PHYSICAL REVIEW D VOLUME 22, NUMBER 6 15 SEPTEMBER 1980

Massless, half-integer-spin fields in de Sitter space

J. Fang and C. Fronsdal
Department of Physics, University of California, Los Angeles, California 90024
(Received 28 January 1980)

Gauge-invariant wave equations are obtained for massless fields in de Sitter space, with arbitrary, half-integer spin. Massless quanta with spins $s = 1/2, 3/2, \ldots$ carry a series of unitary, irreducible representations of the de Sitter group. The special gauge fields carry another series of unitary, irreducible representations, hence ghost fields as well as the Goldstone and the associated special gauge field are actually massive in de Sitter space. Fields (properly: potentials) do not admit chirality for spins other than 1/2, but the field strengths associated with spin 3/2 do admit duality.

I. INTRODUCTION AND SUMMARY

Our motivation for investigating massless fields in de Sitter space has been given in Ref. 1. The case of half-integral spins, presented here, is a necessary step in our study of the role of singletons as elementary constituents of massless fields.[1,2] However, in summarizing our results we shall emphasize those features that may be of wider interest, especially to supergravity.

To begin with, we stress once again that no ambiguity exists concerning the meaning of masslessness in de Sitter space. The main point is that gauge invariance is associated with a unique representation of the de Sitter group, for each spin ≥ 1. This was shown earlier in the case of integer spins.[1] Deser and Zumino[3] pointed out that the spin-$\frac{3}{2}$ field of de Sitter supergravity "has the number of degrees of freedom appropriate to the massless case," without identifying the representation that is involved. The representations associated with elementary particles will be denoted $D(E_0, s)$, where E_0 is the lowest energy (in some very small units) and s is the spin. Here is a list of the most important cases:

$D(E_0, s)$ with $E_0 > s+1$, massive fields,
$D(s+1, s)$, massless fields (Sec. II),
$D(s+3, s)$, special gauge fields (Sec. II),
$D(\frac{1}{2}, 0)$ and $D(1, \frac{1}{2})$, singletons (Ref. 2).

In particular, $D(\frac{5}{2}, \frac{3}{2})$ is the spin-$\frac{3}{2}$ field of supergravity; $D(\frac{7}{2}, \frac{1}{2})$ is the associated special gauge field. It is remarkable that the two inequivalent representations, namely the neutrino $D(\frac{3}{2}, \frac{1}{2})$ and the special gauge field $D(\frac{7}{2}, \frac{1}{2})$, become degenerate in the limit of vanishing curvature. In general, in de Sitter space, *the gauge parameter field is not massless*, an interesting fact that may alleviate infrared difficulties and that may also be of some significance for quantization. In flat space the various ghost fields (i.e., b fields and Faddeev-Popov ghosts) that form a part of the new quantization schemes for gauge fields are themselves massless, and one wonders whether this necessitates the introduction of second-generation ghosts. In de Sitter space that problem does not arise. [The identification of $D(3, 0)$ with the spinless Goldstone field was first made by Castell.[4] Quantization of high-spin fields was discussed in Refs. 5 and 6.]

Gauge-invariant field equations for half-integral spins are obtained in Sec. III, in a five-dimensional notation that stresses both the group-theoretical meaning and the formal similarity with the flat-space wave equations. The propagator is calculated in the Feynman gauge in Sec. IV. Interactions with fixed, external sources are studied in Sec. V, where it is proved that only the physical quanta associated with $D(s+1, s)$ propagate ("helicity theorem"). All this work was carried out in the five-dimensional notation, which makes comparison with the usual covariant formulation difficult.

In Secs. VI and VII we transform our results to intrinsic notation, in terms of covariant derivatives and vierbein fields. The relationship between the five-dimensional spinor-tensors $k_{\alpha}\ldots$ of Secs II–IV and the usual Rarita-Schwinger spinor-tensors $h_{\mu}\ldots$ is given by (Sec. VI)

$$h_{\mu}\ldots(x) = M^{-1}(x)\, y^{\alpha}{}_{\mu}\ldots k_{\alpha}\ldots(y)\,. \qquad (1.1)$$

Here (y^{α}_{μ}) are the differential coefficients of the embedding map of de Sitter space into pseudo-Euclidean five-space and $M(x)$ is a 4-by-4 matrix. This matrix must be so chosen that a recognizable form of spinor calculus in curved space is obtained. The wave equation for $D(E_0, \frac{1}{2})$, which in five-dimensional notation is[7,8]

$$(\kappa - E_0 + \tfrac{7}{2})k = 0 \qquad (1.2)$$

[κ is the Dirac operator, see Eq. (2.1)] is reduced to

$$[i\not{D} - \rho^{1/2}(E_0 - \tfrac{3}{2})]h = 0\,, \qquad (1.3)$$

where ρ is the curvature. Comparison with the work of Zumino[9] shows that the Volkov-Akulov field in de Sitter space is associated with $D(\frac{7}{2},\frac{1}{2})$, as expected. Hence we confirm the identification of this field with the special gauge field $D(\frac{7}{2},\frac{1}{2})$ and with the gauge parameter of spin-$\frac{3}{2}$ massless fields.

