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New renormalization program for broken gauge theories

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We derive a new renormalization-group theorem which expressly relates the parameters of the symmetric high-energy theory to the parameters of the broken, low-energy theory. The relation is an analytic one. It summarizes the threshold effects by leapfrogging across each threshold and provides a connection formula. The relation is derived for an arbitrary group hierarchy structure.

I. INTRODUCTION

In the grand unification theory[1] of strong, electromagnetic, and weak interactions, a very large mass scale of the order of 10^{15} GeV naturally appears.[2] This mass scale is associated with the so-called leptoquark gauge bosons that mediate between the leptons and quarks of the grand unified multiplet. These same leptoquarks lead in turn to a new effective four-fermion interaction that can convert a proton into leptons. A central issue in the grand unification theory is the estimation of this mass scale M_X and its consequent effect on the proton lifetime.[3-6]

Previous attempts at this have largely ignored the problem and essential complications of the mass-dependent renormalization-group equations. As we shall show in Sec. II, because the mass scale here arose as a result of spontaneous symmetry breaking, the treatment of the mass-dependent renormalization-group equation[7,9] must be radically different from that used for the study of heavy quark masses in deep-inelastic scattering.[10,11] In the latter case, the gauge symmetry, SU(3) color, remained strictly unbroken.

The essential complication which arises in the case of broken symmetry comes in through the proliferation of new couplings. In addition to the expected g_1, g_2, g_3, and g_X [respectively, the fermion couplings to the U(1), SU(2), SU(3), and X gauge bosons], new quadrupole-moment couplings appear which involve the X-gauge-boson trilinear interaction with U(1), SU(2), and SU(3). These additional couplings are important in the threshold region and influence the estimation of M_X.[4]

One way to handle this complication is of course to enlarge the system of mass-dependent renormalization-group equations to include these new couplings. For SU(5), the minimal candidate for grand unification, there are at least six new couplings to be considered. The modifications which result are hardly inconsequential.

The alternate way is to analyze anew the renormalization of a spontaneously broken gauge theory. This we have done in the context of the minimal renormalization scheme of 't Hooft and Veltman.[12,13] With this we are able to develop a new renormalization-group equation for broken gauge theories which avoids completely this complication. More precisely, we have been able to derive a new set of rules for the calculation of all low-energy matrix elements in terms of the parameters of the high-energy theory.

The rules are very simple. For the calculation of fully renormalized matrix elements, with external momenta $p \equiv p^0 e^t$,

(1) all vertices are those generated by the underlying, symmetric Lagrangian with, however, $\bar{g}(t)$ replacing the original coupling constant,

(2) massive particles, such as the X, propagate with an $\bar{M}(t)$, rather than the bare or renormalized mass, and

(3) the overall matrix element is to be multiplied by an external factor which depends on the canonical dimension of the matrix element [see Eq. (3.14)].

Here $\bar{g}(t)$ and $\bar{M}(t)$ are the running coupling constant and running mass, as calculated in the symmetric theory. Since $\bar{M}(t)$ has an exponential behavior e^{-t}, in the high-energy limit, the effect of the masses disappears. Conversely, in the low-energy limit, the running mass $\bar{M}$ blows up exponentially. In comparison, therefore, with graphs that have zero-mass gauge bosons running around, the graphs with M_X effectively decouple. Our rules thus manifest, in a mathematically well-defined way through our Theorem 1, the expected Appelquist-Carazzone decoupling.[14]

As we shall show in Sec. V, the theorem provides the needed connection between the low-energy coupling constants and those of the original unbroken gauge theory. It is through this connection that an estimate for M_X can be correctly made.

The theorem can also be used to study the proton-decay probability amplitude. Our analysis differs from earlier attempts by including in addition the mass-renormalization effects of X on the low-energy decay. The simple-minded propagator

$$1/(p^2+M_X^2)$$

is valid only in the neighborhood of the pole, while for p^2 of the order of 1 GeV^2, self-energy effects of M_X are no longer negligible.

Throughout this paper, where applications of the theorem are made, we refer to an asymptotically free SU(5) model,[15] in which all the quartic couplings $\lambda_1, \ldots, \lambda_5$ as well as the Yukawa couplings are fixed relative to the overall gauge constant. In this model asymptotic freedom[16] is preserved for the entire theory. As a result of the eigenvalue conditions,[17-21] the mass-renormalization effects become directly computable. In the standard SU(5) theory where the λ's and Yukawa couplings are arbitrary, the mass renormalization cannot reliably be estimated.

The plan of this paper is as follows.

In Sec. II, we discuss the mass-dependent renormalization-group approach used by Ross to study the threshold effect. We point out the need for the inclusion of additional effective coupling constants in the system of equations. In Sec. III, we adapt the minimal renormalization scheme of 't Hooft and Veltman.

In Sec. IV we apply the theorem to a simple SO(3) gauge theory in which $W^\pm$ acquires mass while A_3 remains massless. The detailed subtraction scheme used is exhibited in that section. Section V discusses the relation between the high-energy renormalization-group equation and the low-energy parameters. Section VI generalizes it to arbitrary gauge hierarchies for the study of the coupling constants of the subgroups. Section VII presents the result for g_X, the coupling to the massive bosons.

II. REVIEW

Early attempts at the estimation of M_X used as input the knowledge of the low-energy renormalization-group equations for g_3, g_2, g_1, respectively:

$$16\pi^2\frac{dg_3}{dt}=-\tfrac{1}{3}(33-4n_f)g_3{}^3\,,$$

$$16\pi^2\frac{dg_2}{dt}=-\tfrac{1}{3}(22-4n_f)g_2{}^3\,, \qquad (2.1)$$

$$16\pi^2\frac{dg_1}{dt}=+\tfrac{4}{3}n_f g_1{}^3\,,$$

where n_f = number of light fermion generations ($ude\nu$ being the first generation, $cs\mu\nu_\mu$ the next, and so on). In Eq. (2.1) the effect of the leptoquark interactions has been neglected. By a straightforward extrapolation from low-energy data, relying on the asymptotic freedom of g_3 and g_2, it was estimated that the grand unification scale where g_3 becomes equal to g_2 is of order 10^{16} GeV.

A more refined estimate would obviously have to involve the leptoquark exchanges. This was done by a detailed study of the mass-dependent renormalization-group equation. In this approach, the renormalized coupling constant is defined to be the value of the full three-point vertex function at the symmetric point with all $p^2=Q^2$. The study of the change in the g_R, as the subtraction point Q^2 is varied, gives the mass-dependent renormalization-group equation.

For $Q^2\to\infty$, the effects of M_X are negligible and the renormalized coupling constants g_3, g_2, g_1 all approach a common $\bar{g}$. For finite Q^2 the effects of M_X reflect upon the broken gauge symmetry and g_3, g_2, g_1 are no longer equal. The original study looked at the broken coupling constants g_3, g_2, g_1, and g_X and wrote down the system of equations involving them. In the equation for g_3, for example, are the graphs of Fig. 1(a), which involve, by definition $g_3{}^3$, while Figs. 1(b) and 1(c) involve again by definition $g_2{}^2g_3$ and $g_1{}^2g_3$, respectively. Figure 1(d), in which the gauge-boson self-energy involves the gluon loop, will contribute a $g_3{}^3$ term to the renormalization-group equation. This is as a result of SU(3) gauge invariance which remains unbroken and serves to relate the gluon trilinear coupling to the fermion-gluon coupling.

Figure 1(e), however, in which the gluon self-energy involves the leptoquark boson loop, cannot be said to contribute to $g_X{}^2g_3$ nor $g_3{}^3$. Indeed even the space-time structure of the GXX coupling is now broken into a minimal "charge" interaction and a quadrupole-moment coupling.[22] More precisely, the broken-symmetry interaction of the X reads ($i=1,2,3,\ a=1,2$)

$$ig_3\kappa_3X^{\dagger a}_{\nu i}\{\partial_\mu G^i_{\nu j}-\partial_\nu G^i_{\mu j}-ig_3[G_\mu,G_\nu]^i_j\}X^j_{\mu a}-\tfrac{1}{2}\left|\partial_\mu X^i_{\nu a}-ig_3G^i_{\mu j}X^j_{\nu a}+ig_2X^i_{\nu b}W^b_{\mu a}-(i\sqrt{5}/2\sqrt{3})g_1B_\mu X^i_{\nu a}-(\mu\leftrightarrow\nu)\right|^2$$

$$+ig_2\kappa_2X^{\dagger a}_{\nu i}\{\partial_\mu W^b_{\nu a}-\partial_\nu W^b_{\mu a}-ig_2[W_\mu,X_\nu]^b_a\}X^i_{\mu b}-(i\sqrt{5}/2\sqrt{3})g_1\kappa_1(\partial_\mu B_\nu-\partial_\nu B_\mu)X^{\dagger a}_{\mu i}X^i_{\nu a}\,. \qquad (2.2)$$

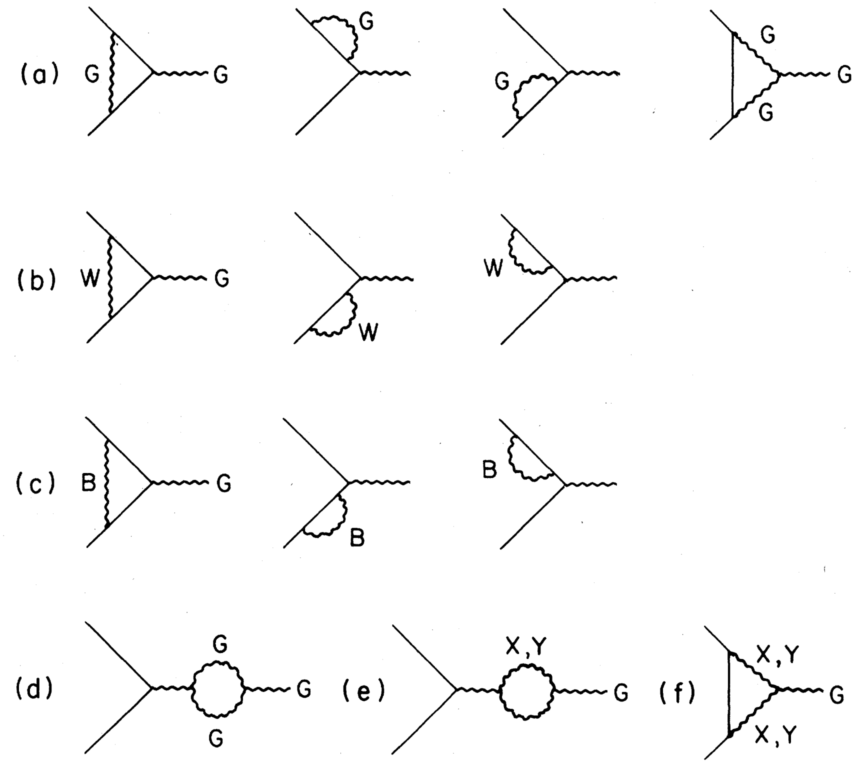


FIG. 1. Corrections to the g_3 vertex due to gauge bosons.

