PHYSICAL REVIEW D VOLUME 22, NUMBER 1 1 JULY 1980

Polarization analysis of reactions with four spin-1/2 particles

Michael J. Moravcsik
Department of Physics and Institute of Theoretical Science, University of Oregon, Eugene, Oregon 97403
(Received 24 August 1979)

The complete polarization structure is developed for a reaction with four spin-1/2 (plus any number of spin-zero) particles, using the general optimal formalism. The aim is twofold: present practical results for this particular reaction, which is a commonly occurring one in nuclear and particle physics; and to exhibit, on a concrete example, the enormous simplification brought about by the use of this formalism as compared to the more traditional formalisms. The optimal-type formalism, being an infinite class of formalisms, includes, as special cases, helicity- and transversity-type formalisms which can be obtained from the general case by a mere interpretation of the spin indices. With an eye on utility also for inelastic reactions (e.g., $N+N\to N+N+\pi$), besides the general case with only rotation (or Lorentz) invariance, the structure with the sole additional constraint of identical particles is also given. The results are formulated both in terms of entirely polarized and partially unpolarized observables. The experiments needed to obtain information on the amplitudes are described, and it is shown that one can define a "distance" between amplitudes so that the complexity of the experiments needed to obtain information about amplitudes is related to this distance. Besides being applicable to a large array of specific reactions, the results are of interest for the most general particle reaction (involving particles with arbitrary spins), since it can be shown that this latter consists of links of structures of the kind exhibited by the reaction with four spin-1/2 particles.

I. INTRODUCTION

With the enormous upswing in recent years in the utilization of polarization experiments as a tool to probe nuclear and elementary-particle reactions, much has been written about the structure of these experiments in terms of the dynamical or nondynamical quantities of interest in such analyses. Yet, there is continued need for further developments along this line, because the richness and complexity of these polarization experiments, even in relatively low-spin cases, is so considerable that even a modest improvement in the coordination of the theoretical formalism with experimental arrangements can result in a very significant amount of additional accessible information on the reactions.

The purpose of the present paper is a composite one. I will consider the reaction

$$\tfrac{1}{2}+\tfrac{1}{2}\to\tfrac{1}{2}+\tfrac{1}{2}+\text{any number of spin-0 particles} \quad (1.1)$$

and describe its polarization structure in the optimal formalism.[1] Such a description will accomplish several objectives. First, it will explicitly provide practical results for the many reactions of the above form, such as

$$\begin{aligned} &p+p\to p+n+\pi,\\ &e+p\to e+p+\gamma,\\ &p+{}^3\mathrm{He}\to p+{}^3\mathrm{He},\\ &p+p\to p+p+X \text{ (inclusive)}, \text{ etc.} \end{aligned} \quad (1.2)$$

Second, the paper, formulated in the most general optimal language, immediately yields results in the helicity[2] or in the transversity[3] formalisms for such reactions, by our simply interpreting the indices in the optimal formalism as being the + and − of the helicity or transversity formalisms.

Third, the results of this paper are directly related also to the optimal structure of the most general reaction containing particles of arbitrary spins. It will be shown briefly how the structure of such a most general reaction can be thought of as being decomposed into linked sections, each of which being the structure of the reaction considered in this paper. The details of the analysis of the most general reaction, however, must be postponed to another paper soon to appear.

The optimal formalism (including an infinite set of specific formalisms) has indeed been shown to represent the ultimate in simplicity in describing the relationships between reaction amplitudes and experimental observables, and so what remains is only to exhibit explicitly the features of this formalism and spell out in detail the ways amplitudes and observables should be chosen to maximally further a specific objective in atomic, nuclear, or particle physics. As suggested recently,[4] the full utilization of such simplicity and generality will entail a judicious, *coordinated* choice of amplitudes and observables, and hence will influence the way experimental arrangements are designed for polarization measurements. Thus the seemingly very abstract considerations of polarization formalisms are directly connected to the very mundane matters of how to design, build, and arrange the extensive and expensive equipment used in polarization studies.

The outline of this paper is as follows. In Sec. II a brief summary will be given of the optimal formalism,[1] in order to specify the notation used in subsequent sections. This is followed by a description of the polarization structure of a reaction with four spin-1/2 (and an arbitrary number of spin-zero) particles, using polarized observables (Sec. III) and unpolarized observables (Sec. IV). Following this, the thus far unconstrained polarization structure will be restated with the constraints of identical particles (Sec. V). In Sec. VI a specific numerical example is worked out to illustrate how the results of the previous sections can be used in practice. Finally, Sec. VII summarizes the main points of the paper.