Chirality is a concept that can be defined in de-Sitter space, but curvature makes a difference. The chirality operator may be defined as an involution that flips the parity, or equivalently as an operator the eigenspaces of which carry irreducible representations of the conformal group SO(4, 2). We say that a field admits chirality if the chirality operator exists and acts locally on the field. In flat space massless fields are chiral only if the spin is half-integral; then the chirality operator is the constant matrix γ_5. (In the present context what we mean by fields should more properly be called potentials.) For spins 1 and 2 chirality is usually called duality and is admitted by the field strengths of electromagnetism (but not by the potentials) and by the curvature tensor of gravitation (but not by the metric tensor). In de Sitter space the situation is quite the same as far as the integer-spin case is concerned, but very different for half-integral spins. Among the representations $D(E_0,s)$ only one, $D(\frac{3}{2},\frac{1}{2})$, has the property that the associated field admits chirality. The chirality operator is,[10] in the notation of Sec. II,

$$\beta \equiv \rho^{1/2}\gamma_0\gamma_1\gamma_2\gamma_3 \not{y} \quad (\not{y} \equiv \gamma_\alpha y^\alpha) .$$

Transformed to conventional notation this becomes the matrix γ_5. The operator β anticommutes with $\kappa+2$, and γ_5 anticommutes with $\not{D}$, so one sees either from (1.2) or (1.3) that the field associated with $D(E_0,s)$ admits chirality only if $E_0=\frac{3}{2}$. But $D(\frac{3}{2},n+\frac{1}{2})$ is unitary only if $n=0$; hence no fields (properly speaking, no potentials) with spin higher than $\frac{1}{2}$ admit chirality. It is noteworthy that the case of spin-$\frac{3}{2}$ field strengths is the first one in which the related concepts of chirality and duality occur together, and that in this case they coincide for free fields: $\tilde{F}_{\alpha\beta}=\beta F_{\alpha\beta}$. (A more complete discussion of chirality and duality in supergravity was given by Deser, Kay, and Stelle.[11])

At least three important problems are left for future study: (i) indefinite-metric quantization of massless fields, (ii) Higgs-Kibble mechanism, (iii) certain anomalies connected with the role of chirality projection operators[10] in weak-interaction phenomenology.

II. THE "MASSLESS" REPRESENTATIONS

The group of motions of de Sitter space is the universal covering of the connected part of SO(3,2). The compact subgroup of SO(3,2) is SO(3) ⊗ O(2); the first factor is associated with angular momentum, and the covering group of O(2) is the group of time translations. Any irreducible representation that can be associated with an elementary particle is characterized by an extremal weight (E_0,s), where E_0 is the lowest eigenvalue of the energy and s is the angular momentum of the ground state.[12] Here we are interested in spins $s=n+\frac{1}{2}$, $n=$positive integer; the case $n=0$ has been investigated previously.[11]

The representation $D(E_0,s)$ may be constructed by reduction of the tensor product $D(E_0,\frac{1}{2})\otimes D(n)$, where $D(n)$ is a finite-dimensional representation. Although $D(n)$ is not unitary for $n\neq 0$, $D(E_0,s)$ is unitary for $E_0>s+1$. The representation $D(n)$ is the irreducible component with highest weight contained in the nth tensor power of the five-dimensional vectorial representation.

The carrier for $D(E_0,\frac{1}{2})$ is a spinor field ϕ satisfying Dirac's wave equation[7,11]

$$(\kappa-E_0+\tfrac{7}{2})\phi=0\,, \qquad \kappa\equiv 2i\Sigma_{\alpha\beta}\,y_\alpha\partial_\beta=\hat{N}-\tilde{\not{y}}\not{\partial}\,,\quad \hat{N}\equiv y_\alpha\partial_\alpha\,. \tag{2.1}$$

The carrier for $D(E_0,s)$ is therefore a spinor-tensor k of rank n that satisfies the above wave equation and all covariant subsidiary conditions:

$$(\kappa-E_0+\tfrac{7}{2})k_{\alpha_1\cdots\alpha_n}=0\,, \tag{2.2}$$

$$y\cdot k=\partial\cdot k=0\,,\quad k'\equiv\gamma\cdot k=0\,. \tag{2.3}$$

If $E_0>s+1$, then the solutions of these equations carry the irreducible representation $D(E_0,s)$. (More details are given in the Appendix.)