Only as $Q^2 \to \infty$ will all the g_i approach a common $\bar{g}$ and the κ_i approach unity.

In principle, therefore, the original system of equations must be enlarged to include the new quadrupole couplings κ_1, κ_2, κ_3 together with the quartic self-couplings involving the X leptoquarks, etc. This complicates considerably the original analysis of the low-energy mass-dependent renormalization-group equations.

For $Q^2 \ll M_X^2$, the effect of Fig. 1(e) is of order Q^2/M_X^2 compared with the other graphs and the misidentification of the new coupling is not a severe error. In the threshold region, however, with $Q^2 \sim M_X^2$, the misidentification could lead to an appreciable effect on the estimation of M_X.

III. NEW RENORMALIZATION SCHEME

To study anew then the renormalization-group equation in the presence of spontaneous symmetry breaking, we recall first the minimal renormalization scheme of 't Hooft and Veltman.[12,13] For this we start from L_B, the bare Lagrangian that includes the kinetic and interaction terms, gauge fixing and ghosts, but no counterterms.

This L_B is the shifted Lagrangian, having been obtained from the original symmetric theory by shifting the Higgs field σ. All the coupling vertices in this L_B are specified by the original coupling constants g, λ. In spite of the spontaneous symmetry breaking, the SU(3), SU(2), U(1) coupling vertices remain symmetric. In the language of Eq. (2.2), the constants that appear in L_B are $g_1 = g_2 = g_3 = g$, $\kappa_1 = \kappa_2 = \kappa_3 = 1$.

The shift

$$\sigma \to \sigma + v$$

has to be done consistent with the quantum requirement that the new vacuum maintain itself in the presence of radiative corrections.[23,24] To maintain

$$\langle \sigma \rangle = 0\,, \tag{3.1}$$

v cannot be the classical vacuum expectation value $m/\sqrt{\lambda}$ but must be (m is the Higgs-boson pseudomass scale here)

$$v = \frac{m}{\sqrt{\lambda}} + \frac{T}{2m^2}\,, \tag{3.2}$$

where T is the total tadpole contribution as calculated with L_B. Apart from the terms thus obtained by shifting, L_B has no added counterterms.

Let $\Gamma_u^{(n)}$ be the one-particle-irreducible n-point

Green's function calculated with L_B. Γ_u as usual has infinities which can be regularized in a gauge-invariant way by working in complex n-dimensional space.[25]

The minimal scheme consists in absorbing the $1/\epsilon$ parts in coupling, wave-function renormalizations.[26] Let $\hat{Z}_3$ and $\hat{Z}_1^{-1}$ be the minimal divergent parts of the gauge-boson wave-function and vertex renormalization, and let α specify the gauge. Define

$$g_r = \hat{Z}_3^{3/2}\hat{Z}_1^{-1}g_B\,,\quad M_r = g_r v_r\,,\quad \alpha_r = \hat{Z}_3^{-1}\alpha_B\,, \tag{3.3}$$

then

$$\Gamma_u(p, g_B, M_B, \alpha_B) = (\hat{Z}_e)^{-n/2}\Gamma_r(p, g_r, M_r, \alpha_r, \mu) \tag{3.4}$$

and Γ_r is a finite function of the renormalized parameters g_r, M_r, α_r. $\hat{Z}_e$ would be $\hat{Z}_3$ if the external leg is a gauge boson, $\hat{Z}_2$ if it is a fermion, etc.

By construction, we have

$$\hat{Z}_3 = \hat{Z}_3(g_B, \alpha_B, \Lambda, \mu)\,,\quad \hat{Z}_1 = \hat{Z}_1(g_B, \alpha_B, \Lambda, \mu)\,, \tag{3.5}$$

where we have understood $1/\epsilon$ term as $\ln(\Lambda/\mu)$ in terms of the regulator mass Λ. What is important for us is the observation that the $\hat{Z}$ constants are independent of M_B and are completely given by the unbroken theory.

Γ_r while finite is not yet fully normalized for on-shell matrix elements. To perform the remaining finite renormalization, we must extract further the finite external wave-function renormalization through the relation

$$\Gamma_r(p, g_r, M_r, \alpha_r, \mu) = (Z_e)^{-n/2}\Gamma_R(p, g_r, M_r, \alpha_r, Q, \mu)\,, \tag{3.6}$$

where

$$Z_e = Z_e(Q, g_r, M_r, \alpha_r, \mu)\,, \tag{3.7}$$

Q being the subtraction point.

As is usual, we can derive a lemma for the Γ_R from the simple observation that Γ_u does not explicitly depend on μ,

$$\mu\frac{d}{d\mu}\Gamma_u = 0 = \mu\frac{d}{d\mu}(\hat{Z}_e^{-n/2}\hat{Z}_e^{-n/2}\Gamma_R) \tag{3.8}$$

or

Lemma:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g_r} - \gamma_M M_r\frac{\partial}{\partial M_r} - 2\gamma\alpha_r\frac{\partial}{\partial\alpha_r} - n\tilde{\gamma}\right)\times\Gamma_R(p, g_r, M_r, \alpha_r, Q, \mu) = 0\,, \tag{3.9}$$

$$\begin{aligned}
\beta &\equiv \mu\frac{\partial}{\partial\mu}g_r\Big|_{g_B,\Lambda}\,,\\
\gamma_M M_r &\equiv -\mu\frac{\partial}{\partial\mu}M_r\Big|_{g_B,\Lambda}\,,\\
\gamma &\equiv \tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln\hat{Z}_3\Big|_{g_B,\Lambda}\,,\\
\tilde{\gamma} &\equiv \tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln\hat{Z}_e\Big|_{g_B,\Lambda} + \tfrac{1}{2}\mu\frac{d}{d\mu}\ln Z_e(Q, g_r, M_r, \alpha_r, \mu)\,.
\end{aligned} \tag{3.10}$$

Note that $\tilde{\gamma}$, in contrast with the usual anomalous dimension γ, is of order g^4 here.

By the usual dimensional analysis we know that

$$\Gamma_R(\kappa p, g_r, M_r, \alpha_r, \kappa Q, \mu) = \mu^{d_n}\Gamma_R\left(\frac{\kappa p}{\mu}, g_r, \frac{M_r}{u}, \alpha_r, \frac{\kappa Q}{\mu}, 1\right),$$

d_n = canonical dimension of the n-point function (3.11)

or

$$\kappa\frac{\partial}{\partial\kappa}\Gamma_R(\kappa p, g_r, M_r, \alpha_r, \kappa Q, \mu) = \left(-\mu\frac{\partial}{\partial\mu} + d_n - M_r\frac{\partial}{\partial M_r}\right)\Gamma_R(\kappa p, g_r, M_r, \alpha_r, \kappa Q, \mu)\,. \tag{3.12}$$

Combining this with our lemma, we obtain the new renormalization-group equation for a broken gauge theory

$$\left[-\kappa\frac{\partial}{\partial\kappa} - (1+\gamma_M)M_r\frac{\partial}{\partial M_r} + \beta\frac{\partial}{\partial g_r} - 2\gamma\alpha_r\frac{\partial}{\partial\alpha_r} + d_n - n\tilde{\gamma}\right]\Gamma_R(\kappa p, g_r, M_r, \alpha_r, \kappa Q, \mu) = 0\,, \tag{3.13}$$

whose solution can be presented as a theorem.[27]

Theorem 1:

$$\Gamma_R(p^0 e^t, g_r, M_r, \alpha_r, Qe^t, \mu) = \exp\left(d_n t - n\int_0^t dt'\tilde{\gamma}\right)\Gamma_R(p^0, \bar{g}, \bar{M}, \bar{\alpha}, Q, \mu)\,, \tag{3.14}$$

$$\begin{aligned}
\frac{d\bar{g}}{dt} &= \beta\,, \quad \bar{g}(0) \equiv g_r\,,\\
\frac{d\bar{M}}{dt} &= -(1+\gamma_M)\bar{M}\,, \quad \bar{M}(0) \equiv M_r\,,\\
\frac{d\bar{\alpha}}{dt} &= -2\gamma\bar{\alpha}\,, \quad \bar{\alpha}(0) = \alpha_r\,.
\end{aligned} \tag{3.15}$$

Theorem 1 incorporates the new set of Feynman rules for the calculation of low-energy matrix elements in terms of the parameters of the high-energy theory, viz., L_B. Note that the β calculated this way is identical to the β function of an unbroken, symmetric theory. This is the direct result of the use of a minimal renormalization scheme. Similarly, the γ and γ_M are identical to that calculated in a symmetric theory.

