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Hamiltonian of the massive Yang-Mills theory

Adel da Silveira
Centro Brasileiro de Pesquisas Físicas, Avenida Wenceslau Braz, 71-Botafogo, CEP-22290-Rio de Janeiro, Brazil
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The Schrödinger equation for a Yang-Mills massive field is written by using a physically useful formulation of the Kemmer-Duffin-Petiau equation due to Sakata and Taketani. It is shown that the linear terms correspond to the usual spin-1 equation. An interaction between the spin and the Yang-Mills field is obtained, even in the absence of the electromagnetic field.

I. INTRODUCTION

It is known that the non-Abelian Yang-Mills system displays many similarities to its classical version as was recently emphasized by Maciejko.[1] It seems natural to investigate similarities between Yang-Mills and Kemmer equations for massive particles of spin 1. The procedure to be followed was developed nearly forty years ago by Sakata and Taketani,[2] who used the Peierce decomposition. This method was described recently by Krajcik and Nieto,[3] and earlier by Heitler.[4] Sakata and Taketani were able to obtain a physically useful formulation of the Kemmer equation by separating out the $(2s+1)\times 2$ components into one Schrödinger equation and a subsidiary condition. The equation obtained by eliminating the redundant components of the wave function exhibits the occurrence of charge operators, with the two observables of the charge ±1, as well as the fact that the Hamiltonian is essentially not Hermitian, a fact that was also pointed out by Giambiagi and Tiomno.[5] Nevertheless, if the wave function is normalized, then the resulting new Hamiltonian will automatically become Hermitian.[6] In a recent publication of Okubo and Tosa,[7] the Duffin-Kemmer formulation of gauge theories was made by using an approach different from the one used here.

In the present paper we shall use the procedure of Sakata and Taketani to obtain the Hamiltonian of the SU(2) Yang-Mills fields over Euclidean four-space E_4, for the sourceless case. The first step is to write the Yang-Mills equations in the form of the Kemmer equation, by introducing a conveniently chosen mass term. The introduction of the minimal electromagnetic coupling is standard. After some calculations the Schrödinger equation is obtained. The Hamiltonian is found to consist of the linear part that already appears in the usual spin-1 formulation and, of course, a nonlinear part that depends on some components of the Yang-Mills field. To each term of the usual electromagnetic interaction there corresponds a term where the four-potential A_μ is substituted by the Yang-Mills field B_μ. We obtain, in this way, a term containing the three-dimensional scalar product of the spin-vector s_μ and the spatial components of the Yang-Mills field $\epsilon_{ijk}\phi_{ij}$ $(i,j,k=1,2,3)$, even in the absence of the electromagnetic field.

II. FUNDAMENTAL RELATIONS

We shall consider the Yang-Mills field $\phi_{\mu\nu}$ over Euclidean four-space E_4, given by

$$\partial_\mu B_\nu - \partial_\nu B_\mu - im\phi_{\mu\nu} = [B_\mu, B_\nu]\,. \tag{1}$$

The following equation is satisfied:

$$\partial_\nu \phi_{\mu\nu} - imB_\mu = [B_\nu, \phi_{\mu\nu}] \tag{2}$$

for the sourceless case. If we introduce the wave function Ψ defined by

$$\Psi = (\phi_{12}, \phi_{13}, \phi_{14}, \phi_{23}, \phi_{24}, \phi_{34}, B_1, B_2, B_3, B_4)^T \tag{3}$$

and Ψ' defined by

$$\Psi' = ([B_1, B_2], [B_1, B_3], [B_1, B_4], [B_2, B_3], [B_2, B_4] \times [B_3, B_4], [B_\nu, \phi_{1\nu}], [B_\nu \phi_{2\nu}], [B_\nu, \phi_{3\nu}], [B_\nu, \phi_{4\nu}])^T, \tag{4}$$

we can write the relations (1) and (2) as

$$(\beta_\nu \partial_\nu + m)\Psi = i\Psi'\,, \tag{5}$$

where the matrices β_ν satisfy the relations

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \beta_\mu \delta_{\nu\lambda} + \beta_\lambda \delta_{\nu\mu}\,. \tag{6}$$

$i\Psi'$ can be written as

$$i\Psi' = (B_\mu \beta_\mu - F)\Psi\,, \tag{7}$$

where F is a 10×10 matrix defined by

$$F = -i \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \tag{8a}$$

with A being given by

$$A=\begin{bmatrix} 0 & -\phi_{12} & -\phi_{13} & -\phi_{14} \\ \phi_{12} & 0 & -\phi_{23} & -\phi_{24} \\ \phi_{13} & \phi_{23} & 0 & -\phi_{34} \\ \phi_{14} & \phi_{24} & \phi_{34} & 0 \end{bmatrix}. \qquad (8b)$$

In (7), B_μ commutes with β_μ. Here B_μ represents a 10×10 matrix which has B_μ in the principal diagonal, the other elements being null. Therefore, in (7) each term of the sum $B_\mu\beta_\mu$ is a 10×10 matrix whose nonzero elements are obtained by substituting each i of β_μ by iB_μ.