The limiting case $E_0=s+1$ is of special interest. Because this is the lowest value of E_0 for which $D(E_0,s)$ is unitary, the limit of $D(E_0,s)$ as $E_0\to s+1$ from above is reducible:

$$\lim_{E_0\to s+1} D(E_0,s)=D(s+1,s)\oplus D(s+2,s-1)\,. \tag{2.4}$$

(The limit is, with respect to the usual topology, in terms of matrix coefficient on K-finite vectors.) The fact that the critical point is at $E_0=s+1$ may be seen easily as follows.

Consider special fields of the form

$$k=\Sigma_1\partial\phi+\Sigma_1\tilde{\gamma}\eta+\Sigma_1 y\chi\,, \tag{2.5}$$

where ϕ, η, χ are symmetric spinor tensors of rank $n-1$. Substituting (2.5) into (2.2) and (2.3) we find that $\phi=\eta=\chi=0$, unless $E_0=s+1$, in which case η and χ can be expressed in terms of ϕ and (2.5) can be written

$$k=\Sigma_1(\partial_T\tilde{\not{y}}+y\not{y}^{-1})\xi\,, \tag{2.6}$$

$$\partial_T\equiv\partial+y^{-2}y(n-2-\hat{N})\,. \tag{2.7}$$

The field ξ is of rank $n-1$ and must satisfy

$$y\cdot\xi=0\,,\quad \gamma\cdot\xi=0\,, \tag{2.8}$$

as well as

$$\partial\cdot\xi=0\,,\quad (\kappa-n+1)\xi=0\,. \tag{2.9}$$

Such solutions, of the form (2.6), form an invariant subspace $\mathcal{V}_0$ of a space $\mathcal{V}$ of solutions of (2.2) and (2.3), and (2.8) and (2.9) show that $\mathcal{V}_0$ carries the representation $D(s+2,s-1)$. The representation by (2.2) and (2.3) is nondecomposable and $D(s+1,s)$ is carried by the quotient space $\mathcal{V}/\mathcal{V}_0$.

The quotient space $\mathcal{V}/\mathcal{V}_0$ is the space of physical states. Fields of the form (2.6) with ξ satisfying (2.8) and (2.9) will be called "special gauge fields." The analogy with massless fields in Minkowski space is very close, and justifies referring to $D(s+1,s)$ as a "massless representation" and to the associated field k as a "massless field." [Additional justification comes from the fact that these representations may be extended to unitary representations of the conformal group SO(4, 2).[13]]

(Notation: $(\gamma^\alpha)=(i,\gamma^0,\gamma^1,\gamma^2,\gamma^3)$, $\tilde\gamma^\alpha=(-i,\gamma^0,\gamma^1,\gamma^2,\gamma^3)$, and $\Sigma_{\alpha\beta}=(i/4)(\tilde\gamma_\alpha\gamma_\beta-\tilde\gamma_\beta\gamma_\alpha)$. The anticommutation relations take the form $\tilde\gamma^\alpha\gamma^\beta+\tilde\gamma^\beta\gamma^\alpha=2\delta^{\alpha\beta}$. The diagonal metric is $\delta^{00}=\delta^{55}=1$, $\delta^{11}=\delta^{22}=\delta^{33}=-1$. The tilde on $\tilde{\not{y}}$ and $\tilde{\not\partial}$ means that, e.g., $\tilde{\not{y}}=\tilde\gamma^\alpha y_\alpha$; however, it will usually be omitted since the presence of a tilde on γ_α can be inferred from its position in the formula. The operator ∂_T defined in (2.7) is convenient. Useful formulas are

$$\not{y}\not\partial_T+\not\partial_T\not{y}=2n\,,\quad [\partial_T,y^2]=0\,,$$

$$[\partial_T,\not{y}]=[\not\partial_T,y]=\gamma_T\equiv\gamma-y/\not{y}\,.$$

The weight diagram of $D(s+1,s)$ is given in the Appendix.)

III. GAUGE-INVARIANT WAVE EQUATIONS

The analogy with Minkowski space[14] suggests the following ansatz for gauge-invariant wave equations in de Sitter space:

$$Lk=0\,,\quad L=BL_0\,, \tag{3.1}$$

$$L_0k\equiv\not\partial_Tk-\Sigma_1\partial_Tk'\,, \tag{3.2}$$

$$Bk\equiv k-\tfrac12\Sigma_1\gamma_Tk'-\tfrac12\Sigma_2\delta_Tk''\,, \tag{3.3}$$

$$\gamma_T\equiv\gamma-y/\not{y}\,,\quad \delta_T\equiv\delta-yy/y^2\,. \tag{3.4}$$

The operator L is symmetric with respect to the indefinite inner product $\int dy\,\delta(y^2-\rho^{-1})\bar kk$, $\bar k\equiv k^\dagger\gamma_0\not{y}$, and (3.1) can be derived from the following Lagrangian:

$$\mathcal{L}=\int dy\,\delta(y^2-\rho^{-1})[\bar k\not\partial_Tk+n\bar k'\not\partial_Tk'-\tfrac14n(n-1)\bar k''\not\partial_Tk''-n(\bar k'\partial\cdot k-\partial\cdot\bar kk')+\tfrac12n(n-1)(\bar k''\partial\cdot k'-\partial\cdot\bar k'k'')]\,. \tag{3.5}$$

The field k is symmetric and satisfies

$$y\cdot k=0\,,\quad k'''=0\,; \tag{3.6}$$

and of course the variation of $\mathcal{L}$ is made with due regard to these restrictions.