For our applications of the theorem, we shall always choose Q to be equal to μ in the calculation of the right-hand side of Eq. (3.14).

IV. EXAMPLE: SO(3)→U(1)

To better exhibit the two-step renormalization scheme, we shall apply the procedure to the simple SO(3) gauge theory with one set of Higgs bosons and n_f generation of fermions, both being in the triplet representation.[28] For simplicity, we have ignored the Yukawa couplings so that the theory is not asymptotically free. Since we shall be interested only in the low-energy behavior here as a general example of the decoupling theorem, we have suppressed the appearance of the Yukawa coupling. In our complete treatment in the next sections, Yukawa couplings will be included.

The unshifted, bare, Lagrangian reads

$$-\tfrac{1}{4}(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu)^2 - \tfrac{1}{2}(\partial_\mu \vec{\phi} + g\vec{A}_\mu \times \vec{\phi})^2 - \vec{\bar{\psi}}\cdot\gamma_\mu(\partial_\mu\vec{\psi} + g\vec{A}_\mu \times \vec{\psi}) + \frac{m^2}{2}\vec{\phi}^2 - \frac{\lambda}{4}(\vec{\phi}^2)^2$$

$$-\tfrac{1}{2}(\partial_\mu A_\mu^1 - M\phi^2)^2 - \tfrac{1}{2}(\partial_\mu A_\mu^2 + M\phi^1)^2 - \tfrac{1}{2}(\partial_\mu A_\mu^3)^2 + \mathcal{L}_{\text{ghost}} \,. \tag{4.1}$$

Because of spontaneous symmetry breaking, the correct perturbation theory is with respect to the field operators ϕ_1, ϕ_2, σ, where σ is related to the original ϕ_3 field by

$$\phi_3 = \sigma + v \,.$$

As already pointed out in Sec. III, v is *not* the classical vacuum expectation value but must include radiative corrections in order that the condition (3.1) remain true in the presence of quantum corrections. The tadpole contribution T can be calculated self-consistently from the shifted Lagrangian. Its explicit form will be important for mass renormalization but not for g-coupling-constant renormalization.

Consider the two-point function for the fermion $\Gamma_u^{(2)}(f)$. From L_B, and to lowest order in g^2, we find ($C \equiv -0.577\ldots$)

$$\Gamma_u^{(2)}(f|p,g,M) = \gamma\cdot p\left\{1 + \frac{2g^2}{16\pi^2}\left[\frac{2}{\epsilon} + C + \ln 4\pi - 1 - \int_0^1 x\,dx \ln\frac{M^2x + x(1-x)p^2}{\mu^2} - \int_0^1 x\,dx\ln\frac{x(1-x)p^2}{\mu^2}\right]\right\}. \tag{4.2}$$

In Eq. (4.2), $1/\epsilon$ can be read as $\ln(\Lambda/\mu)$, with Λ as the cutoff needed to make the theory finite. Therefore, (4.2) in spite of appearances really does not depend on μ^2. It actually, of course, depends logarithmically on Λ.

Now minimal renormalization consists in the extraction of the infinite wave-function renormalization constant through the relation

$$\Gamma_u^{(2)}(p,g,M) = \hat{Z}_2^{-1}\Gamma_r^{(2)}(p,g_r,M_r)\,, \tag{4.3}$$

where the remaining function Γ_r is a finite function of the renormalized coupling constants and masses.

For our example, we have clearly

$$\hat{Z}_2 = 1 - \frac{4g^2}{16\pi^2}\frac{1}{\epsilon} - \frac{2g^2}{16\pi^2}(C + \ln 4\pi) \tag{4.4}$$

and

$$\Gamma_r^{(2)}(f|p,g_r,M_r) = \gamma\cdot p\left\{1 - \frac{2g_r^2}{16\pi^2}\left[1 + \int_0^1 x\,dx\ln\frac{M_r^2x + x(1-x)p^2}{\mu^2} + \int_0^1 x\,dx\ln\frac{x(1-x)p^2}{\mu^2}\right]\right\}. \tag{4.5}$$

$\Gamma_r^{(2)}$ while finite is not yet properly renormalized for the calculation of S-matrix elements. To extract now the finite external wave-function renormalization, we perform the usual subtraction at $p^2 = Q^2$, so that

$$\Gamma_r^{(2)}(p,g_r,M_r,\mu) \equiv Z_e^{-1}(Q,g_r,M_r,\mu)\Gamma_R^{(2)}(p,g_r,M_r,Q) \tag{4.6}$$

and

$$\Gamma_R^{(2)}(p,g_r,M_r,Q)\big|_{p^2=Q^2} = \gamma\cdot p \,. \tag{4.7}$$

From Eq. (4.5), therefore, we find

$$Z_e(f\,|\,Q,g_r,M_r,\mu)=1+\frac{2g_r^2}{16\pi^2}\left[1+\int_0^1 x\,dx\ln\frac{M_r^2x+x(1-x)Q^2}{\mu^2}+\int_0^1 x\,dx\ln\frac{x(1-x)Q^2}{\mu^2}\right]. \tag{4.8}$$

The explicit dependence of $Z_e(f)$ on $\ln\mu$ is by inspection identical to that of $\hat{Z}_2$ [recall that $1/\epsilon=\ln(\Lambda/\mu)$]. Therefore, in the calculation of $\tilde{\gamma}$, the "anomalous dimension" in Eq. (3.10), the $O(g^2)$ terms cancel while the additional terms obtained by $\mu d/d\mu$ acting on g_r^2 and M_r^2 will be of $O(g^4)$.

According to our theorem, for the study of the low-energy coupling constant, where $p\equiv p^0e^t$, we need only to calculate $\Gamma_R(p^0,\bar{g},\bar{M},Q,\mu)$. The choice of p^0 is arbitrary. For convenience we can take p^0 to be at Q. Therefore, in the calculation of Γ_R we need $Z_e(p^0,\bar{g},\bar{M},\mu)$.

Recall that

$$\bar{M}(t)=e^{-t}\tilde{M}(t)\,,\quad \bar{M}(0)\equiv M_r\,, \tag{4.9}$$

where

$$\frac{d}{dt}\tilde{M}(t)=-\gamma_M\tilde{M}\,. \tag{4.10}$$

$\tilde{M}$ can at most grow with some power of t, so that the exponential behavior dominates in $\bar{M}$. For low energies, t being large and negative, $\bar{M}$ is much larger than M_r. It is, therefore, clear that the approximation

$$\bar{M}\gg p^0,\ \mu \tag{4.11}$$

is a valid one for Z_e and we have the result

$$Z_e(p^0,\bar{g},\bar{M},\mu)=1-\frac{2\bar{g}^2}{16\pi^2}\left(\ln\frac{\bar{M}}{\mu}+\ln\frac{p^0}{\mu}-\frac{1}{2}\right). \tag{4.12}$$

Next we turn to the photon two-point function. Since the residual U(1) gauge invariance is unbroken, the photon remains strictly massless. The general form of $\Gamma_u^{(2)}(\gamma)$ reads

$$\Gamma_u^{(2)}(\gamma\,|\,p,g,M)=-ip^2\delta_{\mu\nu}-i(\delta_{\mu\nu}p^2-p_\mu p_\nu)\tilde{f}(p,g,M) \tag{4.13}$$

with

$$\tilde{f}(p,g,M)=\frac{g^2}{16\pi^2}\left(-\frac{6}{\epsilon}+\frac{8}{3}\frac{n_f}{\epsilon}+\text{finite}\right). \tag{4.14}$$

The coefficient of $-ip^2\delta_{\mu\nu}$ is directly related to $\hat{Z}_3^{-1}$. In fact, by minimal renormalization, we find that

$$\hat{Z}_3=1-\frac{g^2}{16\pi^2}\left[-\frac{(6-\frac{8}{3}n_f)}{\epsilon}-(3-\tfrac{4}{3}n_f)(C+\ln4\pi)\right]+O(g^4). \tag{4.15}$$

The coefficient of $p_\mu p_\nu$ is taken care of by renormalization of α, the gauge-fixing parameter. Although we have taken α to be 1, in higher orders α is renormalized. This can be seen by rewriting ($\alpha\equiv\alpha_{\text{bare}}=1$)

$$\begin{aligned}\Gamma_u^{(2)}(\gamma\,|\,p,q,M,\alpha)&=-i(p^2\delta_{\mu\nu}-p_\mu p_\nu)-\frac{i}{\alpha}p_\mu p_\nu\\&\quad-i(\delta_{\mu\nu}p^2-p_\mu p_\nu)\tilde{f}(p,g,M)\\&\equiv\hat{Z}_3^{-1}\Gamma_r^{(2)}(\gamma\,|\,p,g_r,M_r,\alpha_r,\mu)\end{aligned} \tag{4.16}$$

with

$$1/\alpha_r=\hat{Z}_3/\alpha\,. \tag{4.17}$$

The operational definition of $\hat{Z}_3^{-1}$, therefore, is simply to calculate the coefficient of $-ip^2\delta_{\mu\nu}$ in the photon two-point function. To perform the finite external wave-function renormalization we again define

$$\begin{aligned}\Gamma_r^{(2)}(\gamma)\,|\,p,g_r,M_r,\alpha_r,\mu)&\equiv Z_e(\gamma\,|\,Q,g_r,M_r,\alpha_r,\mu)\\&\quad\times\Gamma_R^{(2)}(\gamma\,|\,p,g_r,M_r,\alpha_r,Q)\,,\end{aligned} \tag{4.18}$$

with

$$\Gamma_R^{(2)}\big|_{p^2=Q^2}=-i\left[Q^2\delta_{\mu\nu}-\left(1-\frac{1}{\alpha_r}\right)p_\mu p_\nu\right]. \tag{4.19}$$

