Asymptotically free, one-coupling-constant, one-mass-scale SU(5) model

Ngee-Pong Chang
Physics Department, City College of the City University of New York, New York, New York 10031

Ashok Das
Physics Department, University of Maryland, College Park, Maryland 20742

Juan Perez-Mercader
Physics Department, City College of the City University of New York, New York, New York 10031

(Received 26 November 1979; revised manuscript received 31 July 1980)

We describe an SU(5) model that is an example of a true one-coupling-constant, one-mass-scale, asymptotically free grand unified theory. We present the derivation of the renormalization-group equations in a convenient tabular form.

Grand unification[1] is an exciting attempt at synthesis of the strong, electromagnetic, and weak interactions. At present, the leading candidate for a theory at energies of order 100 GeV or less is the Yang-Mills[2] gauge theory involving $SU(3)\times SU(2)\times U(1)$ which, apart from the three gauge couplings, has nine Yukawa couplings for the three generations of fermion masses and one quartic coupling and one mass scale for the Higgs bosons. In a total synthesis the three gauge couplings will be absorbed into one coupling of the grand unified group G. In such a grand unification, a leading minimal candidate for which is SU(5), there is naturally again a question of Higgs bosons and their couplings.

If we entertain the minimal scenario, which admits a $\underline{5}$ and a $\underline{24}$ Higgs multiplet, and we include three generations of light fermion multiplets in the $\underline{5}^*$ and $\underline{10}$ representations, then we again must have two mass scales, five quartic couplings, one gauge coupling, and six Yukawa couplings. A grand unified theory with 14 arbitrary parameters in the original Lagrangian is less a grand unification than a grand synthesis of the original patchwork of couplings and constants.

One suggestion for the economy of grand unification is to invoke dynamical symmetry breaking as the source of the mass generation in this theory and remove the Higgs boson altogether from the ultimate theory. This is the hope that heavy-color theory addresses itself to.[3]

The other suggestion is to take advantage of the arbitrariness and fix all the couplings by a new eigenvalue principle. While the plethora of Higgs bosons that seems to be necessary continues to proliferate, the asymptotic freedom of the original Yang-Mills theory has been destroyed. To restore the asymptotic freedom,[4] eigenvalue conditions have to be applied on the Yukawa and Higgs-boson quartic couplings.[5] The advantage of this approach is the total lack of arbitrariness in coupling constants for the resulting grand unified theory. It is a true one-coupling-constant, one-mass-scale, asymptotically free grand unified theory.

The fact that the couplings are calculated rather than arbitrary evidently reflects on the "compositeness" of the Higgs field. It would be nice to speculate that someday when dynamical symmetry breakdown is better understood, the "induced" Yukawa couplings and quartic couplings of the "composite" Higgs field can be calculated. Presumably in this fundamental version of the theory, the asymptotic freedom of the theory remains true.

At the present phenomenological level, we address ourselves to the following question. Can a SU(5) model with a $\underline{5}$ and $\underline{24}$ Higgs boson and an unspecified n_f generation of light fermions (in the $\underline{5}^*$ and $\underline{10}^*$ representation) be asymptotically free? An immediate answer to this is in the negative.

As has already been pointed out by Cabibbo, Maiani, Parisi, and Petronzio,[6] it is not impossible to live with this lack of asymptotic freedom. The quartic couplings λ_i are arranged to be $\lesssim 1$ for all energies less than 10^{15} GeV and since there are no experimental constraints on Higgs-boson masses the λ_i are essentially five arbitrary constants subject to some loose constraints.

If we believe in the true asymptotic freedom of the theory, then the next question is whether the SU(5) model can at all be made asymptotically free with the addition of new fermions. These fermions must necessarily be superheavy, i.e., of the same mass scale as that of the X gauge bosons. This is because the loss of asymptotic freedom is associated with the λ_i coupling constants and the only way to reduce the $d\lambda_i/dt$ is to introduce additional fermion loops that couple to all the Higgs bosons. Through symmetry breakdown, these new fermions acquire the same mass scale as the M_X.

In this search for asymptotic freedom, a prime issue is the fermion content of the theory. On the grounds of simplicity, we have chosen a set of superheavy fermions to be in the $\underline{5}$ and $\underline{24}$ representation, so that it is supersymmetric with the spin-0 $\underline{5}$ and $\underline{24}$ Higgs bosons. The result of our computer search is that indeed an asymptotically free SU(5) model exists with the following particle spectrum:

1 set of $\underline{24}$ gauge bosons,
1 set of $\underline{5}$, $\underline{24}$ superheavy Higgs bosons,
1 set of $\underline{5}$, $\underline{24}$ superheavy fermions,
7 sets of $\underline{5}^*$, $\underline{10}$ light fermions.