Some well-known relations to be used in what follows will be considered now. We have from (6)

$$\beta_\mu{}^3=\beta_\mu \quad (\mu=1,2,3,4). \qquad (9a)$$

If we define η_μ, η, and τ by

$$\eta_\mu=(2\beta_\mu{}^2-1), \quad \eta=\eta_1\eta_2\eta_3, \quad \tau=\beta_4{}^2, \qquad (9b)$$

we obtain

$$\tau^2=\tau, \quad \tau\beta_4=\beta_4\tau, \quad \beta_4(1-\tau)=0, \qquad (9c)$$

$$\tau(1-\tau)=(1-\tau)\tau=0 \qquad (9d)$$

as well as

$$\beta_k\tau=(1-\tau)\beta_k \quad (k=1,2,3). \qquad (9e)$$

We use also

$$\beta_4\eta_k=-\eta_k\beta_4, \qquad (9f)$$

$$\eta_i\eta_k=+\eta_k\eta_i, \qquad (9g)$$

and

$$\eta_i{}^2=1, \quad \eta_k\beta_k{}^2=\beta_k{}^2 \ \text{(without sum)}, \qquad (9h)$$

$$\beta_4\eta=-\eta\beta_4, \qquad (9i)$$

$$\tau\eta=\eta\tau, \quad \tau\eta\tau=\tau, \qquad (9j)$$

and

$$\tau\beta_4\tau=\beta_4. \qquad (9k)$$

In Ref. 7 Eq. (2) has m^2 instead of im, so that the matrices β_μ are real.

We can write F_μ as

$$F\Psi=\beta_\mu F_\mu\Psi, \qquad (10)$$

where F_μ are 10×10 matrices. In F_1 only the seventh column is different from zero and this column is given by

$$(\phi_{12},\phi_{13},\phi_{14},0,0,0,0,0,0,0)^T. \qquad (11a)$$

In F_2 only the eighth column has elements different from zero, and is given by

$$(\phi_{12},0,0,\phi_{23},\phi_{24},0,0,0,0,0)^T. \qquad (11b)$$

In F_3 the column different from zero is the ninth:

$$(0,\phi_{13},0,\phi_{23},0,\phi_{34},0,0,0,0)^T. \qquad (11c)$$

Finally, the only column of F_4 different from zero is the tenth, and this one is given by

$$(0,0,\phi_{14},0,\phi_{24},\phi_{34},0,0,0,0)^T. \qquad (11d)$$

It can be seen from (11) that F_1, F_2, F_3, and F_4 contain the elements of the last four columns of the matrix [Eqs. (8a) and (8b)].

If we use (5), (7), (10), and (11), we obtain

$$[\beta_\mu(\partial_\mu-C_\mu)+m]\Psi=0, \qquad (12)$$

where

$$C_\mu=B_\mu-F_\mu. \qquad (13)$$

The introduction of the minimal electromagnetic coupling is standard. We obtain

$$(\beta_\mu D_\mu+m)\Psi=0, \qquad (14)$$

where

$$D_\mu=\partial_\mu-C_\mu-ieA_\mu \qquad (15)$$

satisfies the commutation relations

$$[D_\mu,D_\nu]=-T_{\mu\nu}-ieF_{\mu\nu} \qquad (16)$$

and

$$[D_\mu,\beta_\nu]=-[C_\mu,\beta_\nu]. \qquad (17)$$

$T_{\mu\nu}$ is defined by

$$T_{\mu\nu}=\partial_\mu C_\nu-\partial_\nu C_\mu-[C_\mu,C_\nu], \qquad (18)$$

and $F_{\mu\nu}$ is the electromagnetic field.

