suited to the study of the infrared regime. In theories with a mass shell the renormalized on-shell vertex is just γ_μ; again, corrections to this vanish for small k.

Radiative corrections to the photon propagator do not change the infrared structure of the theory (even for zero-bare-mass fermions), so in the Dyson equation for $S(p)$ we use just the bare photon propagator. Remarkably, with (3.1) the apparently nonlinear Dyson equation linearizes. With the bare fermion mass set equal to zero, the Dyson equation is (in the Feynman gauge)

$$1=\not pS(p)+\frac{ie^2}{(2\pi)^3}\int\frac{d^3k}{k^2} \times\int dW\,\rho(W)\gamma_\mu\frac{1}{\not p-\not k-W}\gamma_\mu\frac{1}{\not p-W}\,, \qquad (3.3)$$

with $S(p)$ given by (2.1). The integrals over k are easily evaluated to give

$$1=W\int dW'\frac{\rho(W')}{W-W'}-\int dW'\frac{\rho(W')}{W-W'}\Sigma(W,W')\,, \qquad (3.4)$$

where $\Sigma(W,W')$ is given by the *one-loop* Feynman graph $\Sigma(\not p,M)$ for a fermion of mass M and $\not p \to W$, $M\to W'$. Equation (3.4) holds for all the propagator models of this paper and will be used again in Sec. IV. The explicit form for $d=2+1$ QED is

$$\Sigma(W,W') = \frac{\alpha}{2}\left\{\frac{W'}{W}+\left(\frac{3W'}{W}-\frac{W'^2}{2W^2}-\frac{1}{2}\right)\ln\left(\frac{W'+W}{W'-W}\right)\right\}, \qquad (3.5)$$

where $\alpha=e^2/4\pi$; note that α has dimensions of mass.

We digress to the entanglements of perturbation theory, to illustrate the assertion made in Sec. II. Define the perturbative fermion mass (for zero bare mass) M_p as the solution of the gap equation

$$M_p=\Sigma(M_p,M_p)\,.$$

This quantity is infrared divergent, so replace the photon propagator k^{-2} in (3.3) by $(k^2-\mu^2)^{-1}$, where μ is infinitesimal (in particular, $\mu \ll \alpha$). Then find[4]

$$M_p=\alpha\ln\frac{2M_p}{\mu}=\alpha\ln\frac{2\alpha}{\mu}+O\left(\alpha\ln\ln\frac{\alpha}{\mu}\right).$$

This divergence reveals the confining nature of the theory, for the static potential between a fermion and an antifermion is also divergent:

$$V(r)=\frac{e^2}{(2\pi)^2}\int\frac{d^2k}{k^2+\mu^2}e^{i\vec k\cdot\vec r}=2\alpha\ln\mu r\,,$$

but $2M_p+V(r)$ is free of (leading) infrared divergences. If $V(r)$ were not confining, there would be no infrared singularities in it and no need for a fermion mass to cancel them in the charge-singlet sector.

To return to (3.5), observe that $\Sigma(W,W')$ behaves no worse than a constant for large W, so (3.4) at $W=\infty$ yields $\int dW\rho(W)=1$, as expected. Far more interesting is the behavior for small W:

$$\Sigma(0,W')=3\alpha\,. \qquad (3.6)$$

When (3.4) is evaluated at $W=0$, we find

$$1=-3\alpha S(0)\,, \qquad (3.7)$$

which directly shows that chiral symmetry is broken [$S(\not p=0)$ either vanishes or is infinite for a Wigner-Weyl realization of chiral symmetry]. For purposes such as calculating F_π, which do not depend so much locally on $S(p)$ as on integrals over p of this function (see the Appendix), it should be a good approximation to take $S^{-1}(\not p)\simeq\not p-3\alpha$, since this $S(p)$ has both the correct large-p and small-p behavior. In this sense the effective mass is 3α, but this is not a real mass since $S(p)$ is entire in the Feynman gauge.

This can be seen somewhat obscurely in (3.4) and (3.5) because if $\rho(W)$ has a normal threshold W_0 [where $\rho(W_0)\neq 0$ in $d=3$], it is impossible to satisfy (3.4) due to the logarithmic term in (3.5). A more direct demonstration follows from writing (3.4) in terms of the weight function $F(s)$ in (2.7). After some algebra, we find

$$\frac{dF}{ds}=\frac{1}{(2\pi)^2}\int_0^\infty ds'F(s')\int_{-\infty}^{\infty}dW \times\int_{-\infty}^{\infty}dW'e^{iW's'-iWs} \times\frac{\Sigma(W,W')}{W-W'}\,, \qquad (3.8)$$

with the boundary condition $F(0)=1$ [so that $S(W) \sim W^{-1}$ for large W]. The same equation holds for $\hat{F}(s)$, which does not vanish for $s<0$. In order that $F(s)$ vanish for $s<0$, an appropriate branch of $\Sigma(W,W')$ must be chosen so that Σ is analytic when W is in the upper-half plane and W' is in the lower-half plane. The integrals over W and W' in (3.8) can easily be evaluated, but there is no need to give details here; we need only note that (3.8) becomes an integro-differential equation for F. The fact that the support of the integral over s' does not collapse to the point $s'=s$ can be traced directly to the coupling between particles ($W, W' > 0$) and antiparticles. An approximation which does lead to a local differential equation for F is to save only those terms in $\Sigma(W,W')$ which are singular functions of the difference variable $W'-W$, otherwise replacing W' by W. The resulting function we term $\hat{\Sigma}(W-W')$ and the equation for F is

$$\frac{dF}{ds}=\frac{F(s)}{2\pi}\int_{-\infty}^{\infty} dW\,\frac{\hat{\Sigma}(W)}{W}\,e^{-iWs}\,. \tag{3.9}$$

For $d=2+1$ QED, $\hat{\Sigma}(W)=\alpha \ln W$, and we find

$$F(s)=e^{-i\alpha s \ln s}\,, \tag{3.10}$$

which is precisely the eikonal answer given earlier.[4] The scale of $\ln s$ is undetermined, but it is obvious that $\ln s$ is shorthand for $\ln(\eta\alpha s)$, where the dimensionless number η can only be determined from the full equation (3.8). In effect, we have determined it through (3.7) already. In the next section we use these techniques in four-dimensional confinement models.