That an asymptotically free SU(5) model exists at all with a given particle spectrum is far from obvious. Fradkin and Kalashnikov[7] reported on an example with an involved particle spectrum. Our example is perhaps simpler in structure being supersymmetric. It is not, by any means, unique. However it may be a good example upon which to study detailed gauge hierarchy questions such as the generation of the m_W mass, etc. In this paper we limit ourselves simply to the search for asymptotic freedom itself. We reserve the applications to future publication.

Let us exhibit the complete SU(5) Lagrangian

$$\begin{aligned}\mathfrak{L} = &-\tfrac{1}{2}\mathrm{Tr}(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])^2 - \tfrac{1}{2}\mathrm{Tr}(\partial_\mu \phi - ig[A_\mu, \phi])^2 - |\partial_\mu H - igA_\mu H|^2 + \frac{\mu^2}{2}\mathrm{Tr}(\phi^2) - \frac{\lambda_1}{4}(\mathrm{Tr}\phi^2)^2 \\ &- \frac{\lambda_2}{2}\mathrm{Tr}(\phi^4) + \frac{\nu^2}{2}H^\dagger\cdot H - \frac{\lambda_3}{4}(H^\dagger\cdot H)^2 - \frac{\lambda_4}{2}H^\dagger\cdot H\,\mathrm{Tr}(\phi^2) - \frac{\lambda_5}{2}H^\dagger_\alpha(\phi^2)^\alpha_\beta H^\beta - \overline{\psi}_R\gamma_\mu(\partial_\mu - igA_\mu)\psi_R \\ &- \overline{\psi}_{L\alpha\beta}\gamma_\mu(\partial_\mu\psi_L^{\alpha\beta} - igA^\alpha_{\mu\gamma}\psi_L^{\gamma\beta} - igA^\beta_{\mu\gamma}\psi_L^{\alpha\gamma}) - \overline{\chi}_\alpha\gamma_\mu(\partial_\mu\chi^\alpha - igA^\alpha_{\mu\beta}\chi^\beta) - \overline{B}^\alpha_\beta\gamma_\mu(\partial_\mu B^\beta_\alpha - ig[A_\mu, B]^\beta_\alpha) \\ &- (\sqrt{2}\,h\overline{\psi}_{L\alpha\beta}\psi_R^\alpha H^\beta + \mathrm{H.c.}) - k_2\overline{\chi}_\alpha\chi^\beta\phi^\alpha_\beta - (k_4\overline{B}^\alpha_\beta\chi^\beta H^\dagger_\alpha + \mathrm{H.c.}) - k_5\overline{B}^\alpha_\beta B^\beta_\gamma\phi^\gamma_\alpha - k_6\overline{B}^\beta_\gamma B^\alpha_\beta\phi^\gamma_\alpha\,. \end{aligned} \tag{1}$$

As we shall show, asymptotic freedom of the theory forces upon the theory the following set of eigenvalues:

$$\begin{gathered} h = -0.06658\,g,\quad k_2 = -0.90392\,g,\quad k_4 = -1.3384\,g,\quad k_5 = -0.96576\,g,\quad k_6 = 0.69265\,g, \\ \lambda_1 = 0.01720\,g^2,\quad \lambda_2 = 0.66275\,g^2,\quad \lambda_3 = 2.87232\,g^2,\quad \lambda_4 = -0.06013\,g^2,\quad \lambda_5 = 2.40814\,g^2\,. \end{gathered} \tag{2}$$

With these eigenvalues there still remains a degree of freedom in the choice of two mass scales μ^2 and ν^2. In fact, the renormalization-group analysis of μ^2 and ν^2 shows[8] that they are coupled equations even when the eigenvalues are employed, viz.,

$$\begin{aligned} 16\pi^2\frac{d\mu^2}{dt} &= 27.2117\,\bar{g}^2\mu^2 + 2.1075\,\bar{g}^2\nu^2\,, \\ 16\pi^2\frac{d\nu^2}{dt} &= 20.2321\,\bar{g}^2\mu^2 + 37.4740\,\bar{g}^2\nu^2\,. \end{aligned} \tag{3}$$