In order to obtain the Hamiltonian from Eq. (14), we shall proceed as follows. We multiply Eq. (14) by $\beta_\nu\beta_\lambda D_\nu$ and use (17) to put D_μ and D_ν together. We obtain

$$[\beta_\nu\beta_\lambda\beta_\mu D_\nu D_\mu-\beta_\nu\beta_\lambda(C_\nu\beta_\mu-\beta_\mu C_\nu)D_\mu+m\beta_\nu\beta_\lambda D_\nu]\Psi=0. \qquad (19)$$

If we interchange μ with ν and use (16), we can write (19) as

$$[\beta_\mu\beta_\lambda\beta_\nu D_\nu D_\mu-\beta_\mu\beta_\lambda\beta_\nu(T_{\mu\nu}+ieF_{\mu\nu}) -\beta_\nu\beta_\lambda(C_\nu\beta_\mu-\beta_\mu C_\nu)D_\mu+m\beta_\nu\beta_\lambda D_\nu]\Psi=0. \qquad (20)$$

If we add (19) and (20) and use (6) we obtain

$$[\beta_\mu D_\lambda D_\mu+\beta_\nu D_\nu D_\lambda-2\beta_\nu\beta_\lambda(C_\nu\beta_\mu-\beta_\mu C_\nu)D_\mu +2m\beta_\nu\beta_\lambda D_\nu-\beta_\mu\beta_\lambda\beta_\nu(T_{\mu\nu}+ieF_{\mu\nu})]\Psi=0. \qquad (21)$$

We then can write

$$\{-2mD_\lambda+2[C_\lambda,\beta_\mu]D_\mu-\beta_\mu(T_{\mu\lambda}+ieF_{\mu\lambda}) -2\beta_\mu\beta_\lambda([C_\mu,\beta_\nu]D_\nu-mD_\mu) -\beta_\mu\beta_\lambda\beta_\nu(T_{\mu\nu}+ieF_{\mu\nu})\}\Psi=0. \qquad (22)$$

With the help of (9a), (9b), and the antisymmetry of $T_{\mu\nu}$ and $F_{\mu\nu}$, the expression for $D_4\Psi$ can be written as

$$\{-2m(1-\beta_4{}^2)D_4+2(1-\beta_4{}^2)[C_4(\beta_4D_4+\beta_kD_k)+\beta_kC_kD_4-\beta_kC_4D_k-\beta_k(T_{k4}+ieF_{k4})]$$
$$+\beta_k\beta_4[2mD_k+2\beta_lC_kD_l-\beta_l(T_{kl}+ieF_{kl})-2C_k(\beta_4D_4+\beta_lD_l)]\}\Psi=0. \quad (23)$$

If we multiply (14) by $-2m\beta_4$, add the result to (23), and use (14) we obtain

$$\left\{-i\partial_t-B_4+F_4-ieA_4+m\beta_4+(1-\beta_4{}^2)\left[C_4+\frac{1}{m}\beta_kC_4D_k-\frac{1}{m}\beta_kC_kD_4+\frac{1}{m}\beta_k(T_{k4}+ieF_{k4})\right]\right.$$
$$\left.+\beta_k\beta_4\left[-D_k-C_k-\frac{1}{m}\beta_lC_kD_l+\frac{1}{2m}\beta_l(T_{kl}+ieF_{kl})\right]+\beta_4\beta_kD_k\right\}\Psi=0. \quad (24)$$

III. SAKATA-TAKETANI METHOD

We shall now use the Sakata and Taketani method. A matrix H can be written as

$$H=H_1+H_2, \quad (25)$$

where

$$H_1=\tau H\tau+\tau H(1-\tau) \quad (26a)$$

and

$$H_2=(1-\tau)H\tau+(1-\tau)H(1-\tau). \quad (26b)$$

If we use (9) we see that

$$2H\Psi=0$$

yields

$$H_1\Psi=0,$$

so that from (24) we obtain

$$\tau(-i\partial_t-B_4+F_4-ieA_4+\beta_4\beta_kD_k+m\beta_4)\tau\Psi$$
$$+\tau(F_4+\beta_4\beta_kD_k)(1-\tau)\Psi=0. \quad (27)$$

We shall not consider the relation corresponding to (26b), for it gives the development in time of the components $(1-\tau)\Psi$ of Ψ, which are entirely expressible by $\tau\Psi$, as we shall see. As was pointed out by Heitler,[4] an equation similar to (27) holds for $(\tau\Psi)^\dagger$ and densities of the type $(\Psi^\dagger A\Psi)$ can be transformed into an expression containing $\tau\Psi$ and operators of the form $\tau A\tau$ only.