Just as before we set $Q=p^0$ and calculate $Z_e(\gamma\,|\,p^0,\bar{g},\bar{M},\bar{\alpha},\mu)$ in the limit $\bar{M}\gg p^0,\ \mu$ and find

$$\begin{aligned}Z_e(\gamma\,|\,p^0,\bar{g},\bar{M},\bar{\alpha},\mu)\\&=1-\frac{\bar{g}^2}{16\pi^2}\left(6\ln\frac{\bar{M}}{\mu}-\frac{8}{3}n_f\ln\frac{p^0}{\mu}+\frac{20}{9}n_f-1\right).\end{aligned} \tag{4.20}$$

Finally, we turn our attention to the three-point function which defines the fermion vertex function for the emission or absorption of photons. According to our L_B, the calculation of $\Gamma_u^{(3)}$ yields the general form, at the symmetric point,

$$\begin{aligned}\Gamma_u^{(3)}(p,g,M,\alpha)&=-g\gamma_\lambda\left\{1+\frac{g^2}{16\pi^2}[8/\epsilon+4(C+\ln4\pi)+\text{finite}]\right\}+\Delta_{\lambda\rho}(q)p_\rho\gamma\cdot pf_1\\&\quad+\Delta_{\lambda\rho}(q)\,p_\rho\gamma\cdot qf_2+\Delta_{\lambda\rho}(q)\gamma_\rho f_3 \qquad (4.21)\\&\equiv-g\gamma_\lambda\hat{Z}_1^{-1}+\text{finite}\,. \qquad (4.22)\end{aligned}$$

In Eq. (4.21), f_1, f_2, f_3 are all finite functions of g, M, α and $\Delta_{\lambda\rho}$ is the transverse projection for the photon momentum q. In Eq. (4.21), p is the incoming fermion momentum, and at the symmetric point,

$$p^2 = q^2 = (p+q)^2 = -2p\cdot q\,. \tag{4.23}$$

By the renormalization theorem,

$$\Gamma_u^{(3)}(p,g,M,\alpha) = \hat{Z}_2^{-1}\hat{Z}_3^{-1/2}\Gamma_r^{(3)}(p,g_r,M_r,\alpha_r,\mu)\,, \tag{4.24}$$

we find

$$g_r = \hat{Z}_2\hat{Z}_3^{1/2}/\hat{Z}_1 g\,, \tag{4.25}$$

$$g_r = g - \frac{g^3}{16\pi^2}\frac{1}{\epsilon}(-7+\tfrac{4}{3}n_f) + O(g^5)\,. \tag{4.26}$$

From Eq. (4.26), recalling that $1/\epsilon = \ln(\Lambda/\mu)$, we find the symmetric renormalization-group equation

$$16\pi^2\frac{d\bar{g}}{dt} = -(7-\tfrac{4}{3}n_f)\bar{g}^3 \equiv -\tfrac{1}{2}\bar{b}\bar{g}^3(16\pi^2)\,. \tag{4.27}$$

To study the low-energy coupling constant, we need the relation

$$\Gamma_R^{(3)}(p^0,\bar{g},\bar{M},\bar{\alpha},\mu) = Z_e(f|p^0,\bar{g},\bar{M},\bar{\alpha},\mu)Z_e^{1/2}(\gamma|p^0,\bar{g},\bar{M},\bar{\alpha},\mu)\hat{Z}_2\hat{Z}_3^{1/2}\Gamma_u^{(3)}(p^0,\bar{g},\bar{M}) \tag{4.28}$$

$$= -\bar{g}\gamma_\lambda\left\{1-\frac{\bar{g}^2}{16\pi^2}\left[7\ln\frac{\bar{M}}{\mu}-\tfrac{4}{3}n_f\ln\frac{p^0}{\mu}-\tfrac{1}{3}+\tfrac{10}{9}n_f\right]+O(\bar{g}^4)\right\}+\Delta_{\lambda\rho}(q)F_\rho\,. \tag{4.29}$$

In Eq. (4.29) F_ρ is finite and of order $\bar{g}^3$. The coefficient of $-\gamma_\lambda$ is, by definition, the low-energy charge e_R, and so we have derived

$$e_R = \bar{g} - \frac{\bar{g}^3}{16\pi^2}\left(7\ln\frac{\bar{M}}{\mu}-\frac{4}{3}n_f\ln\frac{p^0}{\mu}-\frac{1}{3}+\frac{10}{9}n_f\right) + O(\bar{g}^5)\,. \tag{4.30}$$

Equation (4.30) exhibits a useful relation between the coefficient of the $\ln(\bar{M}/\mu)$ term and that of $\ln(p^0/\mu)$, viz., the two add up to the coefficient $\bar{b}/2$ in the renormalization-group equation for $\bar{g}$ [see Eq. (4.27)]. The $\ln(\bar{M}/\mu)$ terms come from graphs involving massive particles in the virtual loop while $\ln(p^0/\mu)$ come from graphs whose loop integral involves only massless particles. Because they add up to $\bar{b}/2$, it is a very useful numerical check on the arithmetic addition of the many graphs which contribute to Γ_R, especially for SU(5).

Equation (4.30) is still too general for our needs. The reference point p^0, in general, may be different from the renormalization scale μ. The theorem [Eq. (3.14)] holds regardless of the relation between p^0 and μ. We shall, in our applications, choose p^0 equal to μ and set μ equal to M_r.

V. "UNDRESSING"

In this section we study the "undressing" that takes place in the renormalization-group equation as the energy scale is lowered from $p \gg M_r$ to $p \ll M_r$. At high energies, the coupling constants are

$$\Gamma_R(p^0e^t,g_r,M_r,\mu)\xrightarrow[t\to\infty]{}-\bar{g}\gamma_\lambda \tag{5.1}$$

and

$$\frac{d\bar{g}^2}{dt} = -(14-\tfrac{8}{3}n_f)\frac{\bar{g}^4}{16\pi^2} \equiv -\bar{b}\bar{g}^4\,, \tag{5.2}$$

while at low energies, Γ_R approach the ordinary charge e_R of QED, which should satisfy

$$\frac{de_R^2}{dt} = \frac{8}{3}n_f\frac{e_R^4}{16\pi^2} \equiv -Be_R^4\,. \tag{5.3}$$

For simplicity of notation we shall from this point on refer only to the $(-\gamma_\lambda)$ part of Γ_R, but continue to write Γ_R for it. Our perturbative result reads ($p^0 = \mu = M_r \ll \bar{M}$)

$$[\Gamma_R(p^0e^t,g_r,M_r,\mu)]^2 = \bar{g}^2 - \frac{\bar{g}^4}{16\pi^2}[14\ln(\bar{M}/\mu)-\tfrac{2}{3}+\tfrac{20}{9}n_f] + O(\bar{g}^6) \tag{5.4}$$

$$\equiv \bar{g}^2 + \bar{g}^4(-\tilde{a}+bt) + O(\bar{g}^6)\,. \tag{5.5}$$

In principle, Eq. (5.4) is only the first two terms of a perturbation series which must be summed to give e_R^2 at low energies. To do this sum, it is convenient to study the differential equation satisfied by e_R^2, viz.,

$$\frac{d}{dt}e_R^2 = \frac{d}{dt}\bar{g}^2 + b\bar{g}^4 + O(\bar{g}^6)$$

$$= -(\bar{b}-b)\bar{g}^4 + O(\bar{g}^6) \tag{5.6}$$

or

$$\frac{d}{dt}e_R^2 = -Be_R^4\,. \tag{5.7}$$

But by the remark following Eq. (4.30) in the last section, the B coefficient precisely represents the contribution of graphs in which only massless particle loops occur, so that as far as e_R^2 is concerned, the massive bosons have, to one-loop approximation, completely decoupled from the low-energy renormalization-group equation.

Still another way to look at this decoupling, at the one-loop level, comes through the realization that Eq. (5.7) has as its solution

$$e_R^2 = \frac{A}{1+AB\tau}, \tag{5.8}$$

where $\tau = 0$ refers to a low-energy point m and is related to t by

$$t = -T + \tau. \tag{5.9}$$

Here T is the logarithmic distance from M_r to the low-energy point.

But Eq. (5.2) also has the solution

$$\bar{g}^2 = \frac{c}{1+c\bar{b}\tau}, \tag{5.10}$$

where c measures the symmetric $\bar{g}^2$ at the same low-energy point. Substituting (5.10) into (5.5), we find

$$\frac{A}{1+AB\tau} = \bar{g}^2 + \bar{g}^4(a + b\tau) + \cdots, \tag{5.11}$$

with

$$a = -\tilde{a} - bT. \tag{5.12}$$

Clearly our perturbation series, in the one-loop approximation, must sum up as a geometric series in order to obtain Eq. (5.8). As a result, we find the immediate relation between the high- and low-energy parameters, viz.,

$$A = \frac{c}{1-ac}, \quad B = \bar{b} - b. \tag{5.13}$$

Reexpressed in terms of the coupling constants, Eq. (5.13) reads

$$\frac{1}{e_R^2(m)} = \frac{1}{\bar{g}^2(m)} + b \ln\frac{M_r}{m} + \tilde{a}. \tag{5.14}$$

Equation (5.14), *without* the constant $\tilde{a}$, may be easily inferred from the well-known result of Georgi, Quinn, and Weinberg.[2] The inclusion of the constant summarizes, within the context of our theorem, the so-called threshold effect. The theorem allowed us to leapfrog across the threshold region, where the subtraction was done ($p^0 = \mu = M_r$), directly into the low-energy region. The constants that appear are remnants of the typical logarithms that occur in the massive loops in the limit $\bar{M} \gg p^0$. In the $t \to +\infty$ region, of course, our Γ_R reduce automatically to simply $\bar{g}(t)$.