IV. CHIRAL-SYMMETRY BREAKDOWN IN QCD

Here we give three different ways of estimating effective quark masses M (defined precisely below) from which follow values for F_π (roughly, $F_\pi=0.3M$). The first is a literal transcription of the work of the last section to a logarithmic potential in $d=3+1$ dimensions, as advocated for charmonium and $b\bar{b}$ states by several authors.[14] The numbers which emerge are very large, e.g., $M \simeq 1$ GeV, $F_\pi \simeq 330$ MeV, so that this exercise is not a success. But there is no reason to believe that logarithmic confinement is a good model of QCD over the entire range of masses from pions to heavy quarkonium, and no reason to be dismayed. We do the exercise only because it follows so easily from the work of the last section.

The second way is also a propagator theory, but this time the phenomenology[5] is that of linear confinement: The product of the coupling constant $\bar{g}^2(k)$ and the quark Casimir eigenvalue $C_F=\frac{4}{3}$ is written $\bar{g}^2(k)C_F=-m^2k^{-2}$, with $m\simeq 1.9$ GeV. When one attempts to use the machinery of Sec. III, there is a new problem: A logarithmic infrared divergence persists in $\Sigma(W,W')$. We show that this divergence can be gauged away, and that the infrared mass regulator should be replaced by a mass scale characteristic of hadronic bound states (essentially the inverse radius). The precise value of this mass depends on in what *color-singlet* process the propagator will be used; that is, in a confining theory with a linear potential, the quark propagator simply cannot be given an independent status. It has meaning only as part of a whole: the color-singlet process. We are hardly justified in calculating the consequences of this theory to better than, say, 30% accuracy, so we replace the now-finite logarithm by unity and estimate $M\simeq m(2\pi)^{-1}\simeq 300$ MeV, $F_\pi \simeq 100$ MeV, very reasonable values.

In these propagator theories, which so closely resemble potential models, where is the signal that a tunneling process between particles and antiparticles is involved? We will see that this resides in the ostensibly infrared-divergent logarithm of the linear-confinement theory, but it is not quite as clear where the tunneling signal is for logarithmic confinement, perhaps because this theory is on the boundary between confining and nonconfining.

The third method for estimating M and F_π is the most interesting because it reveals and clarifies the deficiencies of the other two models. It uses a confinement phenomenology based on an area law (continued to Minkowski space) for Wilson loops, such as might arise from a condensate of vortices.[19] The Wilson loops are converted to color-singlet quark propagator products by proper-time and path integrals; the essential difference between confining and nonconfining theories is that the dominant paths are periodic and restricted to a finite region of three-space, as one expects for a bound state.[5] In many interesting applications the infrared properties of such an area-law theory are virtually (in $d=2$, exactly[4]) the same as the propagator theory with $\bar{g}^2\sim k^{-2}$ ($\sim$ constant in $d=2$). For a single hadron, they are the same (or nearly so) when no two elements of the quark paths stand in spacelike separation to each other. This means that all quark proper velocities $\dot{x}_\mu(s)$ are forward timelike, which is to say that all quark paths are pure particle and all antiquark paths are pure antiparticle. This seems to be a reasonable approximation for heavy hadrons ($\rho, N, J/\psi, \ldots$). But the propagator theory breaks down in two respects, which are related: It has long-range van der Waals forces, and it does not give quite the same results as an area-law theory does for chiral-symmetry breakdown, which necessarily requires consideration of non-FTL paths. (The area-law theory has no van der Waals forces even if vortices are present.[19])

Chiral-symmetry breaking comes from tunneling between a quark and nearby virtual $q\bar{q}$ pairs (as evidenced by paths which have non-FTL segments). Propagator theories prescribe forces between the quark and the pair which would become illegal van der Waals forces if the separation between the quark and the pair became large. Of course, long-distance tunneling is severely suppressed, so there are no gross differences between the area-law theory and propagator theories for chiral-symmetry breakdown; that is why we are able to give consistent estimates for F_π in different models.

It turns out that in the propagator theory with $\bar{g}^2 \sim k^{-2}$, the entire effect of the infrared-singular propagator can be gauged away *for FTL paths*. This is not essential for further developments, but it helps us to understand how to find (in the area-law theory) a term linear in s in the proper-time exponent of (2.2). We treat the tunneling amplitudes associated with finite-length non-FTL segments analogously to the handling of barrier-penetration instantons in quantum mechanics.[20] Unfortunately, it is not easy to calculate an accurate value of M (even if we believed the underlying theory justified high accuracy). Values for M range from 50 to 500 MeV, with a "best" value of around 300 MeV, so $F_\pi \simeq 100$ MeV.

How are these effective masses precisely defined and how are they converted to values for F_π? It is, of course, useless to define the mass as the position of a propagator pole, since there are no such poles (in most gauges), and even if there were such a pole F_π depends on more than just the pole position.

As discussed in the Appendix, F_π depends on a momentum-dependent quantity $M(p^2)$, defined as

$$\{\gamma_5, S^{-1}(p)\} \equiv -2\gamma_5 M(p^2)\,, \tag{4.1}$$

where $S(p)$ is the effective quark propagator in the Feynman gauge. (In principle, any gauge could be used for calculating F_π, but we have given reasons in Sec. II for believing that the Feynman gauge is best suited to the approximations we make.) The effective quark mass M is defined as

$$M = M(p^2 = 0) = -S^{-1}(0)\,. \tag{4.2}$$

If $M(p^2) = M$ is substituted in the formula (A10) for F_π^2 there is an ultraviolet logarithmic divergence, so we need to say something about how $M(p^2)$ behaves for large p^2. This is not hard to do because the Dyson equation for large p^2 has a known kernel, given by renormalization-group-improved perturbation theory. It has been known for a long time[17] that this equation has two solutions, an ultraviolet-dominated one $M(p^2) \sim (\ln p^2)^{-A}$, and an infrared-dominated solution $\sim p^{-2}(\ln p^2)^{+A}$. As Pagels[18] and Lane[17] have emphasized, this second solution is the one relevant for chiral-symmetry breakdown because it is consistent with operator-product expansions in the presence of Goldstone bosons and also consistent with quark-counting rules for the pion form factor.