In general, this coupled system of equations has the asymptotic behavior

$$\frac{\nu^2}{\mu^2} \xrightarrow[t\to\infty]{} 6.3752\,. \tag{4}$$

However, with the Higgs potential as given in Eq. (1) the masses of the H^i $(i=1,2,3)$ SU(3) triplet and H^a $(a=4,5)$ SU(2) doublet are given by

$$\begin{aligned} m^2(H^i) &= -\frac{\nu^2}{2} + \frac{\mu^2}{2}\,\frac{\lambda_4 + \frac{2}{15}\lambda_5}{\lambda_1 + \frac{7}{15}\lambda_2} \\ &= \tfrac{1}{2}(-\nu^2 + 0.7993\,\mu^2)\,, \\ m^2(H^a) &= -\frac{\nu^2}{2} + \frac{\mu^2}{2}\,\frac{\lambda_4 + \frac{3}{10}\lambda_5}{\lambda_1 + \frac{7}{15}\lambda_2} \\ &= \tfrac{1}{2}(-\nu^2 + 2.0286\,\mu^2)\,, \end{aligned} \tag{5}$$

and in the $t\to\infty$ limit, both the SU(3) and SU(2) groups would be spontaneously broken, a highly undesirable state of affairs.

Fortunately, there exists a *special* solution to the system of equations, viz.,

$$\frac{\nu^2}{\mu^2} = -1.5058 \text{ for all } t\,. \tag{6}$$

With this solution, both H^i and H^a have positive masses and they preserve the stability of our original vacuum even at high energies.

The set of eigenvalues in Eq. (2) preserves the positivity of the mass squared of the ϕ 24-plet as well $[v^2=\mu^2/(\lambda_1+\frac{7}{15}\lambda_2)]$,

$$m^2(\varphi^i_j)=\mu^2+(\lambda_1+\tfrac{4}{5}\lambda_2)v^2=0.6766\ \mu^2\,,$$
$$m^2(\varphi^a_b)=-\mu^2+(\lambda_1+\tfrac{9}{5}\lambda_2)v^2=2.7066\ \mu^2\,, \tag{7}$$
$$m^2(\sigma)=2\ \mu^2\,,$$

and for completeness we record the value for the X-boson mass

$$M^2=\tfrac{5}{12}g^2v^2=1.2762\ \mu^2\,. \tag{8}$$

The set of eigenvalues in Eq. (2), together with the special solution Eq. (6), makes our example of an SU(5) model into (i) a truly *one*-coupling-constant theory, with (ii) *one* mass scale, and (iii) one which preserves the asymptotic freedom of the theory.

It is an open question at this point whether with this as a starting point we can generate the second stage of hierarchy, at the right place, as we go down in energy. At the high-energy end, in view of Eq. (6), Eq. (3) implies that

$$16\pi^2\frac{d\mu^2}{dt}=24.0382\,\bar{g}^2\mu^2\,, \tag{9}$$

so that $[\,m_3{}^2\equiv m^2(H^i),\ m_2{}^2\equiv m^2(H^a)]$

$$16\pi^2\frac{dm_3{}^2}{dt}=27.7052\,\bar{g}^2\mu^2\,,$$
$$16\pi^2\frac{dm_2{}^2}{dt}=42.4803\,\bar{g}^2\mu^2\,. \tag{10}$$

It is clear by inspection that as t decreases, $m_2{}^2$ decreases much faster than $m_3{}^2$ so that chances are better than SU(2) will be broken first as t decreases. This would of course lead to a W mass being spontaneously generated. Detailed study of this question involves a study of the broken-symmetry renormalization-group analysis. This work is in progress.

With all the analyses and conclusions out of the way we now present the details of the renormalization-group analysis, at the one-loop level, that we have made on the Lagrangian (1). It is convenient, for this calculation, to work in the Landau gauge. In the minimal renormalization scheme,[9] the renormalization-group analysis is not at all sensitive to the masses of the gauge bosons. This is certainly true of energies $p\gg M_X$. We study the renormalization group in this domain and look for asymptotic freedom. The strategy, once we have found an asymptotically free solution, would be to resort to the broken-symmetry renormalization program to study the low-energy behavior of the theory.