It is worth noting that in contrast with the mass-dependent low-energy renormalization-group approach, we have in Eq. (5.14) an *analytic* relation between high- and low-energy parameters that take into account all one-loop effects including the threshold effect.

VI. GENERALIZATION

In this section we extend the calculation of Sec. IV to a general broken gauge theory. Let G be the grand unified group, with its symmetric coupling constant $\bar{g}$ satisfying

$$\frac{d\bar{g}^2}{dt} = -\bar{b}g^4, \tag{6.1}$$

where

$$16\pi^2\bar{b} = \tfrac{22}{3} C_2(G) - \tfrac{2}{3} \sum_{\substack{\text{Higgs}\\ \text{bosons}}} T_H(R) - \tfrac{4}{3} \sum_{\text{fermions}} T(R). \tag{6.2}$$

In Eq. (6.2), the group-theory coefficients are the standard ones,

$$C_2(G) = \begin{cases} N & \text{if } G = SU(N) \\ 2(N-2) & \text{if } G = SO(N) \end{cases} \tag{6.3}$$

while $T_H(R)$, $T(R)$ depend on the representation:

	representation	$T_H(R)$	$T(R)$
SU(N):	fundamental	$\frac{1}{2}$	$\frac{1}{2}$
	adjoint	$N/2$	N
	antisymmetric rank 2	$(N-2)/2$	$(N-2)/2$

(6.4)

SO(N):	vector	1	2
	adjoint	$N-2$	$2(N-2)$
	spinor (N = even)	$2^{(N-6)/2}$	$2^{(N-6)/2}$, $N \neq 6$
	spinor (N = odd)	$2^{(N-5)/2}$	$2^{(N-5)/2}$, $N \geq 3$.

(6.5)

Let G be broken down to the subgroup

$$\mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \times \cdots$$

by a mass scale M. We focus our attention on the calculation of the coupling constant associated with the $\mathcal{G}_i$ subgroup:

$$C_2(\mathcal{G}_i) \equiv \sum_{jk} f_{ijk} f_{ijk}$$

(summed over the massless bosons of $\mathcal{G}_i$),

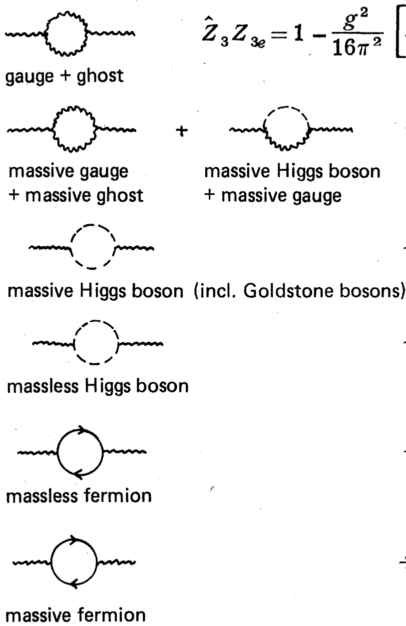
$$\bar{C}_2(\mathcal{G}_i) \equiv \sum_{X,Y} f_{iXY} f_{iXY}$$

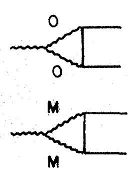
(summed over the massive bosons of G).

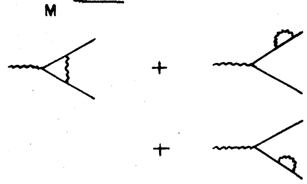
Then

$$C_2(\mathcal{G}_i)+\overline{C}_2(\mathcal{G}_i)=C_2(G)\,. \tag{6.6}$$

In terms of these coefficients, we can tabulate in a convenient fashion the results needed for our calculation. Because the $1/\epsilon$ are good arithmetic checks we have included them here for completeness, but for brevity we have absorbed the constants $\frac{1}{2}(C+\ln 4\pi)$ into $1/\epsilon$ and called them $1/\tilde{\epsilon}$. We have







$$\begin{aligned}
\hat{Z}_3 Z_{3e} = 1 - \frac{g^2}{16\pi^2}\Big[& -C_2(\mathcal{G}_i)\left(\frac{10}{3\tilde{\epsilon}}-\frac{10}{3}\ln\frac{p^0}{\mu}+\frac{31}{9}\right) \\
& -\overline{C}_2(\mathcal{G}_i)\left(\frac{10}{3\tilde{\epsilon}}-\frac{10}{3}\ln\frac{M}{\mu}+\frac{1}{3}\right) \\
& +T_H(R)\left(\frac{2}{3\tilde{\epsilon}}-\frac{2}{3}\ln\frac{M}{\mu}\right)-\tilde{T}_H^i(R)\,\tfrac{2}{3}\ln\frac{M_H}{M} \\
& +T_H(R)\left(\frac{2}{3\tilde{\epsilon}}\right)+T_H(R)\left(-\tfrac{2}{3}\ln\frac{p^0}{\mu}+\frac{8}{9}\right) \\
& +T(R)\left(\frac{4}{3\tilde{\epsilon}}-\tfrac{4}{3}\ln\frac{p^0}{\mu}+\frac{10}{9}\right) \\
& +T(R)\left(\frac{4}{3\tilde{\epsilon}}-\tfrac{4}{3}\ln\frac{M_F}{\mu}\right)\Big] \\
-\frac{g^2}{16\pi^2}\Big[& C_2(\mathcal{G}_i)\left(-\frac{3}{\tilde{\epsilon}}+3\ln\frac{p^0}{\mu}-3+\frac{5s}{3\sqrt{3}}\right) \\
& +\overline{C}_2(\mathcal{G}_i)\left(-\frac{3}{\tilde{\epsilon}}+3\ln\frac{M}{\mu}+\frac{1}{4}\right) \\
& +C_2(\mathcal{G}_i)\left(\frac{1}{\tilde{\epsilon}}-\ln\frac{p^0}{\mu}+\frac{1}{2}\right) \\
& +\overline{C}_2(\mathcal{G}_i)\left(\frac{1}{\tilde{\epsilon}}-\ln\frac{M}{\mu}-\frac{1}{4}\right)\Big]
\end{aligned}$$

$(s=2.029884\ldots)$.

Upon putting it all together, we find ($p^0=\mu=M_r\ll\overline{M}$)

$$\begin{aligned}
\Gamma_R(\mathcal{G}_i)=\overline{g}-\frac{\overline{g}^3}{16\pi^2}\Big\{ & C_2(\mathcal{G}_i)[-\tfrac{38}{9}+(5s/3\sqrt{3})]+\overline{C}_2(\mathcal{G}_i)\left[\frac{21}{6}\ln\frac{\overline{M}(t)}{\mu}-\frac{1}{6}\right]+\sum_H \tilde{T}_H^i(R)\left[-\frac{1}{3}\ln\frac{\overline{M}_H(t)}{\mu}\right] \\
& +\sum_h T_H^i(R)(\tfrac{4}{9})+\sum_F T(R)\left[-\frac{2}{3}\ln\frac{\overline{M}_F(t)}{\mu}\right]+\sum_f T(R)(\tfrac{5}{9})\Big\}.
\end{aligned} \tag{6.7}$$

where H (h) and F (f) refer to the heavy (light) Higgs bosons and fermions, respectively.

In Eq. (6.7) the group-theoretic reduction from $T_H(R)$ to $\tilde{T}_H^i(R)$ is given by

$$T_H(R)=\tilde{T}_H^i+\tfrac{1}{2}\,\overline{C}_2(\mathcal{G}_i)\,, \tag{6.7a}$$

if the breaking of G into $\mathcal{G}_i$ is due to this particular Higgs-boson representation, and

$$T_H(R)=\tilde{T}_H^i(R) \tag{6.7b}$$

otherwise. In Eq. (6.7a) the $\overline{C}_2$ coefficient summarizes the contribution due to the Goldstone bosons associated with the symmetry breakdown, and in the 't Hooft gauge they have the same mass as the massive gauge bosons. The $\tilde{T}_H^i(R)$ summarizes the contributions due to the *physical*, massive Higgs bosons which result.[29] In writing down Eq. (6.7) we have ignored the "fine structure" among the physical Higgs bosons, although in each particular case the fine structure in the $\tilde{T}_H^i(R)$ for different physical Higgs-boson submultiplets can be easily incorporated.