We have no justification to do anything more elaborate than introduce a parameter Λ to describe the transition from infrared to ultraviolet:

$$M(p^2) = \frac{M\Lambda^2}{\Lambda^2 - p^2}\,. \tag{4.3}$$

When (4.3) is used in (A10) we find

$$F_\pi = \frac{3^{1/2}M}{2\pi} F\left(\frac{\Lambda^2}{M^2}\right), \tag{4.4}$$

with $F(x) \simeq 1$. For $x > 10$, $F^2(x)$ is well approximated by $\ln x - 1.2$. A plot of $F(x)$ is given in Fig. 1.

What do we use for Λ? In the first place, it is very reasonable that Λ scales with M (or with F_π), but it is not necessarily the case that $\Lambda \simeq M$. For example, consider the QCD calculations[22] of the asymptotic pion electromagnetic form factor $F_{\text{em}}(t)$:

$$F_{\text{em}}(t) \sim \frac{2F_\pi^2}{b}\,\frac{\ln t}{t}\,, \tag{4.5}$$

where $b = (48\pi^2)^{-1}(33 - 2N_f)$ is the lowest-order coefficient in the β function. It cannot be far off the mark to take $\Lambda^2 \simeq 2b^{-1}F_\pi^2$, since the pion Beth-Salpeter amplitude used to construct $F_{\text{em}}(t)$ is essentially $M(p^2)$. Highly accurate values of Λ^2 are not needed because $F(\Lambda^2/M^2)$ in (4.4) is not rapidly

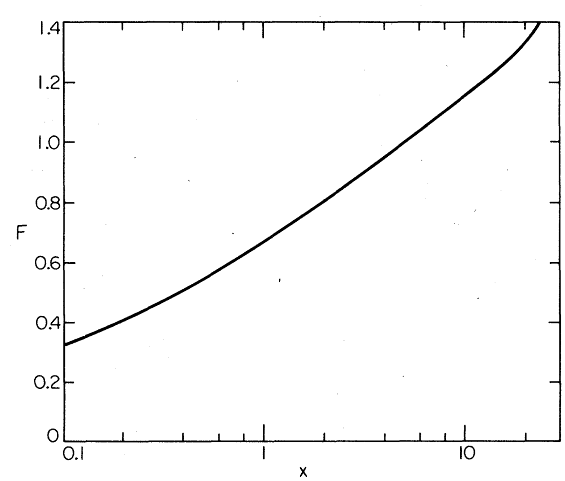

FIG. 1. A plot of $F(x = \Lambda^2/M^2)$ which determines the ratio of F_π to the effective quark mass M.

varying (see Fig. 1). With this estimate for Λ^2 ($\Lambda \simeq 600$ MeV for the experimental value of F_π) we have

$$\frac{F_\pi}{M} = \frac{3^{1/2}}{2\pi} F\left(\frac{2}{b}\frac{F_\pi^{\,2}}{M^2}\right) \simeq 0.3 . \qquad (4.6)$$

One could as well argue that Λ should be the ρ mass, or more accurately the mass used in the dipole fit to nuclear electromagnetic form factors. For $M \simeq 300$ MeV, (4.4) continues to predict $F_\pi \simeq 100$ MeV. But for $M \simeq 1$ GeV (as in the logarithmic model), F_π is about 200 MeV instead of over 300 MeV, as found earlier.

A. Logarithmic confinement

We use the formulas of Sec. III (in four dimensions) with e^2k^{-2} replaced by $C_F\bar{g}^2(k)k^{-2}$ and

$$C_F\bar{g}^2(k) = \hat{m}(-k^2)^{3/2} . \qquad (4.7)$$

Here $C_F = \frac{4}{3}$ is the quark Casimir eigenvalue. The corresponding static potential, appropriately regulated, is

$$V(r) = \frac{\hat{m}}{2\pi^2}\ln \mu r . \qquad (4.8)$$

According to workers[14] who have fitted such a potential to charmonium and $b\bar{b}$ states, $\hat{m}$ has the astonishingly large value of 14 GeV.

The kernel of the integral equation (3.4) is easily found:

$$\Sigma(W, W') = \frac{\hat{m}}{2\pi^2}\int_0^1 d\beta \frac{(1-\beta)^{1/2}[-2W(1-\beta)+4W']}{[\beta W'^2 + (1-\beta)\mu^2 - \beta(1-\beta)W^2]^{1/2}} . \qquad (4.9)$$

Just as in $d = 2+1$ QED, the perturbative self-mass M_p is infrared divergent:

$$M_p \equiv \Sigma(M_p, M_p) \simeq \frac{\hat{m}}{4\pi^2}\ln\left(\frac{2M_p}{\mu}\right) . \qquad (4.10)$$

As before, the sum of the mass term and the $q\bar{q}$ potential is free of leading infrared divergences. A new feature arises for QCD where three-quark states are color singlets. The qq potential is just one-half the $q\bar{q}$ potential:

$$V_{12} = \frac{1}{2}\frac{\hat{m}}{2\pi^2}\ln \mu |\vec{r}_1 - \vec{r}_2| . \qquad (4.11)$$

Clearly, $3M_p + V_{12} + V_{13} + V_{23}$ is also free of leading divergences.

Define an effective quark mass M by

$$M = -S^{-1}(W=0) = \Sigma(0, W') = \frac{\hat{m}}{4\pi} \simeq 1.1 \text{ GeV} . \qquad (4.12)$$

Things look bad for F_π if the effective quark mass is this large; Eq. (4.6) gives $F_\pi \simeq 300$ MeV. We will not pursue this model further.