In the Landau gauge, then, we quote the wave-function renormalization for each one of the fields. For generality, we quote our result for an SU(N) theory although our interest is ultimately in SU(5). Let n_f denote the number of generations ($ude\nu$ being the first generation, $cs\mu\nu_\mu$ the second, etc.), while n_F denotes the number of sets of heavy fermions, each set consisting of $\underline{5}$ and $\underline{24}$ of all four Dirac helicities:

$$Z(H)=1-\frac{1}{16\pi^2}\ln\frac{\Lambda}{\mu}\left\{-3g^2\frac{(N^2-1)}{N}+2h^2n_f(N-1)+4k_4{}^2n_F\frac{(N^2-1)}{N}\right\}\,, \tag{11}$$

$$Z(\psi^\alpha_R)=1-\frac{1}{16\pi^2}\ln\frac{\Lambda}{\mu}\{h^2(N-1)\}\,, \tag{12}$$

$$Z(\psi^{\alpha\beta}_L)=1-\frac{1}{16\pi^2}\ln\frac{\Lambda}{\mu}\{2h^2\}\,, \tag{13}$$

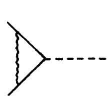
$$Z(\chi^\alpha)=1-\frac{1}{16\pi^2}\ln\frac{\Lambda}{\mu}\left\{\frac{N^2-1}{N}(k_2{}^2+k_4{}^2)\right\}\,, \tag{14}$$

$$Z(B^\alpha_\beta)=1-\frac{1}{16\pi^2}\ln\frac{\Lambda}{\mu}\left\{k_4{}^2+\frac{N^2-2}{N}(k_5{}^2+k_6{}^2)-\frac{4}{N}k_5k_6\right\}\,, \tag{15}$$

$$Z(\phi^\alpha_\beta)=1-\frac{1}{16\pi^2}\ln\frac{\Lambda}{\mu}\left\{-6g^2N+4k_2{}^2n_F+4\frac{(N^2-2)}{N}(k_5{}^2+k_6{}^2)n_F-\frac{16}{N}n_Fk_5k_6\right\}\,. \tag{16}$$

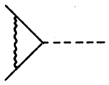
Next we tabulate the contribution of each vertex renormalization graph to the corresponding renormalization-group equation[10]:

Contribution to $16\pi^2dh/dt$:

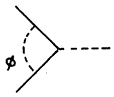


$$-3g^2h\frac{(N^2-N-2)}{N}\,. \tag{17}$$

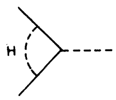
Contribution to $16\pi^2 dk_2/dt$:



$$\frac{3g^2}{N} k_2 , \tag{18}$$

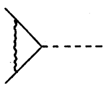


$$-\frac{2}{N} k_2^3 , \tag{19}$$

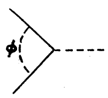


$$-\frac{4}{N} k_4^2 k_5 + \frac{2(N^2-2)}{N} k_4^2 k_6 . \tag{20}$$

Contribution to $16\pi^2 dk_4/dt$:

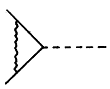


$$-3N g^2 k_4 , \tag{21}$$

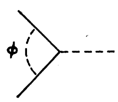


$$-\frac{4}{N} k_2 k_4 k_5 + \frac{2(N^2-2)}{N} k_2 k_4 k_6 . \tag{22}$$

Contribution to $16\pi^2 dk_5/dt$:

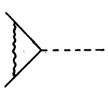


$$-3g^2 N k_5 , \tag{23}$$

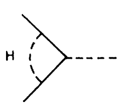


$$-\frac{2}{N} k_5^3 - \frac{8}{N} k_5^2 k_6 + \frac{2(N^2-5)}{N} k_5 k_6^2 - \frac{4k_6^3}{N} . \tag{24}$$

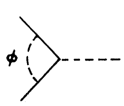
Contribution to $16\pi^2 dk_6/dt$:



$$-3g^2 N k_6 , \tag{25}$$

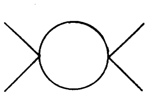


$$2k_4^2 k_2 , \tag{26}$$

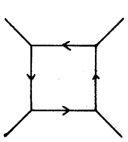


$$-\frac{4}{N} k_5^3 + \frac{2(N^2-4)}{N} k_5^2 k_6 - \frac{6}{N} k_5 k_6^2 - \frac{2}{N} k_6^3 . \tag{27}$$

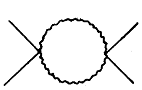
Contribution to $16\pi^2 d\lambda_1/dt$:

 + permutations

$$2(N^2+7)\ \lambda_1^2 + 8\lambda_1\lambda_2 \frac{(2N^2-3)}{N} + \lambda_2^2 \frac{24(N^2+3)}{N^2} + N\lambda_4^2 + 2\lambda_4\lambda_5 , \tag{28}$$

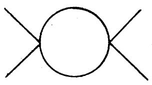
 + permutations

$$-\frac{16}{N^2} n_F k_5^4 - \frac{64}{N^2} n_F k_5^3 k_6$$
$$-\frac{k_5^2 k_6^2}{N^2} (96+48N^2) n_F - \frac{64}{N^2} n_F k_5 k_6^3 - \frac{16}{N^2} n_F k_6^4 , \tag{29}$$

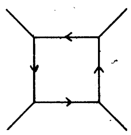
 + permutations

$$9g^4 . \tag{30}$$

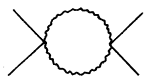
Contribution to $16\pi^2 d\lambda_2/dt$:


+ permutations

$$24\lambda_1\lambda_2+8\lambda_2^2\frac{(N^2-9)}{N}+\tfrac{1}{2}\lambda_5^2\,, \tag{31}$$

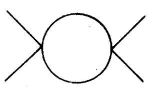

+ permutations

$$-4k_5^4\frac{(N^2-4)}{N}n_F+\frac{64}{N}k_5^3k_6n_F+\frac{96}{N}k_5^2k_6^2n_F$$
$$+\frac{64}{N}k_5k_6^3n_F-\frac{4(N^2-4)}{N}k_6^4n_F-4k_2^4n_F\,, \tag{32}$$

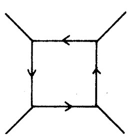

+ permutations

$$\frac{3N}{2}g^4\;. \tag{33}$$

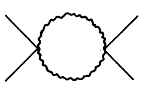
Contribution to $16\pi^2 d\lambda_3/dt$:


+ permutations

$$\lambda_3^2(N+4)+2\lambda_4^2(N^2-1)+4\lambda_4\lambda_5\frac{(N^2-1)}{N}+\lambda_5^2\left(\frac{N^2+2}{N^2}+\frac{N^2-4}{N}\right), \tag{34}$$

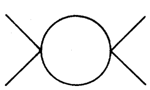

+ permutations

$$-8n_f h^4(N-1)-16n_F\frac{(N^3-2N+1)}{N^2}k_4^4\,, \tag{35}$$

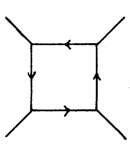

+ permutations

$$3g^4\left(\frac{N^2+2}{N^2}+\frac{N^2-4}{N}\right). \tag{36}$$

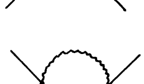
Contribution to $16\pi^2 d\lambda_4/dt$:


+ permutations

$$4\lambda_4^2+\lambda_5^2+\lambda_3\lambda_4(N+1)+\lambda_3\lambda_5+2\lambda_1\lambda_4(N^2+1)$$
$$+4\lambda_2\lambda_4\frac{(2N^2-3)}{N}+2\lambda_1\lambda_5\frac{(N^2-1)}{N}+4\lambda_2\lambda_5\frac{(N^2+3)}{N^2}\,, \tag{37}$$

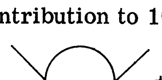

+ permutations

$$-16n_F k_4^2k_2^2-16n_F k_4^2k_2k_6-\frac{16}{N^2}n_F k_4^2k_5^2$$
$$-16n_F\frac{N^2+1}{N^2}k_4^2k_6^2-\frac{32}{N^2}n_F k_4^2k_5k_6\,, \tag{38}$$

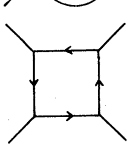

+ permutations

$$3g^4\;. \tag{39}$$

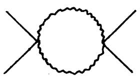
Contribution to $16\pi^2 d\lambda_5/dt$:


+ permutations

$$\lambda_3\lambda_5+4\lambda_1\lambda_5+8\lambda_4\lambda_5+\frac{4(N^2-6)}{N}\lambda_2\lambda_5+\lambda_5^2\frac{(N^2-4)}{N}\,, \tag{40}$$


+ permutations

$$\frac{16}{N}n_F k_4^2k_2^2+\frac{32}{N}n_F k_4^2k_2k_5+\frac{32}{N}n_F k_4^2k_2k_6-\frac{16}{N}n_F k_4^2k_5^2(N^2-3)$$
$$+\frac{96}{N}n_F k_4^2k_5k_6+\frac{48}{N}n_F k_4^2k_6^2\,, \tag{41}$$