Equation (3.1) is satisfied identically by every field k of the type (2.6), provided only that the "gauge parameter" ξ satisfies (2.8). We call such fields "general gauge fields" (or simply "gauge fields") in contrast with the special gauge fields that are subject to (2.9) as well. Note that the general gauge fields satisfy (3.6); hence the Lagrangian and the field equations are gauge invariant.

When $n=1$ the field strengths

$$F_{\alpha\beta}\equiv\partial_{T\alpha}k_\beta-\partial_{T\beta}k_\alpha \tag{3.7}$$

are transverse, $y^\alpha F_{\alpha\beta}=0$, and gauge invariant. We have

$$(L_0k)_\beta=\gamma^\alpha F_{\alpha\beta}\,, \tag{3.8}$$

$$\begin{aligned}(Lk)^\beta&=\gamma^\alpha F_\alpha{}^\beta-\tfrac12\gamma_T^\beta\gamma^\gamma\gamma^\alpha F_{\alpha\gamma}\\&=\tfrac12(\gamma^\alpha\delta^{\beta\gamma}-\gamma^\gamma\delta^{\alpha\beta}+\gamma_T^\beta\gamma^\alpha\gamma^\gamma-\gamma_T^\beta\delta^{\alpha\gamma})F_{\alpha\gamma}\\&=(i/2)\rho^{1/2}\epsilon^{\alpha\beta\gamma\delta\epsilon}\gamma_\alpha y_\gamma\beta F_{\delta\epsilon}\,.\end{aligned} \tag{3.9}$$

The last step is an identity (see the Appendix). The spin-$\frac12$ chirality operator is[10,11]

$$\beta\equiv\rho^{1/2}\gamma_0\gamma_1\gamma_2\gamma_3\not{y}\,. \tag{3.10}$$

As pointed out in Sec. I, the field k cannot be chiral. It is apparently possible, however, for F to be self-dual. Equation (3.9) shows that

$$\tilde F^{\alpha\beta}\equiv(i\rho^{1/2}/2)\epsilon^{\alpha\beta\gamma\delta\epsilon}y_\gamma F_{\delta\epsilon} \tag{3.11}$$

satisfies the same field equation as F; hence the self-dual and anti-self-dual combinations $F_{\alpha\beta}\pm\tilde F_{\alpha\beta}$ also solve the field equations. One may show that $\tilde F_{\alpha\beta}=\beta F_{\alpha\beta}$ for free fields. (An analogous relation for $s>\frac32$ is not known.)

(All the operators defined above: L, L_0, B have the property that they preserve transversality; e.g., $y\cdot k=0$ implies that $y\cdot(Lk)=0$. Also $[B,\not{y}]k$ vanishes if k is transverse. Again, for transverse fields,

$$\partial_T\cdot k=\partial\cdot k\,,\quad (\not\partial_Tk)'=2\partial\cdot k-\not\partial_Tk'\,,$$

$$(L_0k)'=2\partial\cdot k-2\partial_Tk'-\Sigma_1\not\partial_Tk''\,.)$$

IV. THE PROPAGATOR

Adding an interaction term $-\bar{k}t-\bar{t}k$, where t is a fixed external source subject to

$$y\cdot t=0\,,\quad t'''=0\,, \tag{4.1}$$

to the Lagrangian density, leads to the field equation

$$Lk=t\,. \tag{4.2}$$

Because L annihilates the gauge fields, and L is symmetric, the traceless part of $\partial\cdot(Lk)$ vanishes identically. Self-consistency therefore requires the conservation law

$$\Psi\partial\cdot t=0\,, \tag{4.3}$$

where Ψ is the traceless projection operator for spinor-tensors of rank $n-1$. This condition ensures that the expanded Lagrangian is gauge invariant.

We look for a solution of (4.2) of the form

$$h=A(P^2)^{-1}t\,,\quad P^2\equiv\rho(\langle Q\rangle-Q)\,. \tag{4.4}$$

The operator Q is the Casimir operator

$$Q=\tfrac{1}{2}\sum L_{\alpha\beta}L^{\alpha\beta}\,, \tag{4.5}$$

where $(L_{\alpha\beta})$, $\alpha,\beta=0,1,2,3,5$ are the generators of SO(3,2) and $\langle Q\rangle$ is the eigenvalue of Q associated with the representation $D(s+1,s)$, namely $\langle Q\rangle=2s^2-2$. Hence P^2 is an analog of the Klein-Gordon operator on Minkowski space. It is also related to the covariant Klein-Gordon operator introduced by Lichnerowicz[15] (here specialized to a space of constant curvature but generalized to spinor-tensors of arbitrary rank). The operator A/P^2 is related to the Lichnerowicz propagator. In our case the existence of a causal inverse of P^2 follows from the fact that Q is the Casimir operator with eigenvalue $\langle Q\rangle$ in a unitary representation.[8] We find, as in the flat case,[14] that P^2 can be factored in the sense that