B. Propagator theory of linear confinement

It is known that the leading logarithmic singularities of QCD, both in the infrared and in the ultraviolet regime, sum up to Abelian exponentials.[4,16] Quarks in a single hadron interact through an effective gluon propagator just as in QED with the replacement (*modulo* gauge terms)

$$-e^2\frac{g_{\mu\nu}}{k^2} \to -g_{\mu\nu}\frac{\bar{g}^2(k)}{k^2}C_F \qquad (4.13)$$

for gluon exchange between q and $\bar{q}$ or for a gluon beginning and ending on the same quark line; for qqq processes each pair of quarks interacts through (4.31) multiplied by $\frac{1}{2}$. Note that this effective propagator is *not* the (gauge-dependent) canonical gluon propagator in any gauge, but rather a compound object receiving contributions from vertex graphs; it is completely gauge independent.[4,5,16,19] This motivates (but certainly does not justify) the extrapolation of $\bar{g}^2(k)$ to a nonperturbative function which yields a linearly rising potential. (A better motivation will be found in the area-law theory discussed below.) This can be done,[16,23] for example, by using an effective β function of the form

$$\beta(g) = -bg^3(1+bg^2)^{-1} . \qquad (4.14)$$

Rather than use the complicated $\bar{g}^2(k)$ which emerges from (4.14),[16] we simply use the confining part:

$$C_F\bar{g}^2(k) = -m^2k^{-2} , \quad m \simeq 1.9 \text{ GeV} . \qquad (4.15)$$

Work on the nonconfining corrections will be reported later. The gluon propagator defined by (4.13)–(4.15) is rather strange; it has no particle interpretation, and it leads to off-shell infrared singularities in Feynman graphs. The lack of a particle interpretation means that we must suppress processes involving would-be on-shell gluons. This can be done in coordinate space by taking the real part of the propagator,[5] or in momentum space by taking k^{-4} as a principal part.[11] The latter leads to complicated integrals and awkward divergences which cancel in physical processes. Instead, we use the simple tactic of regulating k^{-4} to $(k^2 - \mu^2 + i\epsilon)^{-2}$ and suppressing on-shell gluons "by hand," if necessary.

We appear to be stuck with the infrared divergences as $\mu \to 0$, but in fact this is not the case. The reason is that color-singlet Green's functions in this propagator theory possess an Abelian gauge invariance and the choice of different infinitesimal μ's is just a choice of different gauges. For meson Green's functions this gauge invariance is precisely analogous to that of QED [see (4.28) below]; for baryon Green's functions it is slightly more elaborate,[19] depending on the fact that the qq gluon-exchange potential has one-half the strength of the $q\bar{q}$ potential [see the remarks below (4.11) about cancellation of infrared singularities between quark masses and potentials].

To see that different choices of μ correspond to different gauges, look at the effective gluon propagator in coordinate space:

$$\Delta_{\alpha\beta}(x)=m^2 g_{\alpha\beta}\frac{1}{(2\pi)^4}\int d^4k\frac{e^{-ikx}}{(k^2-\mu^2)^2}$$
$$\underset{\mu\to 0}{\sim}\frac{-im^2 g_{\alpha\beta}}{16\pi^2}\ln[\mu^2(-x^2+i\epsilon)]+O(\mu^2x^2)\,. \quad (4.16)$$

[For future reference we record the real part of the propagator,

$$\mathrm{Re}\Delta_{\alpha\beta}(x)=\frac{m^2 g_{\alpha\beta}}{16\pi}\theta(x^2)+\cdots. \quad (4.17)$$

This is regulator independent.] A different choice of μ, say μ', leads to the propagator

$$\Delta'_{\alpha\beta}(x)=\Delta_{\alpha\beta}(x)-i\frac{m^2}{32\pi^2}\ln\left(\frac{\mu'^2}{\mu^2}\right)\partial_\alpha\partial_\beta x^2\,. \quad (4.18)$$

The existence of such a gauge transformation (real in Euclidean space) means that no color-singlet Green's function can depend on μ (as long as μ is infinitesimal). Of course, a quark propagator is not a color singlet, but to the extent that an effective quark propagator can be defined at all, (4.18) entitles us to replace $\ln\mu$ in this propagator by $\ln\omega$, where ω is no longer infinitesimal but instead is of mass typical of hadrons [e.g., Λ of (4.3)]. One recalls similar arguments used long ago to dispose of spurious infrared divergences in the Lamb shift.

We proceed as before, calculating $\Sigma(W,W')$ and M_p, the perturbatively defined infrared-singular mass:

$$\Sigma(W,W')=\frac{m^2}{8\pi W}\ln\left(\frac{W'^2}{W'^2-W^2}\right)$$
$$+\frac{m^2(2W'-W)}{8\pi^2W^2(\beta_+-\beta_-)}\ln\left[\frac{\beta_-(1-\beta_+)}{\beta_+(1-\beta_-)}\right], \quad (4.19)$$

where

$$\beta_\pm=\tfrac{1}{2}\left\{1+\frac{\mu^2-W'^2}{W^2}\pm\left[\left(1+\frac{\mu^2-W'^2}{W^2}\right)^2-\frac{4\mu^2}{W^2}\right]^{1/2}\right\}.$$

(Note that Σ must be treated as a real quantity in order to avoid problems with would-be gluon emission processes.) For infinitesimal μ the perturbative self-mass is

$$M_p=\frac{m^2}{16\pi\mu}+O(\mu\ln\mu)\,. \quad (4.20)$$

On the other hand, the static $q\bar{q}$ potential is

$$V(r)=-\frac{m^2}{8\pi\mu}+\frac{m^2r}{8\pi}+O(\mu) \quad (4.21)$$

and, as before, $2M+V(r)$ is free of divergences (also for qqq processes). This allows us to evaluate m as $\simeq 1.9$ GeV, based on fits to charmonium spectra.[15]

For logarithmic confinement the integral-equation technique of Sec. III removed all the infrared divergences associated with M_p and made the propagator entire. In the present case, $\Sigma(W,W')$ still has an infrared divergence which we have argued can be gauged away [it is interesting to find the $\ln\mu$ terms in (4.19) by using the gauge transformation (4.18) in momentum space]. When the Dyson equation (3.4) is evaluated at $W=0$ we find, not the propagator at $W=0$, but instead essentially its first derivative:

$$1=+\frac{m^2}{4\pi^2}\int dW'\frac{\rho(W')}{W'^2}\ln\left(\frac{W'^2}{\mu^2}\right). \quad (4.22)$$

We have already argued that μ can be replaced by a finite mass, say of $O(\Lambda)$ in (4.3). At this stage of hadronic physics it would be pointless to do anything more than estimate the logarithm in (4.22) as essentially unity; when this is done (4.22) is simply

$$1=-\frac{m^2}{4\pi^2}\frac{\partial S}{\partial W}(W=0)\,. \quad (4.23)$$

For a normal propagator with the form $S^{-1}(W)=W-M$, this predicts

$$M=\frac{m}{2\pi}\simeq 300\text{ MeV}\,. \quad (4.24)$$

Even though the effective propagator is entire, we use (4.24) to define the effective quark mass. Then (4.6) gives $F_\pi\simeq 100$ MeV.