 + permutations $\qquad 3Ng^4\,.$ (42)

To each of the above equations, the effect of the wave-function renormalization must be added. For each external leg the added term reads

wave-function renormalization contribution to $16\pi^2\, dh/dt$ due to

$$\psi_L^{\alpha\beta}\ \text{leg} = h\left(-\frac{\partial}{\partial \ln\Lambda}\right) Z^{1/2}(\psi_L^{\alpha\beta}) \tag{43}$$

$$= \frac{h}{2}\ \frac{1}{16\pi^2}\{2h^2\}\ . \tag{44}$$

From Eqs. (43) and (44) it is clear that the operational rule for including the external wave-function renormalization in the $16\pi^2\, d/dt$ equation is to include *half* of the quantity inside the curly brackets in the z expression for *each* external leg.

With this rule in mind, we display the full set of renormalization-group equations:

$$16\pi^2 \frac{dg}{dt} = -\left[\tfrac{11}{3}N - \tfrac{1}{6}(N+1) - \tfrac{1}{3}n_f(N-1) - \tfrac{2}{3}(2N+1)n_F\right]g^3\,, \tag{45}$$

$$16\pi^2 \frac{dh}{dt} = h\left\{-\frac{3g^2}{2N}(3N^2-2N-5) + 2n_F k_4^2\,\frac{(N^2-1)}{N}\right\} + h^3\left\{\frac{N+1}{2} + n_f(N-1)\right\}\,, \tag{46}$$

$$\begin{aligned}16\pi^2 \frac{dk_2}{dt} = {}& -3g^2k_2\,\frac{(N^2-1)}{N} + k_2\left\{\frac{N^2-1}{N}k_4^2 + \frac{2(N^2-2)}{N}\,n_F(k_5^2+k_6^2) - \frac{8}{N}\,n_F\,k_5k_6\right\}\\ & -\frac{4}{N}\,k_4^2k_5 + \frac{2(N^2-2)}{N}\,k_4^2k_6 + \frac{N^2-3}{N}\,k_2^3 + 2n_F\,k_2^3\,,\end{aligned} \tag{47}$$

$$\begin{aligned}16\pi^2 \frac{dk_4}{dt} = {}& k_4\left\{-\frac{3g^2}{2N}(3N^2-1) - \frac{4}{N}\,k_2k_5 + \frac{2(N^2-2)}{N}\,k_2k_6 + n_f(N-1)h^2\right.\\ & \left.+\frac{N^2-1}{2N}\,k_2^2 + \frac{N^2-2}{2N}\,(k_5^2+k_6^2) - \frac{2}{N}\,k_5k_6\right\}\\ & + k_4^3\left\{\frac{N^2+N-1}{2N} + 2n_F\,\frac{N^2-1}{N}\right\}\,,\end{aligned} \tag{48}$$

$$\begin{aligned}16\pi^2 \frac{dk_5}{dt} = {}& k_5\left\{-6g^2N + 2n_F\,k_2^2 + \frac{2(N^2-2)}{N}\,n_F\,k_6^2 + \frac{3N^2-12}{N}\,k_6^2 + k_4^2\right\}\\ & + k_5^2\left\{-\frac{12}{N}\,k_6 - \frac{8}{N}\,n_F\,k_6\right\} + k_5^3\left\{\frac{N^2-4}{N} + \frac{2(N^2-2)}{N}\,n_F\right\} - \frac{4}{N}\,k_6^3\,,\end{aligned} \tag{49}$$

$$\begin{aligned}16\pi^2 \frac{dk_6}{dt} = {}& 2k_4^2k_2 - \frac{4}{N}\,k_5^3 + k_6\left\{-6g^2N + 2n_F\,k_2^2 + \frac{2(N^2-2)}{N}\,n_F\,k_5^2 + \frac{3N^2-10}{N}\,k_5^2 + k_4^2\right\}\\ & + k_6^2\left\{-\frac{10}{N}\,k_5 - \frac{8}{N}\,n_F\,k_5\right\} + k_6^3\left\{\frac{N^2-4}{N} + \frac{2(N^2-2)}{N}\,n_F\right\}\,,\end{aligned} \tag{50}$$