II. A SUMMARY OF THE OPTIMAL FORMALISM[1]

We are considering a reaction of the type

$$s_1 + s_2 \to s_3 + s_4 + 0 + 0 + \cdots , \tag{2.1}$$

where on the right-hand side we might have any number of spin-zero particles (denoted by 0), and s_i denotes a particle with spin s_i. In what follows, the spin-zero particles play no role except that the amplitudes to be defined might depend on the kinematic variables describing all particles and hence also those with spin zero.

The initial particles will have Latin indices, and the final particles Greek ones. Also, we will pair off one initial with one final (it does not matter in which way), and one pair will have lower-case indices, the other pair capital ones. Since each particle will have its own density matrix, we will need two indices for each particle. Thus we have

$$\begin{aligned} &u, v = 1, 2, \ldots, 2s_1 + 1, \quad l = 1, 2, \ldots, 2s_1 + 1, \\ &U, V = 1, 2, \ldots, 2s_2 + 1, \quad L = 1, 2, \ldots, 2s_2 + 1, \\ &\xi, \omega = 1, 2, \ldots, 2s_3 + 1, \quad \lambda = 1, 2, \ldots, 2s_3 + 1, \\ &\Xi, \Omega = 1, 2, \ldots, 2s_4 + 1, \quad \Lambda = 1, 2, \ldots, 2s_4 + 1 . \end{aligned} \tag{2.2}$$

The M matrix of reaction (2.1) can then be written as

$$M = \sum_l \sum_\lambda \sum_L \sum_\Lambda D(\lambda l, \Lambda L) \mathbb{S}^{\lambda l} \otimes \mathbb{S}^{\Lambda L} , \tag{2.3}$$

where $\mathbb{S}^{\lambda l}$ is a spin-momentum tensor connecting s_1 and s_3, and hence has $2s_3 + 1$ rows and $2s_1 + 1$ columns; $\mathbb{S}^{\Lambda L}$ is the analogue for s_2 and s_4; and $D(\lambda l, \Lambda L)$ is the corresponding complex amplitude, depending on the various rank-zero tensors that can be constructed out of the momentum variables that describe the reaction.

The initial state of reaction (2.1) can be described by a density matrix that is the outer product of the two density matrices pertaining to s_1 and s_2, respectively, both of which are square matrices of size $(2s_1 + 1) \times (2s_1 + 1)$ and $(2s_2 + 1) \times (2s_2 + 1)$, respectively. The final state of reaction (2.1), on the other hand, is characterized by our measuring the expectation values of the outer product of two spin-momentum tensors, which are both square, and of size $(2s_3 + 1) \times (2s_3 + 1)$ and $(2s_4 + 1) \times (2s_4 + 1)$, respectively. If we denote the initial density matrix by $\rho^{uv} \otimes \rho^{UV}$ and the final operator by $Q^{\xi\omega} \otimes Q^{\Xi\Omega}$, we obtain for the observable $\mathfrak{L}(uv, UV; \xi\omega, \Xi\Omega)$ the following expression:

$$\begin{aligned} \mathfrak{L}(uv, UV; \xi\omega, \Xi\Omega) = &\sum_l \sum_\lambda \sum_{l'} \sum_{\lambda'} \sum_L \sum_\Lambda \sum_{L'} \sum_{\Lambda'} D(\lambda l, \Lambda L) D^*(\lambda' l', \Lambda' L') \\ &\times \sum_a \sum_b \sum_A \sum_B \sum_\alpha \sum_\beta \sum_\Gamma \sum_\Delta (Q^{\xi\omega})_{\beta\alpha} (\mathbb{S}^{\lambda l})_{\alpha a} (\rho^{uv})_{ab} (\mathbb{S}^{\lambda' l'})^*_{b\beta} (Q^{\Xi\Omega})_{\Delta\Gamma} (\mathbb{S}^{\Lambda L})_{\Gamma A} (\rho^{UV})_{AB} (\mathbb{S}^{\Lambda' L'})_{B\Delta} . \end{aligned} \tag{2.4}$$