One expects that chiral-symmetry breakdown in QCD should resemble the generation of fermion masses in a superconductor where the gap (fermion mass) has a factor $e^{-1/g}$. Here g is the coupling constant for the potential which binds Cooper pairs and which is not infrared singular. In a sense, the propagator theory we use does have such a factor; it is signaled by the logarithm in (4.22). The closest we can come to a relativistic superconductor is to define the mass from $\Sigma(M,M)$, as given in (4.19), with the proviso that $\mu\gg M$; now one finds the solution $M=0$ as well as

$$M\simeq\mu\exp(-2\pi^2\mu^2m^{-2})\,, \quad (4.25)$$

with m^2 playing the role of g. The idea of taking μ finite probably makes some sense (for the effective propagator) but it is not possible to have $\mu\gg M$. The physical situation is more like $2\pi^2\mu^2\simeq m^2$, $\mu\simeq M$.

C. An area-law theory

Earlier, we motivated a gluon propagator $\sim k^{-4}$ as a relativistic generalization of a linearly rising potential. But any propagator theory for $d>2$ apparently has trouble with long-range van der Waals forces,[24] a defect not shared by area-law theories. The prototypical continuum area-law theory has confinement of fractionally charged particles by a condensate of vortices; examples are the Abelian Higgs model in $d=2$,[25] and possibly massive-gluon QCD in $d=4$[19] (if the vortices condense). Of course, $d=2$ massive-quark QCD is also a confining theory, but it is a propagator theory (as far as infrared behavior goes). Actually, in $d=2$ the two types of confinement are very similar.[4]

The area-law theory and the propagator theory have the same general structure, so let us first recall the coordinate-space expression for some hadronic Green's function in the propagatory theory discussed immediately above:

$$\langle 0|T(\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y))|\rangle \sim \int_0^\infty ds_1 ds_2 (d\,\text{path}) e^{iA_0+iA_I}, \tag{4.26}$$

$$A_0=-\frac{\hat{M}}{2}\sum_{i=1}^{2}\int_0^{s_i} d\tau[\dot{x}(i)^2+1], \tag{4.27}$$

$$A_I=-\tfrac{1}{2}\sum_{i,j}\int_0^{s_i} d\tau_i \int_0^{s_j} d\tau_j' \eta_{ij}\dot{x}^\mu(i)\dot{x}^\nu(j)\,\text{Re}\Delta_{\mu\nu}[x(i)-x(j)] \quad (\eta_{ii}=1\,;\ \eta_{ij}=-1 \text{ for } i\neq j) \tag{4.28}$$

The path integral is over all quark paths $x^\mu(1)$ going from x to y in proper time s_1 and all antiquark paths $x^\mu(2)$. $\hat{M}$ is the intrinsic quark mass (aside from that produced by chiral-symmetry breakdown); the above Green's function is perfectly well behaved in the limit $\hat{M}=0$ relevant for us.[5] Equations (4.26)–(4.28) are written for spinless quarks, which is not an oversimplification; the main point in chiral-symmetry breakdown is to distinguish particle from antiparticle which is conveniently done with Dirac matrices but which we can do by other means. Moreover, spin-dependent gluon forces are short range,[19] thus outside the scope of this paper. Our area-law theory has the same general structure as (4.26)–(4.28) except that A_I is differently defined.

In (4.28), $\Delta_{\mu\nu}$ is the effective gluon propagator in (4.16) and (4.17). The real part is a θ function prescribing a timelike separation for its argument. It is more convenient to make a gauge transformation of (4.17), which leaves (4.26) unchanged:

$$\text{Re}\Delta_{\alpha\beta}(x)\to\frac{m^2}{16\pi}g_{\alpha\beta}\theta(x^2)-\frac{m^2}{32\pi}\partial_\alpha\partial_\beta x^2 = -\frac{m^2}{16\pi}g_{\alpha\beta}\theta(-x^2) \tag{4.29}$$

because this greatly simplifies the quark self-energy terms [i.e., the terms with $i=j$ in (4.28)]. Indeed, it is easy to see that these vanish for all FTL paths [those for which $\dot{x}^\mu(s_i)$ is FTL everywhere]. There are many circumstances in which it is physically reasonable to approximate (4.26) with only FTL paths, for example, when the quarks are heavy. Then (4.26) has a potential-theory limit in which the dominant FTL paths are periodic and confined to a finite volume of three-space. These paths are easily generalized to relativistic FTL motion and have a sensible $\hat{M}=0$ limit; they have been used[5] to discuss the WKB bound-state spectrum of (4.26) and shown to yield a linearly rising Regge trajectory. But such paths yield zero quark mass and do not produce chiral-symmetry breakdown. Paths which are almost FTL, that is, which have non-FTL segments of finite duration sprinkled on an otherwise FTL path, give rise to quark self-energy terms in A_I which are linear in the s_i and are thus identified as quark masses. Such terms can be calculated for the propagator theory defined by (4.28) and (4.29), but we will calculate them instead in the area-law theory.

The Minkowski-space area-law theory is defined by an obvious continuation of the Euclidean theory. In the Euclidean version (with $s_j\to -is_j$, $x^0\to ix^0$) iA_I in (4.26) is replaced by $-(m^2/8\pi)\mathfrak{a}$, where $\mathfrak{a}$ is the area of a certain surface spanning the $q\bar{q}$ loop. It is a somewhat delicate question what $\mathfrak{a}$ should be for the most general loop, but we will only consider loops where it is apparent that $\mathfrak{a}$ is the absolute minimum area; these loops will be flat and in one space and one time dimension. Such simplification is consistent with earlier crude WKB analysis,[5] but needs to be eliminated in future work.