A word of caution is needed for Eq. (6.7a). It is to be used only in the basis appropriate to the surviving group $\mathcal{G}_i$. Consider, for example, the breakdown of SU(2)× U(1) into U(1). With respect to the photon field A_μ the complex doublet Higgs-boson contribution actually behaves like a real triplet contribution. That is, $T_H(R)=1$ with respect to e^2 even though with respect to $g_2{}^2$ and $g_1{}^2$ we have $T_H(R)=\frac{1}{2}$. This is because $A_\mu=\cos\theta B_\mu+\sin\theta W_\mu^3$, and

$$\sin^2\theta\, g_2{}^2T_H^W(R)+g_1{}^2T_H^B(R)\cos^2\theta=(\tfrac{1}{2}+\tfrac{1}{2})e^2 = (1)e^2 .$$

Before relating to Eq. (5.14), it is clear from our tabulation that in fact our calculation can be ultimately generalized to several gauge hierarchy breakdowns.[30] For at each stage of hierarchy breakdown, $C_2(\mathcal{G}_i)$ will further break into $C_2(g)+\overline{C}_2(g)$, where $g\subset\mathcal{G}_i$ and the $\overline{C}_2(g)$ will, according to the table, be associated with its massive-gauge-boson diagram *including* the emission and absorption of the massive Higgs boson which caused the breakdown. Thus, let us label the hierarchy stage by a, the gauge bosons X_a, Y_a, ... associated with the generators X_a, Y_a, ... having acquired mass scale M_a ($M_1\gg M_2\gg M_3\ldots$), through the Higgs-boson multiplets H_a and let the residual group be

$$\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3\times\cdots .$$

For g_i, the coupling constant associated with the subgroup $\mathcal{G}_i$, we define

$$c_i\equiv C_2(\mathcal{G}_i)$$

and (6.8)

$$c_i^a\equiv\sum_{X_a,Y_a}f_{iX_aY_a},$$

so that

$$c_i+\sum_a c_i^a=C_2(G). \tag{6.9}$$

Then the generalization of Eq. (6.7) reads ($M_r^a\gg m$; $M_{H_a}, M_F\gg m$)

$$\Gamma_R(\mathcal{G}_i)=\bar{g}-\frac{\bar{g}^3}{16\pi^2}\Big\{c_i[-\tfrac{38}{9}+(5s/3\sqrt{3})]+\sum_a c_i^a[\tfrac{21}{6}\ln(M_r^a/m)-\tfrac{1}{6}]+\sum_{H_a}\tilde{T}_H^i(R)[-\tfrac{1}{3}\ln(M_{H_a}/m)]+\sum_h T_H(R)(\tfrac{4}{9}) + \sum_F T(R)[-\tfrac{2}{3}\ln(M_F/m)]+\sum_f T(R)(\tfrac{5}{9})\Big\}, \tag{6.10}$$

where

$$m\equiv M_r^1e^t \tag{6.11}$$

is the low-energy point.

Throughout our discussion we have made the assumption that at each stage of hierarchy breakdown the Goldstone bosons acquire a mass equal to that of the massive gauge bosons. This requires that the gauge-fixing terms have to be prearranged to make this happen.

By the arguments of Sec. V, Eq. (6.10) is but the first two terms of a perturbation series which can be summed to give the low-energy coupling constants. The analog to Eq. (5.14) now reads

$$\frac{16\pi^2}{g_i{}^2(m)}=\frac{16\pi^2}{\bar{g}^2(m)}+c_i\left[\frac{10s}{3\sqrt{3}}-\frac{76}{9}\right]+\sum_a c_i^a\left(\frac{21}{3}\ln\frac{M_a}{m}-\frac{1}{3}\right) +\sum_{H_a}\tilde{T}_H^i(R)\left(-\frac{2}{3}\ln\frac{M_{H_a}}{m}\right)+\sum_h T_H(R)(\tfrac{8}{9}) +\sum_F T(R)\left(-\frac{4}{3}\ln\frac{M_F}{m}\right)+\sum_f T(R)(\tfrac{10}{9}). \tag{6.12}$$

In Eq. (6.12) the $\tilde{T}_H^i(R)$ refer, for each subgroup $\mathcal{G}_i$, to the set of massive physical Higgs bosons which couple to the massless gauge bosons of $\mathcal{G}_i$.

Equation (6.12) is an analytic relation between the low-energy coupling constant g_i and the parameters of the high-energy theory. It is valid, however, only if m satisfies the constraints

$$M_a\gg m\,,$$

$$M_H, M_F\gg m\,,$$

$$m\gg m_i \quad (\text{gauge bosons of subgroup } \mathcal{G}_i),$$

$$m\gg m_f, m_h \quad (\text{light fermions and Higgs bosons}).$$

Therefore, Eq. (6.12) cannot itself be used right at each of those thresholds although it can be used to leapfrog across the thresholds.

Finally, before closing out this section, it should be pointed out that the Yukawa couplings, even when they are present in the theory, do not contribute to the $\Gamma_R(\mathcal{G}_i)$. They cancel out in the $\Gamma_R(\mathcal{G}_i)$. This is as expected since even for low energies the renormalization-group equation for $\mathcal{G}_i$, at the one-loop level, does not depend on Yukawa coupling nor on quartic Higgs-boson self-couplings.

VII. X COUPLING

So far we have only looked at the couplings of the unbroken subgroups $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$. We now proceed with the study of the coupling g_X, which involves the emission and absorption of the massive X gauge boson. Since proton decay involves such emissions, the study is far from being academic.

Let us go back to our discussion of the minimal renormalization scheme in Sec. IV and indicate the changes that now are expected to occur. Equation (4.16) for the X boson now reads

$$\begin{aligned}\Gamma_u^{(2)}(X|p,g,M,\alpha) &= -i(p^2\delta_{\mu\nu}-p_\mu p_\nu)-(i/\alpha)p_\mu p_\nu - iM^2\delta_{\mu\nu} \\ &\quad -i(p^2\delta_{\mu\nu}\tilde{f}_1 - p_\mu p_\nu \tilde{f}_2) - iM^2\delta_{\mu\nu}\tilde{f}_3\,, \qquad (7.1)\end{aligned}$$

where

$$\begin{aligned}\tilde{f}_1 &= \frac{a}{\epsilon}+f_1+\frac{a}{2}(C+\ln 4\pi)\,, \\ \tilde{f}_2 &= \frac{a}{\epsilon}+f_2+\frac{a}{2}(C+\ln 4\pi)\,, \qquad (7.2) \\ \tilde{f}_3 &= \frac{b}{\epsilon}+f_3+\frac{b}{2}(C+\ln 4\pi)\,.\end{aligned}$$

Here f_1, f_2, f_3 are finite functions of p, g, M, μ, with the property that as $M \gg \mu, p$, f_1, f_2 at most grow logarithmically with M, while $M^2 f_3$ in the same limit is independent of p^2.

Then

$$\begin{aligned}\hat{Z}_3 &= 1-\frac{a}{\epsilon}-\frac{a}{2}(C+\ln 4\pi)\,, \\ M_r^2 &= M^2\left[1+\frac{b-a}{\epsilon}+\frac{b-a}{2}(C+\ln 4\pi)\right], \qquad (7.3)\end{aligned}$$

and

$$\Gamma_u^{(2)} = \hat{Z}_3^{-1}\Gamma_r^{(2)}(X|p,g_r,M_r,\alpha_r,\mu)\,. \qquad (7.4)$$

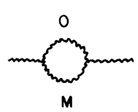
To do the finite external wave-function renormalization one temptation might be to subtract at the physical mass of the X boson. Since the X mass is of order 10^{15} GeV the actual need for normalizing the propagator residue at that energy is not likely to occur soon. Instead it is more convenient to choose a normalization which connects smoothly across gauge hierarchy thresholds, i.e., we choose to define

$$Z_e(X|Q,g_r,M_r,\alpha_r,\mu) = 1+f_1(Q,g_r,M_r,\alpha_r,\mu)\,. \qquad (7.5)$$

For the application we require Z_e for the argument $p^0, \bar{g}, \bar{M}, \bar{\alpha}, p^0$, which implies that we simply read off the coefficient of $-ip^2\delta_{\mu\nu}$ in the limit where $M \gg \mu$, p in Eq. (7.1).

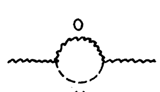
With these preliminaries, we can now again tabulate the graphs for $\Gamma_R(X)$. The Higgs-boson and fermion contributions to vacuum polarization of X are identical to their contributions to $\mathcal{G}_i$ vacuum polarization. For brevity, therefore, we have omitted them in our tabulation.

The contribution to $\hat{Z}_3 Z_e(X)$ is

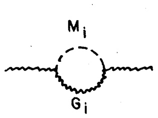


gauge + ghost

$$-\frac{g^2}{16\pi^2}\left[-C_2(G)\left(\frac{10}{3\bar{\epsilon}}-\frac{10}{3}\ln\frac{M}{\mu}+\frac{83}{36}\right)\right],$$



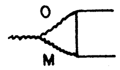
$$-\frac{g^2}{16\pi^2}\left[-C_2(G)(-\tfrac{1}{4})\right],$$



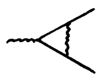
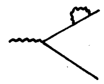
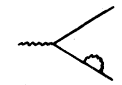
$$-\frac{g^2}{16\pi^2}\left\{\sum_{X\mathcal{G}_i} f_{Xab}f_{Xab}\left[\frac{1+\eta_i}{(1-\eta_i)^2}+\frac{2\eta_i}{(1-\eta_i)^3}\ln\eta_i\right]\right\}.$$

The last entry comes from the emission and absorption of the subset of Higgs bosons belonging to the $\mathcal{G}_i$ subgroup whose masses are $M_i^2 \equiv \eta_i M^2$. For example, in SU(5) the 24 Higgs bosons responsible for the breaking of SU(5) into SU(3)× SU(2)× U(1) is itself broken up into the SU(3) octet φ^i_j, SU(2) triplet ω^a_b, U(1) singlet σ, plus the unphysical ϕ^i_a Goldstone bosons with mass identical to the X bosons. In this embedding

$$\sum_{X\mathcal{G}_i} f_{Xab}f_{Xab} = \begin{cases} \dfrac{n^2-1}{n} & \text{if } \mathcal{G}_i = \mathrm{SU}(n)\,, \\ \dfrac{N}{n(N-n)} & \text{if } \mathcal{G}_i = \mathrm{U}(1) \text{ where } \mathrm{SU}(N)\to \mathrm{SU}(n)\times\mathrm{SU}(N-n)\times\mathrm{U}(1)\,, \end{cases}$$



$$-\frac{g^2}{16\pi^2}\left[C_2(G)\left(-\frac{3}{\bar{\epsilon}}+3\ln\frac{M}{\mu}-\frac{5}{4}\right)\right],$$