So far we have not brought into the picture the "optimal" nature of the formalism. To do so, we choose the following forms for the above quantities:

$$\begin{aligned} &(\mathbb{S}^{\lambda l})_{\alpha a} = \delta_{al}\delta_{\alpha\lambda}, \quad (\mathbb{S}^{\lambda' l'})_{\beta b} = \delta_{bl'}\delta_{\beta\lambda'}, \quad (\mathbb{S}^{\Lambda L})_{\Gamma A} = \delta_{AL}\delta_{\Gamma\Lambda}, \quad (\mathbb{S}^{\Lambda' L'})_{B\Delta} = \delta_{BL'}\delta_{\Delta\Lambda'}, \\ &(\rho^{uv})_{ab} \equiv (\rho^{uvH_p})_{ab} = \tfrac{1}{2}[(1+p) + (1-p)i](\delta_{ub}\delta_{va} + p\delta_{ua}\delta_{vb}), \end{aligned} \tag{2.5a}$$

where if $H_p = R$ ("real"), then $p = 1$, and if $H_p = I$ ("imaginary"), then $p = -1$;

$$(\rho^{UV})_{AB} \equiv (\rho^{UVH_P})_{AB} = \tfrac{1}{2}[(1+P) + (1-P)i](\delta_{UB}\delta_{VA} + P\delta_{UA}\delta_{VB}), \tag{2.5b}$$

where if $H_P = R$, then $P = 1$, and if $H_P = I$, then $P = -1$;

$$(Q^{\xi\omega})_{\beta\alpha} = (Q^{\xi\omega H_q})_{\beta\alpha} = \tfrac{1}{2}[(1+q) + (1-q)i](\delta_{\xi\alpha}\delta_{\omega\beta} + q\delta_{\xi\beta}\delta_{\omega\alpha}), \tag{2.5c}$$

where if $H_q = R$, then $q = 1$, and if $H_q = -I$ then $q = -1$;

$$(Q^{\Xi\Omega})_{\Delta\Gamma} = (Q^{\Xi\Omega H_Q})_{\Delta\Gamma} = \tfrac{1}{2}[(1+Q) + (1-Q)i](\delta_{\Xi\Gamma}\delta_{\Omega\Delta} + Q\delta_{\Xi\Delta}\delta_{\Omega\Gamma}), \tag{2.5d}$$

where if $H_Q = R$, then $Q = 1$, and if $H_Q = I$, then $Q = -1$.

Equation (2.5) is what makes the formalism optimal; that is, makes the relationship between the bilinear combinations of amplitudes and the observables as economical (as nearly diagonal) as possible.

In this representation, we then get the following explicit expression for the observables given in Eq. (2.4):

$$\mathfrak{L}(u\nu H_p, UVH_P; \xi\omega H_q, \Xi\Omega H_Q) = \tfrac{1}{2}\kappa Z Z_2 H_{wW}[D(\xi u, \Xi U)D^*(\omega v, \Omega V) + wD(\omega v, \Xi U)D^*(\xi u, \Omega V) + pD(\xi v, \Xi U)D^*(\omega u, \Omega V)$$
$$+ pwD(\omega u, \Xi U)D^*(\xi v, \Omega V) + PD(\xi u, \Xi V)D^*(\omega v, \Omega U)$$
$$+ PwD(\omega v, \Xi V)D^*(\xi u, \Omega U) + pPD(\xi v, \Xi V)D^*(\omega u, \Omega U)$$
$$+ pPwD(\omega u, \Xi V)D^*(\xi v, \Omega U)], \qquad (2.6)$$

where $w \equiv pq$, $W \equiv PQ$, $Z_1 \equiv 1 + pq - p + q$, $Z_2 \equiv 1 + PQ - P + Q$, and $\kappa = 1$ unless $w = W = -1$, in which case $\kappa = -1$.

Note that we have not said anything at all so far about the quantization direction in which each particle's $2s+1$ indices are to be interpreted. Indeed, these directions are completely arbitrary, and hence the "optimal formalism" is in reality a multiply infinite set of formalisms. Which specific formalism we want to use in a given problem will depend on the particular geometry, on the objectives to be pursued, on the experimental equipment that is available, etc. If the quantization direction of each particle is taken to be its own momentum, we obtain the helicity[2] formalism. If, on the other hand, the quantization direction is taken to be the normal to the reaction plane (if it can be defined), we have the transversity[3] forma-

TABLE I. The observable-bicom matrix structure for an arbitrary reaction in the optimal formalism as given by Eq. (2.6). For the notation, see the text, Secs. II and III.