$$\begin{aligned}16\pi^2 \frac{d\lambda_1}{dt} = {}& 2(N^2+7)\lambda_1^2 + 8\lambda_1\lambda_2\,\frac{(2N^2-3)}{N} + \frac{24(N^2+3)}{N^2}\,\lambda_2^2 + N\lambda_4^2 + 2\lambda_4\lambda_5\\ & + 9g^4 - 12Ng^2\lambda_1 - \frac{16}{N^2}\,n_F\,k_5^4 - \frac{64}{N^2}\,n_F\,k_5^3k_6 - \frac{k_5^2k_6^2}{N^2}(96+48N^2)n_F\\ & - \frac{64}{N^2}\,k_5k_6^3n_F - \frac{16}{N^2}k_6^4n_F + 8n_Fk_2^2\lambda_1 + 8n_F\,\frac{(N^2-2)}{N}\,(k_5^2+k_6^2)\lambda_1 - \frac{32}{N}\,n_Fk_5k_6\lambda_1\,,\end{aligned} \tag{51}$$

$$\begin{aligned}16\pi^2 \frac{d\lambda_2}{dt} = {}& 24\lambda_1\lambda_2 + 8\lambda_2^2\,\frac{(N^2-9)}{N} + \frac{\lambda_5^2}{2} + \frac{3}{2}\,g^4N - 12Ng^2\lambda_2 - 4k_2^4n_F - \frac{4(N^2-4)}{N}\,k_5^4n_F + \frac{64}{N}\,n_F\,k_5^3k_6\\ & + \frac{96}{N}\,k_5^2k_6^2n_F + \frac{64}{N}\,k_5k_6^3n_F - \frac{4(N^2-4)}{N}\,k_6^4n_F + 8n_Fk_2^2\lambda_2 + 8n_F\frac{(N^2-2)}{N}\lambda_2(k_5^2+k_6^2) - \frac{32}{N}\,n_F\,k_5k_6\lambda_2\,,\end{aligned} \tag{52}$$

$$16\pi^2\frac{d\lambda_3}{dt}=(N+4)\lambda_3{}^2+2(N^2-1)\lambda_4{}^2+\frac{4(N^2-1)}{N}\lambda_4\lambda_5+\lambda_5{}^2\left(\frac{N^2+2}{N^2}+\frac{N^2-4}{N}\right)+3g^4\left(\frac{N^2+2}{N^2}+\frac{N^2-4}{N}\right)$$
$$-\frac{6(N^2-1)}{N}g^2\lambda_3-16n_F\frac{(N^3-2N+1)}{N^2}k_4{}^4-8n_f h^4(N-1)+4n_f h^2\lambda_3(N-1)+8n_F k_4{}^2\lambda_3\frac{(N^2-1)}{N}, \quad (53)$$

$$16\pi^2\frac{d\lambda_4}{dt}=(N+1)\lambda_3\lambda_4+\lambda_3\lambda_5+4\lambda_4{}^2+\lambda_5{}^2+2\lambda_1\lambda_4(N^2+1)+4\lambda_2\lambda_4\frac{(2N^2-3)}{N}+2\lambda_1\lambda_5\frac{(N^2-1)}{N}$$
$$+4\lambda_2\lambda_5\frac{(N^2+3)}{N^2}+3g^4-\frac{9N^2-3}{N}g^2\lambda_4-16n_Fk_4{}^2k_2{}^2-16n_Fk_4{}^2k_2k_6-\frac{16}{N^2}n_Fk_4{}^2k_5{}^2$$
$$-16n_F\frac{N^2+1}{N^2}k_4{}^2k_6{}^2-\frac{32}{N^2}n_Fk_4{}^2k_5k_6+4n_Fk_2{}^2\lambda_4+4n_F\frac{(N^2-2)}{N}(k_5{}^2+k_6{}^2)\lambda_4$$
$$-\frac{16}{N}n_Fk_5k_6\lambda_4+2n_f(N-1)h^2\lambda_4+4n_F\frac{(N^2-1)}{N}k_4{}^2\lambda_4, \quad (54)$$