In two dimensions the area-law theory and the propagator theory (which is essentially $d=2$ QCD for spinless quarks) are exactly the same in the infrared regime, provided only FTL paths are kept.[4] The demonstration holds in the light-cone gauge and is assumed to be true in all gauges. Evaluate A_I in the light-cone gauge and in two dimensions. For FTL paths, the $i=j$ terms vanish and we find

$$A_I = -\frac{m^2}{8\pi}\int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2\, \dot{x}_-(1)\dot{x}_-(2)\,|x_+(1)-x_+(2)| \times \delta(x_-(1)-x_-(2)), \quad (4.30)$$

where

$$x_\pm = \frac{1}{\sqrt{2}}(x_0 \pm x_1). \quad (4.31)$$

For FTL paths, $\dot{x}_-$ is always non-negative and the δ function has a unique root, so the integral over one τ can be evaluated:

$$A_I = -\frac{m^2}{8\pi}\int d\tau\, \dot{x}_-|x_+(1)-x_+(2)|, \quad (4.32)$$

where the x_+'s are evaluated at a common τ. It is easily seen that this is the same as the area-law action

$$-\frac{m^2}{8\pi}\int d\sigma\, d\tau [(z_\tau \cdot z_\sigma)^2 - z_\tau^{\,2} z_\sigma^{\,2}]^{1/2} \quad (4.33)$$

[where $z^\mu = z^\mu(\sigma,\tau)$ gives the points of the surface, and subscripts indicate derivatives] in the light-cone gauge $z_- = f(\tau)$. This gauge allows one to do the σ integration explicitly to come to (4.32).

The propagator theory can be evaluated in any gauge and the gauge of (4.17) is just $d=2$ QCD in the Feynman gauge (with $m^2/4\pi$ identified as $C_F g^2$) for FTL paths.

For non-FTL paths, (4.30) and (4.33) do not agree either because $\dot{x}_-$ is negative somewhere, or because the δ function in (4.30) has multiple roots. To see this graphically, consider Figs. 2 and 3. Figure 2 shows a quark and an antiquark in periodic straight-line motion at (nearly) the speed of light.[26] In the propagator theory, the q and $\bar{q}$ interact at a common value of x_- and it is easy to see how (4.30) measures the area. Figure 3 shows a sample of different insertions of non-FTL segments on an otherwise FTL path. In Fig. 3(a), for example, line segments at constant x_- are connected by propagators both inside and outside the area defined by (4.33). It is clear that segments of Fig. 3 which go backwards in x_+ or x_- can be interpreted an antiparticle segments on a particle path; these are like z graphs and exhibit a particle accompanied by a virtual $q\bar{q}$ pair. There is an evident similarity between the non-FTL segments of Fig. 3 and the usual Bardeen-Cooper-Schrieffer BCS gap equation of Fig. 4.

The formation of these pairs is described by a tunneling amplitude, from which we can read off the effective quark mass. Let us take seriously the graphs of Figs. 2 and 3 as a description of meson bound states. The unit step for these paths is $\Delta x_0 = \Delta x_1 = \frac{1}{2}T$, where T is the bound-state period. We have made[5] crude WKB estimates of T earlier based on simple circular classical orbits somewhat like those of Fig. 2. This work used the bound-state frequency $\omega \equiv 2\pi T^{-1}$, which we estimate to be $\omega \simeq 0.3m - 0.35m \simeq 600-700$ MeV. (This number was not reported as such in Ref. 5, but was used in numerical work given there.) For the circular orbits of Ref. 5, ω^{-1} is the meson radius and the $q\bar{q}$ action per half bound-state period is $(m^2/8\pi)2\pi\omega^{-2}$; we use this as the definition of action per unit square of Figs. 2 and 3. (Roughly

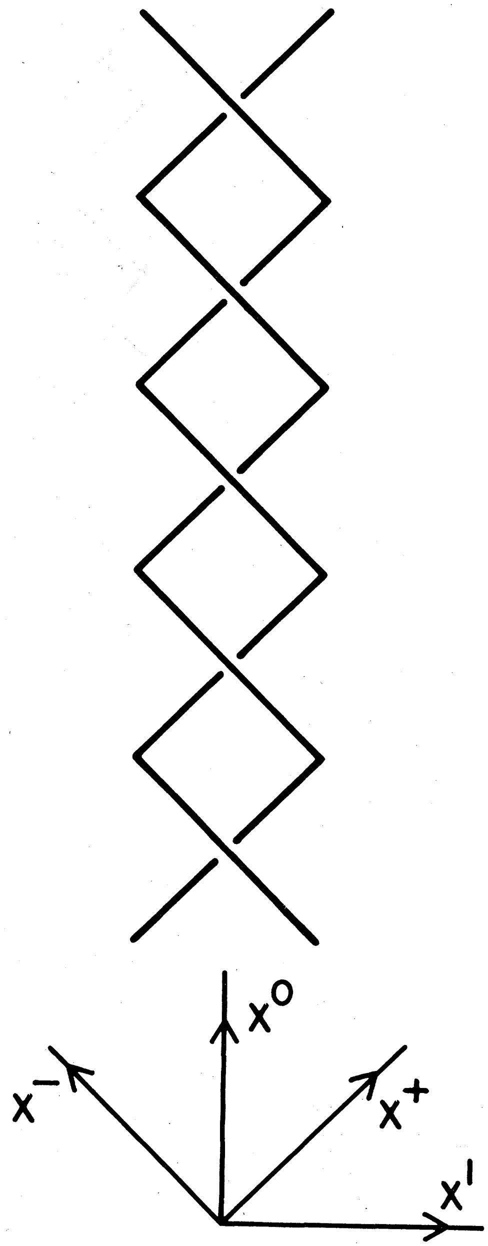

FIG. 2. Approximation to the space-time paths of a quark and an antiquark in a periodic bound-state orbit.

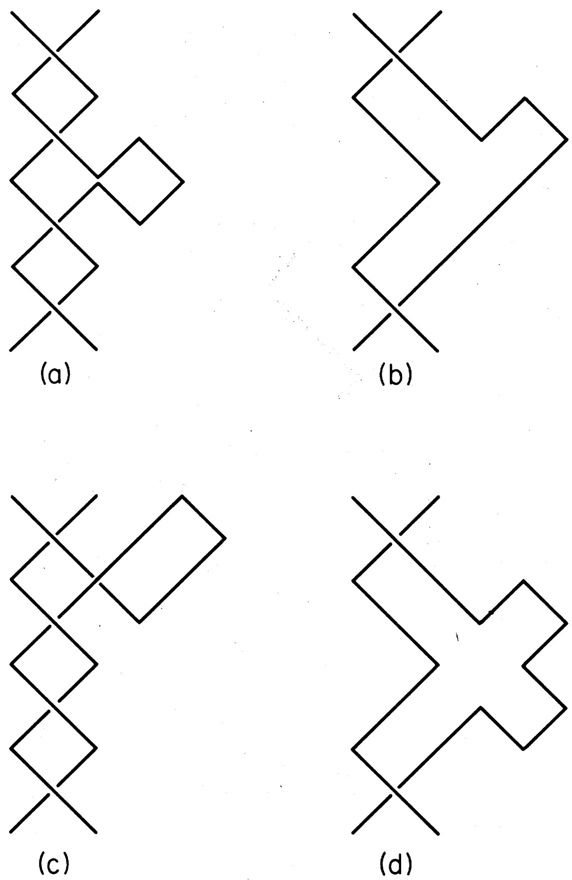

FIG. 3. The FTL paths of Fig. 2, with non-FTL instantons added; an infinite number of such instantons could be drawn.

speaking, this part of the action should be $\simeq\pi$ in the WKB approximation to a bound state.)