$$P^2k=L_0{}^2k+\tilde{k}\,, \tag{4.6}$$

where $\tilde{k}$ is a gauge field. Therefore, if we define the Dirac operator A by

$$A=L_0B^{-1}\,, \tag{4.7}$$

then $(AL-P^2)k$ is a gauge field. Since $L=BL_0$ is symmetric, so is A, and thus we conclude that

$$(LA-P^2)t=0 \tag{4.8}$$

holds for every source t that satisfies (4.3). That is, (4.4) satisfies (4.2). Equation (4.4) is the generalization, to de Sitter space and to arbitrary spin, of the equation $h=\not{p}(p^2)^{-1}t$ for a flat-space neutrino field produced by an external spinor source. The operator

$$G\equiv A/P^2 \tag{4.9}$$

is the propagator of the k field in what may be called "the Feynman gauge," since A is a first-order differential operator.

(The operators $L_{\alpha\beta}$ have the form

$$L_{\alpha\beta}k\ldots=i(y_\alpha\partial_\beta-y_\beta\partial_\alpha)k\ldots+\Sigma_{\alpha\beta}k\ldots$$
$$+i\Sigma_1(\delta_\alpha k_\beta\ldots-\delta_\beta k_\alpha\ldots)\,, \tag{4.10}$$

where δ_α is the "vector" with components $(\delta_\alpha)_\beta=\delta_{\alpha\beta}$. An explicit expression for Q is given by

$$(\langle Q\rangle-Q)k=[(n-2-\hat{N})(n+2+\hat{N})+y^2\partial^2+\not{y}\not{\partial}]k$$
$$+2\Sigma_1 y\partial\cdot k+\Sigma_1\gamma k'+2\Sigma_2\delta k''.) \tag{4.11}$$

V. HELICITY THEOREM

Let $t=t_1+t_2$, where supp t_1 and supp t_2 are disjoint. The one-particle exchange amplitude is

$$t_2(h_1)=\int dy\,\delta(y^2-\rho^{-1})(\bar{t}_2h_1+\bar{h}_1t_2)\,, \tag{5.1}$$

where h_1 is the field produced by t_1:

$$h_1=(A/P^2)t_1+(\text{gauge field})\,. \tag{5.2}$$

Let ψ be any field satisfying $y\cdot\psi=\psi'''=0$; having the same support as t_1. Then $t_2(h_1)$ is unaffected when we replace t_1 by

$$\check{t}_1=t_1+L\psi\,, \tag{5.3}$$

for the new contribution to h_1 is

$$(A/P^2)L\psi=(AL/P^2)\psi$$
$$=\psi+(1/P^2)(\text{gauge field}),\quad\text{by (4.6)}\,. \tag{5.4}$$

The first term vanishes on supp t_2 and the second term is a gauge field; hence neither contributes to $t_2(h_1)$.

Next, we show that ψ can be so chosen that $\check{t}_1'=0$. The condition is

$$2\partial\cdot\psi-2\not{\partial}_T\psi'-\Sigma_1\partial_T\psi''-(1/n)t_1'=0\,, \tag{5.5}$$

which can be satisfied by taking, first, ψ'' arbitrary (subject to $\psi'''=0$, $y\cdot\psi''=0$), then ψ' such that

$$\partial\cdot\psi'=\tfrac{1}{2}\not{\partial}_T\psi''-(1/2n)t_1''\,,$$

and finally ψ such that

$$\partial\cdot\psi=\not{\partial}_T\psi'+\tfrac{1}{2}\Sigma_1\partial_T\psi''+(1/2n)t_1'\,.$$

Thus $\check{t}_1'=0$, and consequently $\partial\cdot t_1'=0$. Since the traceless parts of $\partial\cdot(L\psi)$ and of $\partial\cdot t_1$ both vanish, it follows that also $\partial\cdot\check{t}_1=0$; hence the field $\check{h}_1\equiv(A/P^2)\check{t}_1$ satisfies all the subsidiary conditions. [If we choose the gauge field in (5.2) equal to the second term in (5.4), then $\check{h}_1=h_1+\psi$; hence h_1

satisfies all subsidiary conditions outside supp ψ = supp t_1.] Since the functional t_2 defined by (5.1) is gauge invariant, it follows that t_2 reduces to a linear functional on the quotient space $\mathcal{V}/\mathcal{V}_0$ of physical states. Therefore, only the physical quanta that carry the irreducible representation $D(s+1,s)$ propagate effectively between conserved sources.