$$16\pi^2\frac{d\lambda_5}{dt}=\lambda_3\lambda_5+4\lambda_1\lambda_5+4\lambda_2\lambda_5\frac{N^2-6}{N}+8\lambda_4\lambda_5+\lambda_5{}^2\frac{(N^2-4)}{N}+3Ng^4$$
$$-\frac{9N^2-3}{N}g^2\lambda_5+\frac{16}{N}n_Fk_4{}^2k_2{}^2+\frac{32}{N}n_Fk_4{}^2k_2k_5+\frac{32}{N}n_Fk_4{}^2k_2k_6$$
$$-16n_F\frac{(N^2-3)}{N}k_4{}^2k_5{}^2+\frac{96}{N}n_Fk_4{}^2k_5k_6+\frac{48}{N}n_Fk_4{}^2k_6{}^2+4n_Fk_2{}^2\lambda_5$$
$$+4n_F\frac{(N^2-2)}{N}(k_5{}^2+k_6{}^2)\lambda_5-\frac{16}{N}n_Fk_5k_6\lambda_5+2n_f(N-1)h^2\lambda_5+4n_F\frac{(N^2-1)}{N}k_4{}^2\lambda_5. \quad (55)$$

The solution of this set of coupled system of differential equations is fortunately reducible to a set of algebraic equations. By looking for eigenvalues of the form

$$h,\ k_i\propto g\,,$$
$$\lambda_i\propto g^2\,,$$

the eigenvalues become the roots of a coupled system of polynomial equations. Furthermore, the system decouples into two disjoint sets of five equations, one involving the Yukawa couplings, the other the quartic couplings. On a computer, once the full set of equations has been correctly punched in, the search for roots of a system of five polynomials is not a time-consuming one.[11]

Not all the roots for $N=5$ satisfy the stability conditions

$$\lambda_2>0,\quad \lambda_1+\tfrac{7}{15}\lambda_2>0,\quad \lambda_3>0,$$

except the one reported in Eq. (2). In the search we have left as unspecified both n_f and n_F. There is room for $n_F=2$; however, none of those $n_F=2$ cases have acceptable roots.

Note added in proof. An asymptotically free SU(5) model with *three* generations has recently been found by us [Phys. Rev. D (to be published)].

The work of N.-P.C. and J.P.-M. was supported in part by the Research Foundation of CUNY under PSC-BHE Award No. 12336. The work of N.-P.C. was also supported in part by NSF Grant No. 77-01350. The work of A.D. was supported by the National Science Foundation.

[1]H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974); for a review see J. C. Pati, talk presented at the Einstein Centennial Symposium, Jerusalem, 1979, University of Maryland Report No. 80-006 (unpublished).
[2]C. N. Yang and R. Mills, Phys. Rev. 96, 191 (1954).
[3]S. Dimopoulos and L. Susskind, Phys. Rev. D 18, 4500 (1978).
[4]D. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); Phys. Rev. D 8, 3633 (1973); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).
[5]N. P. Chang, Phys. Rev. D 10, 2706 (1974); M. Suzuki, Nucl. Phys. B83, 269 (1974); E. Ma, Phys. Rev. D 11, 322 (1975); Phys. Lett. 62B, 347 (1976); Nucl. Phys. B116, 195 (1976); E. S. Fradkin and O. K. Kalashnikov, J. Phys. A 8, 1814 (1975); Phys. Lett. 59B, 159 (1975); N. P. Chang and J. Perez-Mercader, Phys. Rev. D 18, 4721 (1978); 19, 2515(E) (1979). For a review see N. P. Chang, in *Proceedings of the 1978 International Meeting on Frontier of Physics, Singapore*, edited by K. K. Phua, C. K. Chew, and Y. K. Lim (Singapore National Academy of Science, Singapore, 1979), Vol. 2.
[6]N. Cabibbo, L. Maiani, G. Parisi, and R. Petronzio, Nucl. Phys. B158, 295 (1979).
[7]E. S. Fradkin and K. O. Kalashnikov, Phys. Lett. 64B,

177 (1976).
[8]N. P. Chang, A. Das, and J. Perez-Mercader, Report No. CCNY-HEP-79-25, 1979 (unpublished); Phys. Lett. 93B, 137 (1980).
[9]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972); G. 't Hooft, *ibid.* B61, 455 (1973); B62, 444 (1973); H. Kluberg-Stern and J. B. Zuber, Phys. Rev. D 12, 467 (1975); 12, 482 (1975); 12, 3159 (1975).
[10]An earlier calculation of the quartic self-coupling constant renormalization-group equations, without any Yukawa couplings, was given by T. P. Cheng, E. Eichten, and L. F. Li, Phys. Rev. D 9, 2259 (1974).
[11]A convenient computer program for this work is the ZONE J routine contained in the dynamic Bell Laboratories PORT subroutine package.