Let us now go to a gauge where the quark self-interaction terms are small or vanishing for FTL paths [such as (4.29), or the light-cone gauge for $d=2$]. The only remaining term in A_I for FTL paths is the $q\bar{q}$ term; let us denote its contribution to the Green's function, for the FTL paths of Fig. 2, as $e^{-N\phi(q\bar{q})}$ (in Euclidean space). Here N is the number of bound-state half periods which have elasped, i.e., the number of diamonds in Fig. 2 from beginning to end of the Wilson loop. The non-FTL insertion of Fig. 3 can be summed just as for instantons,[20] so the Green's function is proportional to

$$e^{-N\phi(q\bar{q})}\left[1+N\sum e^{-\phi_j}+\frac{N(N-1)}{2!}\left(\sum e^{-\phi_j}\right)^2+\cdots\right]$$
$$\simeq \exp\left\{-N\left[\phi(q\bar{q})+\sum e^{-\phi_j}\right]\right\}. \quad (4.34)$$

The summation index j refers to different configurations in Fig. 3. In particular, one should be prepared to sum over non-FTL fluctuations of all sizes (like instantons of all sizes), but very small fluctuations are suppressed by their large kinetic energy and large ones are suppressed by the area law. A combination of analytic and numerical work which is too tedious to report in detail suggests that a reasonable estimate is obtained by taking the size scale of Fig. 3 literally. We identify the quark mass as the coefficient of t via $N=t/\Delta t=\omega t/\pi$ in (4.34), where t is the common quark/antiquark time. Unfortunately, it is impossible to achieve high accuracy both because of the uncertainty in the ratio ω/m (which is exponentiated) and because many graphs must be summed in a series which appears to be rather slowly converging. Saving the twenty or so graphs of the type in Fig. 3 (aside from reflections) of largest value and taking $\omega \simeq 600-700$ MeV gives values for M from 50 to 500 MeV, with a best value in the middle of this range. It does not seem worthwhile to try to improve this spread much until one is more certain that sufficient detail is included in the underlying theory, especially short- and middle-ranged effects. As in the propagator theory, $M\simeq 300$ MeV gives an estimate for F_π of $\simeq$100 MeV. Of course, the closeness to the experimental value is entirely accidental.

V. SUMMARY AND OUTLOOK

We need not detail the deficiencies of our estimates of F_π, which are partly the result of using models which only inaccurately reflect QCD, and partly the result of our inability to deal with the models exactly. But it was not at all obvious when we stated that, say, the propagator model would give a value of F_π within a factor of 3 of the experimental value. As it turns out, we can optimistically hope that all factors of 2π, $3^{1/2}$, and the like, which differ significantly from unity, have been identified in the formula $F_\pi \simeq 2a3^{1/2}(2\pi)^{-3/2}\simeq 88$ MeV for $a=400$ MeV. At the least, it is instructive that F_π, the smallest of all hadronic mass parameters (except for current-quark masses), should be simply given in terms of the parameter a determined from the heaviest-known hadrons.

The same parameter a (or its equivalent, the mass m occurring in the running coupling constant)

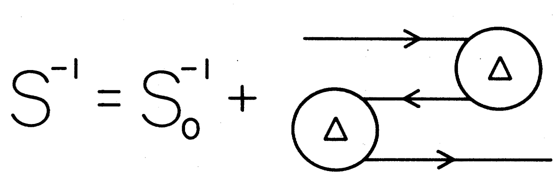

FIG. 4. The gap equation for superconductivity showing the excitation of a particle-hole condensate.

sets the scale for every hadronic mass in the idealization of zero current-quark mass. This means that QCD must yield a set of dimensionless mass ratios varying over at least an order of magnitude. In itself this poses no particular problem; witness the success of the bag model or the more crudely explored propagator model used here [which, e.g., predicts the Regge slope to be $\alpha'(0) \simeq (8a^2)^{-1}$]. But the successes are strictly limited to calculations of the properties of an isolated hadron; we are far from being able to make the crudest estimates of hadronic coupling constants from QCD or even from models of QCD.

Now F_π plays a critical role in determining these coupling constants. *In fact, knowledge of the hadronic mass spectrum, and of F_π, plus unitarity constraints completely determines all couplings of hadrons with spin ≤ 1 (in the zero-quark-mass limit), without further reference to the dynamical properties of QCD.* One could replace the words "unitarity constraints" by a plethora of older relationships, such as Goldberger-Treiman, Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF), and the first Weinberg sum rule, which are derived from a variety of assumptions, but all these relations are consequences of the bound $|\sin\delta| \leq 1$, where δ is a scattering phase shift. The point is that perturbative unitarity for the phenomenological hadron Lagrangian demands that this Lagrangian be that of a spontaneously broken gauge theory.[27]

We briefly discuss a simple example here, deferring details to a later publication. Suppose the meson spectrum (for spin ≤ 1) contains isovector multiplets with $J^p = 0^-, 1^-$, and 1^+ (π, ρ, A_1): In each multiplet the masses are equal and $M_\pi = 0$, $M_{A_1} \neq M_\rho$. It turns out that pions are necessarily massless and $F_\pi \neq 0$ reflects the fact that $M_{A_1} \neq M_\rho$. Then, in addition, there must be three isosinglets, two 0^+ particles and a 0^- state. All these couple to each other and to nucleons. Unitarity tells us that ρ and A_1 must couple in a Yang-Mills fashion under the chiral group $SU(2)_L \times SU(2)_R$; in particular, only one coupling constant g characterizes *all* ρ and A_1 couplings, to themselves and to anything else. It is very instructive to derive, directly from the constraint of tree-graph unitarity, the Goldberger-Treiman, KSRF, and first Weinberg sum rules, plus a few others, which determine completely all the remaining couplings such as $G_{\pi NN}$, once we know F_π. These couplings are strong simply because F_π is small (e.g., the KSRF relation is $g^2 = M_\rho^2/2F_\pi^2$). At this point one may well wish to resurrect the old bootstrap principle, augmented with unitarity constraints, to see how well it works in predicting mass ratios, that is, to see how far we can go in the direction of QCD without QCD. Should this be successful, knowledge of F_π alone becomes supremely important.