VI. TRANSFORMATION TO INTRINSIC COORDINATES

Let V be the space R^5, endowed with coordinates (y^α) and pseudo-Euclidean metric $(\delta^{\alpha\beta})$. Let U be a domain in V, such that the intersection U_0 between U and the hyperboloid $y^2=1/\rho$ is simply connected. Let (x^μ), $\mu=0,1,2,3$ be coordinates for U_0 and $y\to x(y)$ a differentiable mapping of U into R^4 such that its restriction to U_0 is the coordinate map. Let $x\to y(x)$ denote the imbedding map, and set

$$x^\mu_\alpha \equiv \partial x^\mu/\partial y^\alpha\,,\quad y^\alpha_\mu = \partial y^\alpha/\partial x^\mu\,.$$

Then

$$x^\mu_\alpha y^\alpha_\nu = \delta^\mu_\nu\,,\quad x^\mu_\alpha y^\beta_\mu = \delta^\beta_\alpha - \rho y_\alpha y^\beta\,,$$

$$y^\alpha_\mu y^\beta_\nu \delta_{\alpha\beta} = g_{\mu\nu}\,,$$

where $(g_{\mu\nu})$, $\mu,\nu=0,1,2,3$ are the components of the de Sitter metric in these coordinates.

Let a field h be defined on U_0 by

$$h_{\mu\ldots}(x) = M^{-1}(x)\, y^\alpha{}_\mu \cdots k_{\alpha\ldots}(y)\,, \tag{6.1}$$

where $M^{-1}(x)$ is a 4-by-4 matrix. This is a conventional spinor-tensor on de Sitter space, and the function $x\to M(x)$ must be so chosen that the field equations for k take a recognizable form when expressed in terms of h.

Let

$$\hat\gamma_\mu(x) = M^{-1}(i\rho^{1/2}\tilde{y}\gamma_\alpha y^\alpha_\mu)M\,, \tag{6.2}$$

then the anticommutator is

$$\{\hat\gamma^\mu(x),\, \hat\gamma^\nu(x)\} = 2g^{\mu\nu}(x)\,.$$

The choice of the function $x\to M(x)$ is restricted by requiring (i), that

$$M^\dagger \gamma^0 M = \gamma^0 \tag{6.3}$$

and (ii) that $\hat\gamma^\mu(x)$ be a linear combination of the constant matrices (γ^a), $a=0,1,2,3$:

$$\hat\gamma^\mu(x) = e^\mu{}_a(x)\gamma^a \quad (\text{sum } a=0,1,2,3)\,. \tag{6.4}$$

Such a choice actually exists and may be found in the Appendix. Of course, the coefficients $e^\mu{}_a$ satisfy

$$e^\mu{}_a e^\nu{}_b \delta^{ab} = g^{\mu\nu}$$

and may be interpreted as vierbien coefficients.

Next

$$M^{-1}(\partial/\partial x^\mu)M = \partial/\partial x^\mu + i\omega_\mu{}^{\alpha\beta}\Sigma_{\alpha\beta} = \Delta_\mu\,. \tag{6.5}$$

More generally we define covariant differentiation of spinor-tensors tentatively by [compare (6.1)]

$$\Delta_\mu h_{\nu\ldots}(x) \equiv M^{-1} y^\alpha{}_\nu \cdots \partial_\mu k_{\alpha\ldots}(y)$$
$$= (\partial/\partial x^\mu + i\omega_\mu{}^{\alpha\beta}\Sigma_{\alpha\beta})h_{\nu\ldots} - \Sigma_1 \Gamma_{\mu\nu}{}^\lambda h_{\lambda\ldots}\,. \tag{6.6}$$

The connection coefficients $\omega_\mu{}^{\alpha\beta}$ and $\Gamma_{\mu\nu}{}^\lambda$ are given in the Appendix. Although this definition seems quite natural here, it does not quite agree with the spinor-covariant derivative commonly used in the literature. This will be remedied below; for the present it should be noted that the covariant derivative of $\hat\gamma_\mu$ is not zero:

$$\Delta_\mu \hat\gamma_\nu = (i\rho^{1/2}/2)[\hat\gamma_\mu, \hat\gamma_\nu]\,. \tag{6.7}$$

Also, it follows immediately from the definition (6.5) that

$$[\Delta_\mu, \Delta_\nu]h_{\lambda\ldots} = \rho\Sigma_1(g_{\mu\lambda}h_{\nu\lambda\ldots} - g_{\nu\lambda}h_{\mu\lambda\ldots})\,. \tag{6.8}$$