In order to eliminate the factor $(1-\tau)\Psi$ from (27) we shall use Eq. (14). We obtain

$$(1-\tau)\Psi=-\frac{1}{m}(1-\tau)\beta_k(\partial_k-B_k-ieA_k)\Psi$$
$$-\frac{1}{m}(1-\tau)\beta_kF_k\Psi. \quad (28)$$

The last term of (28) can be written as $(i/m)H\tau\Psi$, where H is a 10×10 matrix. The only row of H different from zero is the tenth which is given by

$$(0,0,0,0,0,0,\phi_{14},\phi_{24},\phi_{34},0).$$

If we insert (28) into (27) we obtain

$$2\left[-i\partial_t-B_4+F_4-ieA_4+\beta_4\beta_kD_k+m\beta_4-\frac{1}{m}\tau F_4\beta_k(\partial_k-B_k-ieA_k)+\frac{i}{m}\tau F_4H-\frac{1}{m}\beta_4\beta_l(\partial_l-B_l-ieA_l)\beta_k(\partial_k-B_k-ieA_k)\right.$$
$$\left.-\frac{1}{m}\beta_4\beta_lF_l\beta_k(\partial_k-B_k-ieA_k)+\frac{i}{m}\beta_4\beta_l(\partial_l-B_l-ieA_l)H+\frac{i}{m}\beta_4\beta_lF_lH\right]\tau\Psi=0. \quad (29)$$

Some simplifications can be made in (29). As ∂_k, A_k, and B_k commute with τ, we can write

$$\beta_4\beta_kD_k\tau\Psi=\beta_4\beta_kF_k\tau\Psi.$$

We also have

$$F_4\tau\Psi=0, \quad \tau F_4=F_4.$$

As

$$\beta_4\beta_lF_l(1-\tau)=0,$$

we have

$$-\frac{1}{m}\beta_4\beta_lF_l\beta_k(\partial_k-ieA_k-B_k)\tau\Psi=0.$$

The last term of (29) is null because only the last row of H has elements different from zero, whereas all the elements of the last column of β_lF_l are null.

With these simplifications (29) is reduced to

$$\Big[-i\partial_t - B_4 - ieA_4 + m\beta_4 - \frac{1}{m}\beta_4\beta_l\beta_k(\partial_l - B_l - ieA_l)(\partial_k - B_k - ieA_k) - \frac{1}{m}F_4\beta_k(\partial_k - B_k - ieA_k) + \frac{i}{m}F_4H + \beta_4\beta_kF_k + \frac{i}{m}\beta_4\beta_l(\partial_l - B_l - ieA_l)H\Big]\tau\Psi = 0\,. \quad (30)$$

If we define the quantity S_i by

$$iS_j = \epsilon_{jkl}\beta_k\beta_l\,,$$

we have

$$\beta_i\beta_k + \beta_k\beta_i = -\eta(S_iS_k + S_kS_i)\,,$$

$$\beta_4\eta S_i^{\,2} = -\beta_4\tfrac{1}{2}(-\eta + \eta_i)\,,$$

and

$$\beta_4\beta_i^{\,2} = \beta_4\tfrac{1}{2}(1 + \eta_i)\,.$$

We can write

$$-\frac{1}{m}\beta_4\beta_l\beta_k(\partial_l - B_l - ieA_l)(\partial_k - B_k - ieA_k) = -\frac{1}{m}\left[\left(\beta_4\frac{1+\eta}{2}\right)(\partial_l^- - B_l)(\partial_l^- - B_l)\right] - \frac{i}{2m}\beta_4(1+\eta)\epsilon_{jkl}S_j[(\partial_k^- - B_k)(\partial_l^- - B_l) - (\partial_l^- - B_l)(\partial_k^- - B_k)] + \frac{1}{m}\beta_4\eta[S_jS_i(\partial_j^- - B_j)(\partial_i^- - B_i)]\,, \quad (31)$$

where

$$\partial_j^- = \partial_j - ieA_j\,.$$

If we compare (30) and (31) with the formula (4.17) indicated by Krajcik and Nieto,[3] we see that the last four terms of (30) are entirely of nonlinear origin. The five remaining terms of (30) are analogous to the terms of the above-mentioned relation (4.17) with one difference: Instead of the electromagnetic four-potential we have the sum $ieA_\mu + B_\mu$. We obtain in Eqs. (30) and (31) a term of the form

$$-\frac{i}{2m}\beta_4(1+\eta)\epsilon_{jkl}S_j\phi_{kl}\,,$$

which gives an interaction of the spin S_j with the Yang-Mills ϕ_{kl} even in the absence of the electromagnetic field. The nonlinear terms of (30) depend on ϕ_{i4}, ϕ_{kl}, B_k, and B_4.

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