 +  + 

$$=-\frac{g^2}{16\pi^2}\left[C_2(G)\left(\frac{1}{2\bar{\epsilon}}-\frac{1}{2}\ln\frac{p^0}{\mu}+\frac{1}{4}\right)+C_2(G)\left(\frac{1}{2\bar{\epsilon}}-\frac{1}{2}\ln\frac{M}{\mu}-\frac{1}{8}\right)\right].$$

Or, all told, we find ($p^0=\mu=M_r\ll\bar{M}$)

$$\Gamma_R(X)=\bar{g}-\frac{\bar{g}^3}{16\pi^2}\left[C_2(G)\left(\frac{25}{6}\ln\frac{\bar{M}}{\mu}-\frac{155}{72}\right)+\tfrac{1}{2}\zeta_X+\sum_H T_H(R)\left(-\frac{1}{3}\ln\frac{\bar{M}_H}{\mu}\right)+\sum_h T_H(R)(\tfrac{4}{9})\right.$$

$$\left.+\sum_F T(R)\left(-\frac{2}{3}\ln\frac{\bar{M}_F}{\mu}\right)+\sum_f T(R)(\tfrac{5}{9})\right],$$

$$\zeta_X=\sum_{X\mathcal{G}_i} f_{Xij}f_{Xij}\left[\frac{1+\eta_a}{(1-\eta_a)^2}+\frac{2\eta_a}{(1-\eta_a)^3}\ln\eta_a\right]\ \text{summed over } \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \ldots,$$

where G has been broken down to $\mathcal{G}_1\times\mathcal{G}_2\times\mathcal{G}_3\times\cdots$.

Here we have ignored in the T_H contribution terms of order $\ln(M/M_H)$ as well as terms of order $\ln(M_{h_1}/M_{h_2})$, ratios of masses in the Higgs-boson multiplet having been assumed to be of order unity. Similarly, we have ignored the fine structure in the heavy fermion sector (see the Appendix).

VIII. CONCLUSION

In this paper we have presented in some detail the new renormalization program suitable for the study of hierarchy in broken gauge theories. Equation (6.12) summarizes the result of a massive program of manual calculation of SU(5) grand unification graphs. By means of a well-defined minimal subtraction scheme, spelled out in Sec. IV, we have an analytic relation between the parameters of the high-energy theory [SU(5)-symmetric theory, for example] and those of the broken low-energy theory in which only U(1) $\times$ SU(3)$_c$ is symmetric.

Applications of Eq. (6.12) to SU(5) are done in the Appendix. We merely quote here the result of that analysis. With an input of

$$\alpha_s(m)=0.2 \text{ at } m=6 \text{ GeV},$$

$$\alpha(m)=1/133.058,$$

$$M_W=85 \text{ GeV},$$

the determination of the M_r parameter gives 5.97 $\times 10^{14}$ GeV while $\sin^2\theta(m)$ is 0.2139. These numbers are *independent* of the Higgs-boson and fermion content of the theory. To get an estimate of $\sin^2\theta(m_W)$ (Ref. 31) would require knowledge of the Higgs-boson and fermion content. It is, however, in any case highly encouraging that a two-state hierarchy formula yields a much improved value of $\sin^2\theta$ so that the earlier largely pessimistic view of SU(5) grand unification is no longer justified.

The value of the minimal renormalization parameter M_r is not strictly speaking the physical value of M_X. However, since at that energy $\bar{g}^2/4\pi$ is 0.048, the actual value of M_X is not significantly different from M_r.

The study of proton lifetime involves the question of mass renormalization and will be deferred to a later paper in this series.[32]

Note added in proof. P. Binetruy and T. Schucker [Report No. CERN TH-2857, 1980 (unpublished)] have proposed an alternative two-step renormalization procedure, which they have named the decoupling subtraction (DS) scheme.

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APPENDIX[33]

In Eq. (6.7) we have applied the correct bookkeeping in separating the Goldstone-boson contributions from the physical Higgs boson. As a result of that bookkeeping, $T_H^i(R)$ depends on both the subgroup G_i and on the representation.

An alternate bookkeeping turns out to be more convenient. In the 't Hooft gauge, the Goldstone bosons have mass M, identical to the heavy gauge-boson mass. Recognizing this, we can restore the Goldstone-boson contribution to T_H and obtain the modified Eq. (6.7):

$$\Gamma_R(\mathcal{G}_i)=\bar{g}-\frac{\bar{g}^3}{16\pi^2}\left\{C_2(\mathcal{G}_i)[-\tfrac{38}{9}+(5s/3\sqrt{3})]+\bar{C}_2(\mathcal{G}_i)\left(\frac{11}{3}\ln\frac{\overline{M}(t)}{\mu}-\frac{1}{6}\right)+\sum_H T_H(R)\left(-\frac{1}{3}\ln\frac{\overline{M}_H(t)}{\mu}\right)\right.$$
$$\left.+\sum_h T_H(R)(\tfrac{4}{9})+\sum_F T(R)\left(-\frac{2}{3}\ln\frac{\overline{M}_F(t)}{\mu}\right)+\sum_f T(R)(\tfrac{5}{9})\right\}. \qquad \text{(A1)}$$

In Eq. (A1) we have allowed for the inclusion of mass differences within the Higgs-boson multiplet. An example would make this clear. Let SU(5) be broken down into SU(3)× SU(2)× U(1) by a 24-plet (ϕ_β^α), then with the decomposition

$$\phi_j^i=\varphi_j^i+\sqrt{2/15}\,\delta_j^i\sigma,\quad \phi_b^a=\omega_b^a-\sqrt{3/10}\,\delta_b^a\sigma,\ \phi_a^i,\ \phi_i^a, \qquad \text{(A2)}$$

$$\mathcal{G}_3: T_H(\underline{24})=\tfrac{3}{2}\ \text{with } M_H=\text{mass of } \varphi_j^i \quad (i,j=1,2,3;\ a,b=4,5)$$
$$+\tfrac{1}{2}\bar{C}_2(\mathcal{G}_3)\ \text{with } M_H=\text{mass of } M_X,$$
$$T_H(\underline{5})=\tfrac{1}{2}\ \text{with } M_H=\text{mass of } H^i, \qquad \text{(A3)}$$

$$\mathcal{G}_2: T_H(\underline{24})=\tfrac{2}{2}\ \text{with } M_H=\text{mass of } \omega_b^a$$
$$+\tfrac{1}{2}\bar{C}_2(\mathcal{G}_2)\ \text{with } M_H=\text{mass of } M_X,$$
$$T_H(\underline{5})=\tfrac{1}{2}\ \text{with } M_H=\text{mass of } H^a, \qquad \text{(A4)}$$

$$\mathcal{G}_1: T_H(\underline{24})=\tfrac{1}{2}\bar{C}_2(\mathcal{G}_1)\ \text{with } M_H=\text{mass of } M_X,$$
$$T_H(\underline{5})=\tfrac{1}{2}(\tfrac{2}{5})\ \text{with } M_H=\text{mass of } H^i$$
$$+\tfrac{1}{2}(\tfrac{3}{5})\ \text{with } M_H=\text{mass of } H^a. \qquad \text{(A5)}$$

The advantage of Eq. (A1) is that if one is able to ignore the fine structure in mass splittings within a multiplet, $T_H(R)$ does not depend on the subgroup $\mathcal{G}_i$ and numerical analysis becomes much more convenient.

For applications to SU(5), the fine structure in the heavy fermion multiplets may safely be ignored, as can the fine structure in the split $\underline{24}$-plet. The $\underline{5}$-plet, however, is expected, for phenomenological reasons, to be badly split after two hierarchy breakdowns when m is of order 6 GeV. For this we can single out the mass ratios

$$\rho_3\equiv M(H^i)/M_X,\quad \rho_2\equiv M(H^a)/M_X, \qquad \text{(A6)}$$

and separate it out from the symmetric $T_H(R)$. Thus we find

$$\frac{16\pi^2}{g_3^2(m)}=\frac{16\pi^2}{g_3^2(m)}\bigg|_{\substack{\text{no fine}\\\text{structure}}}-\tfrac{1}{3}\ln\rho_3, \qquad \text{(A7)}$$

$$\frac{16\pi^2}{g_2^2(m)}=\frac{16\pi^2}{g_2^2(m)}\bigg|_{\substack{\text{no fine}\\\text{structure}}}-\tfrac{1}{3}\ln\rho_2, \qquad \text{(A8)}$$

$$\frac{16\pi^2}{g_1^2(m)}=\frac{16\pi^2}{g_1^2(m)}\bigg|_{\substack{\text{no fine}\\\text{structure}}}-\tfrac{2}{15}\ln\rho_3-\tfrac{1}{5}\ln\rho_2, \qquad \text{(A9)}$$

and

$$\frac{16\pi^2}{g_i^2(m)}\bigg|_{\substack{\text{no fine}\\\text{structure}}}=\frac{16\pi^2}{\bar{g}^2(m)}+\sum_a c_i^a\left(\frac{22}{3}\ln\frac{M_a}{m}-\frac{1}{3}\right)+c_i[(10s/3\sqrt{3})-\tfrac{76}{9}]-\sum_H T_H(R)\left(\tfrac{2}{3}\ln\frac{M_H}{m}\right)$$
$$-\sum_h T_H(R)(-\tfrac{8}{9})-\sum_F T(R)\left(\tfrac{4}{3}\ln\frac{M_F}{m}\right)-\sum_f T(R)(-\tfrac{10}{9}), \qquad \text{(A10)}$$

where in Eq. (A10) the $T_H(R)$ now is simply the symmetric unbroken $T_H(R)$ and M_H refers to the average mass of the Higgs-boson multiplet.