Observable \ Bicom	$\frac{R\,R\,R\,R\,R\,R\,R\,R}{\xi u\ \omega v\ \xi v\ \omega u\ \xi u\ \omega v\ \xi v\ \omega u}$ $\frac{\Xi U\ \Xi U\ \Xi U\ \Xi U\ \Xi V\ \Xi V\ \Xi V\ \Xi V}{\omega v\ \xi u\ \omega u\ \xi v\ \omega v\ \xi u\ \omega u\ \xi v}$ $\Omega V\ \Omega V\ \Omega V\ \Omega V\ \Omega U\ \Omega U\ \Omega U\ \Omega U$	
$\frac{1}{2}RRRR$	$+\ +\ +\ +\ +\ +\ +\ +$	$(-)\frac{1}{2}RRRI$
$\frac{1}{2}RRII(-)$	$+\ -\ +\ -\ +\ -\ +\ -$	$(-)\frac{1}{2}RRIR$
$\frac{1}{2}RIRI$	$+\ +\ +\ +\ -\ -\ -\ -$	$\frac{1}{2}RIRR$
$\frac{1}{2}RIIR$	$+\ -\ +\ -\ -\ +\ -\ +$	$(-)\frac{1}{2}RIII$
$\frac{1}{2}IRRI$	$+\ -\ -\ +\ +\ -\ -\ +$	$\frac{1}{2}IRRR$
$\frac{1}{2}IRIR$	$+\ +\ -\ -\ +\ +\ -\ -$	$(-)\frac{1}{2}IRII$
$\frac{1}{2}IIRR(-)$	$+\ -\ -\ +\ -\ +\ +\ -$	$\frac{1}{2}IIRI$
$\frac{1}{2}IIII$	$+\ +\ -\ -\ -\ -\ +\ +$	$\frac{1}{2}IIIR$
	$I\ I\ I\ I\ I\ I\ I\ I$	

$u = v$

Observable \ Bicom	$\frac{R\,R\,R\,R}{\xi u\ \omega u\ \xi u\ \omega u}$ $\frac{\Xi U\ \Xi U\ \Xi V\ \Xi V}{\omega u\ \xi u\ \omega u\ \xi u}$ $\Omega V\ \Omega V\ \Omega U\ \Omega U$	
$\frac{1}{4}DRRR$	$+\ +\ +\ +$	$(-)\frac{1}{4}DRRI$
$\frac{1}{4}DRII(-)$	$+\ -\ +\ -$	$(-)\frac{1}{4}DRIR$
$\frac{1}{4}DIRI$	$+\ +\ -\ -$	$\frac{1}{4}DIRR$
$\frac{1}{4}DIIR$	$+\ -\ -\ +$	$(-)\frac{1}{4}DIII$
	$I\ I\ I\ I$	

$U = V$

Observable \ Bicom	$\frac{R\,R\,R\,R}{\xi u\ \omega v\ \xi v\ \omega u}$ $\frac{\Xi U\ \Xi U\ \Xi U\ \Xi U}{\omega v\ \xi u\ \omega u\ \xi v}$ $\Omega U\ \Omega U\ \Omega U\ \Omega U$	
$\frac{1}{4}RDRR$	$+\ +\ +\ +$	$(-)\frac{1}{4}RDRI$
$\frac{1}{4}RDII(-)$	$+\ -\ +\ -$	$(-)\frac{1}{4}RDIR$
$\frac{1}{4}IDIR$	$+\ +\ -\ -$	$(-)\frac{1}{4}IDII$
$\frac{1}{4}IDRI$	$+\ -\ -\ +$	$\frac{1}{4}IDRR$
	$I\ I\ I\ I$	