The same concepts and means of calculation will also be useful in discussing other strongly interacting systems, e.g., "weak" interactions at $\simeq 1$ TeV. With the assumption of no elementary scalar fields,[28] it may be that an analog of color[29] (but with $a \simeq 1$ TeV) is responsible for the appearance of broken-symmetry physical multiplets, with all masses scaled by the ratio of a's.[30]

We do not know exactly what to say about the effective quark masses M which determine F_π. These are not really constituent masses, which is a misnomer for constituent energies, and they are certainly not any sort of genuine mechanical mass. Their dynamical role is not yet clear because they tend to get swallowed up in what we usually think of as the potential; that is, in the linearly rising potential a term $-m^2/8\pi\mu$ is exactly canceled by the infrared-divergent fermion masses. A related and more cogent puzzle is: Exactly how does it happen (as we know it must) that the pion is massless in QCD? How does this emerge naturally in the $q\bar{q}$ Bethe-Salpeter equation (or similar dynamical scheme)? This seems considerably harder to show for infrared-singular forces than for the ultraviolet-singular forces invoked twenty years ago by Nambu and Jona-Lasinio,[1] but work on this problem is in progress.

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APPENDIX: CALCULATING F_π FROM THE FERMION PROPAGATOR

We collect here the well-known formulas necessary to calculate F_π from (effective) fermion propagators and proper vertex functions. The axial-vector current for quarks is

$$J_\mu^{5a}(x) = \bar{\psi}(x)\tfrac{1}{2}\tau^a\gamma_\mu\gamma_5\psi(x)\,, \qquad \text{(A1)}$$

with a trace over color indices understood. The quark-quark-current proper vertex $\Gamma_\mu^{5a}(p,p')$ obeys the Ward identity

$$q^\mu\Gamma_\mu^{5a}(p,p') = [S^{-1}(p)\gamma_5 + \gamma_5 S^{-1}(p')]\tfrac{1}{2}\tau^a\,, \qquad \text{(A2)}$$

with $q = p - p'$. Define a "mass" $M(p^2)$ by

$$\{\gamma_5, S^{-1}(p)\} \equiv -2\gamma_5 M(p^2). \qquad (A3)$$

If $M \neq 0$, Γ^{5a}_μ has a zero-mass pole and for q sufficiently small,

$$\Gamma^a_{5\mu} = \tfrac{1}{2}\tau^a\left[-2\gamma_5 M(p^2)\frac{q_\mu}{q^2} + \cdots\right], \qquad (A4)$$

where the omitted terms are regular at $q=0$.

The quark-quark-pion proper vertex Γ^{5a} has the kinematical decomposition

$$\Gamma^a_5(p,p') = i\gamma_5\{G_1 + \not{q}G_2 + [\not{q}, \not{p}+\not{p}']G_3 + (\not{p}+\not{p}')G_4\}\tfrac{1}{2}\tau^a. \qquad (A5)$$

(These G_i are not to be identified with the corresponding ϕ_i of Farrar and Jackson,[17] who use the improper vertex with quark legs attached.) Charge conjugation demands that G_4 be odd under $p \leftrightarrow -p'$ and thus vanish at $q=0$, so only G_1 is relevant at $q=0$. With the definition of F_π,

$$\langle \pi^a(q)|J^{5b}_\mu(0)|0\rangle = iq_\mu F_\pi \delta^{ab}, \qquad (A6)$$

we identify the residue of the pole in (A4) at small q with $i^2F_\pi G_1$, thus finding the Goldberger-Treiman relation (for unit axial-vector form factor)

$$G_1(p,p) = \frac{2M(p^2)}{F_\pi}. \qquad (A7)$$

[For pion-nucleon physics we would identify G_1 with $2g_{\pi N}(0)$.]

Unfortunately, the values of the other G_i at $q=0$ depend on the regular terms in the axial-vector vertex and on the exact form of $S(p)$. We will make the following approximations, if only because they are consistent with exact chiral symmetry and because we see no reason why infrared confinement effects should alter them greatly: (1) The regular term in (A4) is given exactly by $\gamma_\mu\gamma_5$; (2) the quark propagator can be replaced by the form

$$S^{-1}(p) = \not{p} - M(p^2). \qquad (A8)$$

Of course, this is not an entire function, but it has three properties of the correct propagator: $S(p)$ behaves like $\not{p}^{-1}$ at large p; $S(0)$ is, by definition, $-M^{-1}(0)$; the anticommutator (A3) is (by definition also) exact. The Ward identity (A2) now yields $G_2 = G_3 = G_4 = 0$, at $q=0$. G_2 will be nonzero if the coefficient of $\gamma_\mu\gamma_5$ is not precisely unity, but we estimate that a ±10% change in this coefficient will not alter F_π by more than 25%.

We return to the evaluation of F_π. Clearly, (A6) can be written in terms of (A5) and quark propagators as

$$iq_\mu F_\pi \delta_{ab} = \int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}S(p)\Gamma^{5a}(p,p-q)S(p-q)\tfrac{1}{2}\tau^b\gamma_\mu\gamma_5, \qquad (A9)$$

where the trace is over Dirac, flavor, and color indices, and on the right the limit is to be taken as $q \to 0$. With the approximations made above, one finds

$$F_\pi^2 = -\frac{12i}{(2\pi)^4}\int d^4p \frac{M(p^2)}{[p^2 - M(p^2)]^2} \times\left[M(p^2) - \tfrac{1}{2}p^2\frac{d}{dp^2}M(p^2)\right]. \qquad (A10)$$

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