Equation (A10) is valid for every range of m that satisfies the constraints

$$M_a\gg m,$$
$$M_F, M_H\gg m,$$
$$m\gg m_i\ \text{(gauge bosons of subgroup } \mathcal{G}_i), \qquad \text{(A11)}$$
$$m\gg m_f,\ m_h\ \text{(light fermion and Higgs bosons)}.$$

Therefore, Eq. (A10) itself cannot be used at each of the threshold regions ($m\sim M_1$, M_2, etc., or $m \sim M_{F_1}, M_{F_2}$, etc.).

For phenomenological applications, an appropriate choice of m might be ~6 GeV which is well below the 10-GeV threshold of the third generation and at the same time is well above the end of the second generation at 3.1 GeV. Equation (A10) can thus be applied, in turn, to $g_3(m)$ and $e(m)$, the renormalized gluon and electromagnetic charges, respectively. For this we need to introduce the SU(2)×U(1) mixing angle as it has been embedded in G. Let

$$\frac{1}{e^2(m)}=\frac{1}{g_2^2(m)}+\cot^2\theta_0\frac{1}{g_1^2(m)}, \qquad \text{(A12)}$$

where $\cot^2\theta_0=\frac{5}{3}$ if all fermions in the fundamental representation have the usual charge assignments. Then

$$\sin^2\theta_0\frac{16\pi^2}{e^2(m)}-\frac{16\pi^2}{g_3^2(m)}=(\sin^2\theta_0 c_2-c_3)[(10s/3\sqrt{3})-\tfrac{76}{9}]+\sum_a(c_2^a\sin^2\theta_0+c_1^a\cos^2\theta_0-c_3^a)\left(\tfrac{22}{3}\ln\frac{M_a}{m}-\frac{1}{3}\right)$$
$$+(\tfrac{1}{5}+\tfrac{2}{15}\sin^2\theta_0)(\ln\rho_3-\ln\rho_2), \qquad \text{(A13)}$$

while

$$\sin^2\theta(m) = \sin^2\theta_0 \frac{e^2(m)}{16\pi^2}\left\{\frac{16\pi^2}{e^2(m)} + c_2\cot^2\theta_0[(10s/3\sqrt{3}) - \tfrac{76}{9}] + \sum_a \cot^2\theta_0(c_2^a - c_1^a)\left(\frac{22}{3}\ln\frac{M_a}{m} - \frac{1}{3}\right)\right.$$
$$\left. + \tfrac{2}{15}\cot^2\theta_2(\ln\rho_3 - \ln\rho_2)\right\}. \qquad \text{(A14)}$$

For SU(5), assigning the Higgs boson to a 5 and 24 representation, the first stage of hierarchy breakdown brings G down to SU(3)× SU(2)× U(1). This is the situation for m in the range ($\sim 10^{15}$ GeV $\gg m \gg 80$ GeV). For m in this range then

$$\begin{aligned} &\mathcal{G}_1 = \mathrm{U}(1), \quad c_1 = 0, \quad c_1^1 = 5, \\ &\mathcal{G}_2 = \mathrm{SU}(2), \quad c_2 = 2, \quad c_2^1 = 3, \quad M_1 = M_X, \\ &\mathcal{G}_3 = \mathrm{SU}(3), \quad c_3 = 3, \quad c_3^1 = 2. \end{aligned} \qquad \text{(A15)}$$

Our input data must necessarily involve low-energy data, however, and in that case, for $m \sim 6$ GeV, there will have been two stages of hierarchy breakdown,

$$\begin{aligned} &G \xrightarrow{1} \mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3) \xrightarrow{2} \mathrm{U}(1)\times\mathrm{SU}(3), \\ &\mathcal{G}_1 = \mathrm{U}(1), \qquad c_1 = 0, \quad c_1^1 = 5, \quad c_1^2 = 0, \\ &\mathcal{G}_2 = \mathrm{U}(1)' \subset \mathrm{SU}(2), \quad c_2 = 0, \quad c_2^1 = 3, \quad c_2^2 = 2, \quad M_1 = M_X, \quad M_2 = M_W, \\ &\mathcal{G}_3 = \mathrm{SU}(3), \qquad c_3 = 3, \quad c_3^1 = 2, \quad c_3^2 = 0. \end{aligned} \qquad \text{(A16)}$$

With the input[34]

$$\alpha_s(m) \equiv g_3^2(m)/4\pi = 0.2 \text{ at } m \sim 6 \text{ GeV},$$

$$\alpha(m) \equiv e^2(m)/4\pi = 1/133.058,$$

$$M_W = 85 \text{ GeV}, \quad p_3 = 10^{-3}, \quad p_2 = 10^{-12}$$

we find

$$M_X = 5.97 \times 10^{14} \text{ GeV}$$

and

$$\sin^2\theta(m) = 0.213\,9 \text{ at } m \sim 6 \text{ GeV}.$$

To get an idea of the variation of $\sin^2\theta$ as the mass scale changes, we have used Eq. (A10) for $m \sim 800$ GeV, in a region where only the first stage of hierarchy is important. Assuming the number of light fermion generation remains at 3 even up to 800 GeV, we find

$$\sin^2\theta(m) = 0.217\,0 \text{ at } m \sim 800 \text{ GeV}$$

with

$$\alpha(m) = 1/127.978.$$

Equation (A10) cannot itself be directly used for $m \sim m_W$, so that we cannot give a rigorous value for $\sin^2\theta$ at m_W. However, based on our study at $m \sim 6$ vs $m \sim 10 m_W$, it is fair to expect a 0.21 value also for $\sin^2\theta$ at m_W. This is then to be compared with the $\sin^2\theta$ experimental value of 0.23 ± 0.02.

Finally, we turn to our estimate for the SO(10) hierarchy chain. If SO(10) is broken down in the sequence

$$\mathrm{SO}(10) \xrightarrow{1} \mathrm{SU}(5) \xrightarrow{2} \mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3) \xrightarrow{3} \mathrm{U}(1)\times\mathrm{SU}(3)$$

then, with $c_2[\mathrm{SO}(10)] = 16$, we have

$$\begin{aligned} &\mathcal{G}_1 = u(1), \quad c_1 = 0, \quad c_1^1 = 11, \quad c_1^2 = 5, \quad c_1^3 = 0, \\ &\mathcal{G}_2 = \mathrm{U}(1)' \subset \mathrm{SU}(2), \quad c_2 = 0, \quad c_2^1 = 11, \quad c_2^2 = 3, \quad c_2^3 = 2, \\ &\mathcal{G}_3 = \mathrm{SU}(3), \quad c_3 = 3, \quad c_3^1 = 11, \quad c_3^2 = 2, \quad c_3^3 = 0. \end{aligned} \qquad \text{(A17)}$$

Equation (A13) immediately yields the result that the low-energy inputs (α and α_s) do *not* determine M_1, while $M_2 \equiv M_X$ is fixed to be the same value as the SU(5) number given earlier. Constraints on M_1 [the bosons responsible for SO(10) break down into SU(5)] will have to come from proton lifetime, and not from grand unification input.

On the other hand, if SO(10) is broken in the sequence

$$\mathrm{SO}(10) \xrightarrow{1} [\mathrm{SO}(6)]\times[\mathrm{SO}(4)] \xrightarrow{2} [\mathrm{U}(1)\times\mathrm{SU}(3)]\times[\mathrm{SU}(2)]$$
$$\xrightarrow{3} \mathrm{U}(1)\times\mathrm{SU}(3),$$

we have (assuming $\sin^2\theta_0 = \frac{3}{8}$ as usual)

$$\begin{aligned} &\mathcal{G}_1 = \mathrm{U}(1) \subset \mathrm{SO}(6), \quad c_1 = 0, \quad c_1^1 = 8, \quad c_1^2 = 8, \quad c_1^3 = 0, \\ &\mathcal{G}_2 = \mathrm{U}(1)' \subset \mathrm{SU}(2) \subset \mathrm{SO}(4), \quad c_2 = 0, \quad c_2^1 = 12, \quad c_2^2 = 2, \quad c_2^3 = 2, \\ &\mathcal{G}_3 = \mathrm{SU}(3) \subset \mathrm{SO}(6), \quad c_3 = 3, \quad c_3^1 = 8, \quad c_3^2 = 5, \quad c_3^3 = 0. \end{aligned} \qquad \text{(A18)}$$

By Eqs. (A12) and (A13), and using as input α_s, α, and $\sin^2\theta(m)$ (assumed to be 0.23), we find, if we ignore all fine structure,

$$M_1 = 1.091 \times 10^{15}\ \text{GeV},$$

$$M_2 = 2.861 \times 10^{14}\ \text{GeV},$$

masses which are potentially at the edge of the allowed range according to proton stability requirements.

In conclusion, we make two remarks on the unification formulas Eqs. (A13) and (A14).

(i) They are valid for every range of m that satisfies the constraints in Eq. (A11) and are *independent* of the Higgs-boson and fermion content of the theory if one can ignore fine-structure effects. The scale of hierarchy masses can be determined independent of the ultimate asymptotic freedom of the theory.

(ii) In using Eqs. (A13) and (A14) we have assumed a linear hierarchy. This is clear from the table in Eq. (A18). In this linear hierarchy the U(1) embedded in SO(6) is not mixed with SU(2) embedded in SO(4) until the third (and last) stage.

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[34]The input value for α at 6 GeV has been referred through the renormalization-group equation to the value 1/137.036 at the m_e scale. For other input values for α_s, see Ref. 32.