If k is a gauge field [of the form (2.6)], then h has the form

$$h = \Sigma_1 \Delta\xi\,,\quad \xi' = 0\,. \tag{6.9}$$

Here and below the prime is defined as in

$$\xi'_{\mu\ldots} \equiv \hat\gamma^\nu \xi_{\nu\mu\ldots}\,.$$

Finally, let us introduce the usual spinor covariant derivative

$$D_\mu = \Delta_\mu + (\rho^{1/2}/2i)\hat\gamma_\mu\,. \tag{6.10}$$

This commutes with $\hat\gamma_\nu$; note, however, that the gauge fields (6.9) are not symmetrized covariant gradients; instead

$$h = \Sigma_1 D\xi - (\rho^{1/2}/2i)\Sigma_1\hat\gamma\xi\,. \tag{6.11}$$

Equations (6.7) and (6.8) give

$$[D_\mu, D_\nu]h_{\lambda\ldots} = \rho\Sigma_1(g_{\mu\lambda}h_{\nu\lambda\ldots} - g_{\nu\lambda}g_{\mu\lambda\ldots})$$
$$- i\rho\Sigma_{\mu\nu}h_{\lambda\ldots} \equiv (-i\rho S_{\mu\nu}h)_{\lambda\ldots}\,. \tag{6.12}$$

This is a reflection of the structure relations of the conformal group SO(4,2); for

$$(i/\rho^{1/2})(D_\mu h)_{\lambda\ldots} = M^{-1}y^\alpha{}_\lambda \cdots (y_\mu{}^\beta L_{6\beta}k)_{\alpha\ldots}\,, \tag{6.13}$$

where

$$L_{6\beta} \equiv \rho^{1/2}[y^\alpha L_{\alpha\beta} - i(s+1)y_\beta]\,,\quad \beta = 0,1,2,3,5\,, \tag{6.14}$$

are the generators of conformal transformations of the field strengths.[11]

VII. INTRINSIC FORMULATION OF THE FIELD THEORY

From now on we leave out the caret on $\hat{\gamma}_\mu$; we also give new meanings to the symbols L, L_0, A, and B.

The wave equations (3.1), etc., become

$$Lh=0\,,\quad L=BL_0\,, \tag{7.1}$$

$$L_0h=i\not{D}h-i\Sigma_1 Dh'-(\rho^{1/2}/2)(2nh+\Sigma_1\gamma h')\,, \tag{7.2}$$

$$Bh=h-\tfrac{1}{2}\Sigma_1\gamma h'-\tfrac{1}{2}\Sigma_2 g h''\,. \tag{7.3}$$

Using (6.12) one easily checks that L_0 annihilates gauge fields of the form (6.11) with $\xi'=0$. Taking $n=1$ one recovers the wave equations and the gauge transformations of supergravity in de Sitter space. The "Klein-Gordon operator" P^2 takes the form

$$P^2h=[\,g^{\mu\nu}D_\mu D_\nu-(\rho/2)S_{\mu\nu}S^{\mu\nu}+2\rho(s^2-1)]h\,, \tag{7.4}$$

where $S_{\mu\nu}$ was defined in Eq. (6.12). The Dirac operator A becomes

$$Ah=L^0B^{-1}h\,, \tag{7.5}$$

where L^0 is given by (7.2) after reversing the sign of the last term. We still have

$$(AL-P^2)h=\text{gauge field}\,. \tag{7.6}$$

The conservation law for t is

$$\Psi\Delta\cdot t=0\,, \tag{7.7}$$

where Ψ is the traceless projection operator and Δ_μ is defined by (6.10). The solution of

$$Lh=t \tag{7.8}$$

in the Feynman gauge is thus, as before,

$$h=(A/P^2)t\,. \tag{7.9}$$

When $n=1$ one can use the formula (3.9) to obtain the simple form for the Lagrangian,

$$\mathfrak{L}=\tfrac{1}{2}\int(-g)^{1/2}d^4x\,\epsilon^{\mu\nu\lambda\rho}\,\bar{h}_\lambda\gamma_5\gamma_\mu F_{\nu\rho}\,, \tag{7.10}$$

in agreement with de Sitter supergravity [Ref. 3, Eq. (11)]. Here

$$F_{\mu\nu}=\Delta_\mu h_\nu-\Delta_\nu h_\mu\,. \tag{7.11}$$

The relation $\tilde{F}_{\alpha\beta}=\beta F_{\alpha\beta}$ for free fields, noted below Eq. (3.11), becomes $\tilde{F}_{\mu\nu}=\gamma_5F_{\mu\nu}$; this formula was noted by Freedman and van Nieuwenhuizen[16] and generalized by Deser and Zumino.[3]

Returning to the general case, we note that the wave operator L_0, Eq. (7.2), may be defined more elegantly by

$$L_0h=i\gamma^\alpha\Gamma_\alpha\,,\quad \Gamma_\alpha\equiv\Delta_\alpha h-\Sigma_1\Delta h_\alpha\,, \tag{7.12}$$

where Γ is (a generalization of) the Cristoffel symbol introduced by de Wit and Freedman.[17]

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation and by the Research Committee of the Academic Senate, UCLA. We thank Luis Urrutia for helpful discussions.

APPENDIX

The identity (Bjorken-Drell conventions)

$$i\gamma_5\gamma_\mu\epsilon^{\lambda\mu\nu\rho}A_{\lambda\nu\rho}=\gamma^\sigma\gamma^\nu\gamma^\tau A_{\nu\sigma\tau}+\gamma^\nu A_{\nu\sigma}{}^\sigma-\gamma^\sigma A_{\nu\sigma}{}^\nu-\gamma^\sigma A^\nu{}_{\nu\sigma}\,, \tag{A1}$$

where all indices run over 0, 1, 2, 3 and the coefficients $(A_{\lambda\rho})$ are arbitrary, has an extension to five-tensors, namely (in our conventions)

$$i\rho^{1/2}\epsilon^{\alpha\beta\gamma\delta\epsilon}\gamma_\alpha y_\gamma\beta A_{\beta\delta\epsilon}=\gamma^\alpha\tilde{\gamma}^\beta\gamma^\gamma A_{\beta\alpha\gamma}+\gamma^\alpha A_{\alpha\beta}{}^\beta-\gamma^\alpha A_{\beta\alpha}{}^\beta-\gamma^\alpha A^\beta{}_{\beta\alpha}\,. \tag{A2}$$

Here all indices run over 0, 1, 2, 3, 5 and the coefficients $(A_{\alpha\beta\gamma})$ are arbitrary except that $y^\alpha A_{\alpha\beta\gamma}=y^\alpha A_{\beta\alpha\gamma}=y^\alpha A_{\beta\gamma\alpha}=0$. This identity was used to derive (3.9). To prove (A2) it is enough to notice that both sides are invariant if A transforms like a tensor and that (A2) reduces to (A1) in the case of $y_5=\rho^{-1/2}$ and the other components of y vanish.

The matrix M introduced in (6.1) was calculated in a special, global coordinate system $(x^\mu)=(t,x^i)$:

$$x^i=y^i\,,\quad \sin(\rho^{1/2}t)=y_0/Y\,,\quad \cos(\rho^{1/2}t)=y_5/Y\,,$$

$$Y=(y_0^2+y_5^2)^{1/2}\,,\quad i,j,\ldots=1,2,3\,.$$

In these coordinates a choice of vierbein coefficients is

$$e^i{}_j=\delta^i{}_j-x^ix_j\,\rho(1+\rho^{1/2}Y)^{-1}\,,$$

$$e^t{}_0=\rho^{-1/2}Y\,,\quad e^t{}_i=e^i{}_0=0\,.$$

We require (6.3) in order to obtain the conventional metric and impose (6.4) on the matrices (6.2); this leads to

$$M=2^{-1/2}(a+ix^i\gamma_i\,\rho^{1/2}/a)\exp(i\gamma^0\rho^{1/2}t/2)$$

with $a=(1+\rho^{1/2}Y)^{1/2}$. With this M we obtain the following expression for the spinor connection in (6.5):

$$i\omega_\mu{}^{\alpha\beta}\Sigma_{\alpha\beta}=\omega_\mu{}^{ab}\sigma_{ab}+(i\rho^{1/2}/2)\hat{\gamma}_\mu\,.$$

The first part is conventional; the second part is just the difference between the operator Δ_μ and the usual spinor covariant derivative D_μ. The other connection coefficients are defined in the obvious way:

$$\Gamma_{\mu\nu}{}^\lambda\equiv y^\alpha{}_{\mu,\nu}\,y^\beta{}_\sigma\delta_{\alpha\beta}\,g^{\sigma\lambda}\,.$$

In the text, references to spaces of solutions of wave equations as "carriers" of representations of SO(3, 2) needs clarification. Of course, what is implied is that there exists a basis of K-finite vectors consisting of normalizable solutions. In the limit $E_0 \to s+1$ the basis vectors associated with the gauge field solutions become nonnormalizable (zero norm), and the representation defined on the solution space becomes nondecomposable. Such a basis of K-finite vectors can be constructed explicitly from the known basis for $D(E_0, \frac{1}{2})$ and the identification of $D(E_0, n+\frac{1}{2})$ with a subrepresentation of $D(E_0, \frac{1}{2}) \otimes D(n)$. The weight diagram of $D(s+1, s)$ is easily calculated from the weight diagrams of $D(E_0, \frac{1}{2})$ and $D(n)$. All the weights are simple and the coordinates are given by

$$\{(E,j)\,;\, E-j=1,2,\ldots\,;\, j-s=0,1,\ldots\},$$

for integer as well as half-integer spins. This allows one to write the simple closed formula for the character of $D(s+1, s)$ that was used in Ref. 2.

The form (2.6) for the gauge field is, strictly, not the only possibility. The equivalence transformation $k \to \beta k$ gives another alternative, but (1.2) must then be replaced by $(\kappa + E_0 + \frac{1}{2})k = 0$.

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