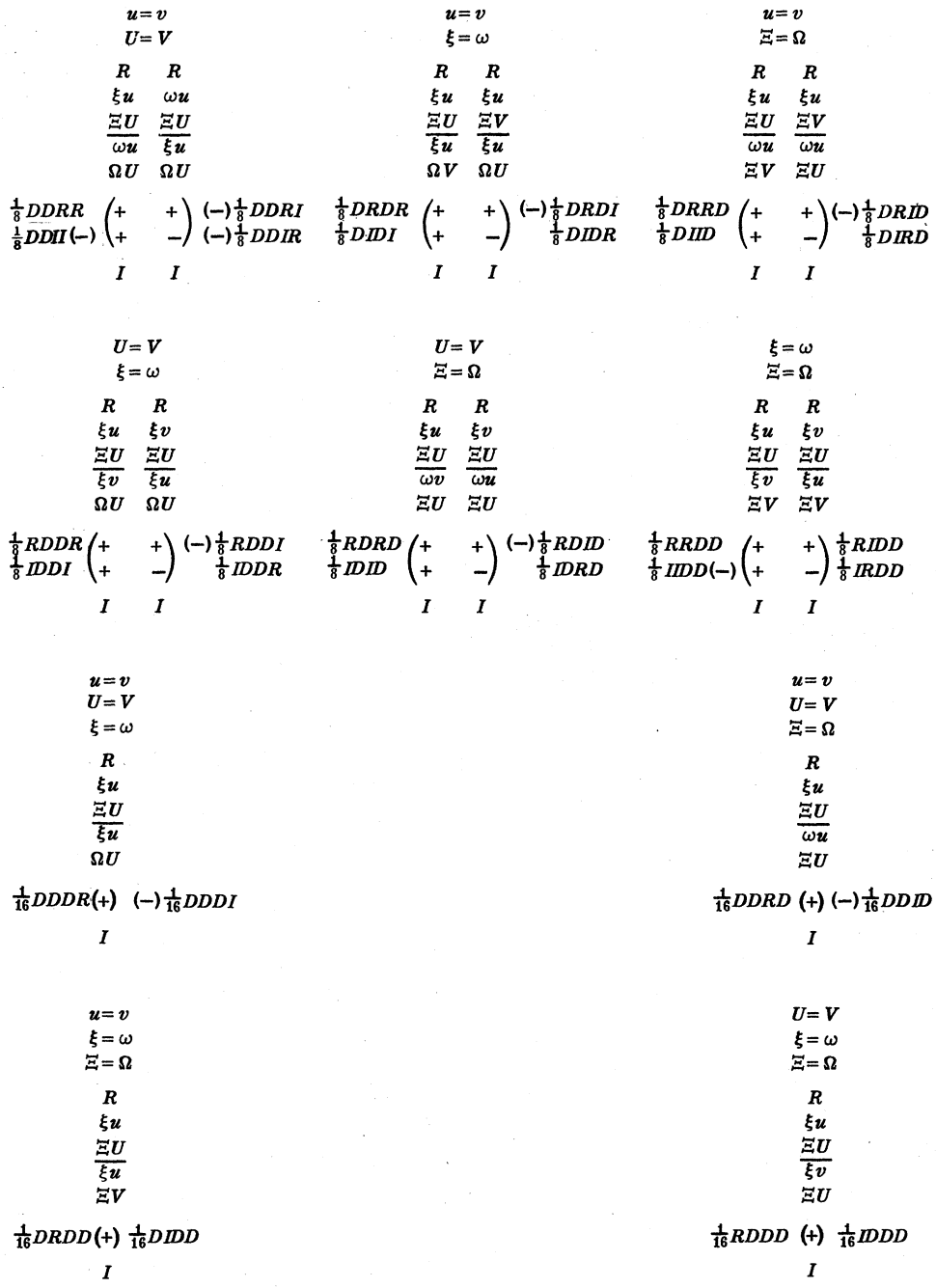
$\xi = \omega$

Observable \ Bicom	$\frac{R\,R\,R\,R}{\xi u\ \xi v\ \xi u\ \xi v}$ $\frac{\Xi U\ \Xi U\ \Xi V\ \Xi V}{\xi v\ \xi u\ \xi v\ \xi u}$ $\Omega V\ \Omega V\ \Omega U\ \Omega U$	
$\frac{1}{4}RRDR$	$+\ +\ +\ +$	$(-)\frac{1}{4}RRDI$
$\frac{1}{4}IRDI$	$+\ -\ +\ -$	$\frac{1}{4}IRDR$
$\frac{1}{4}RIDI$	$+\ +\ -\ -$	$\frac{1}{4}RIDR$
$\frac{1}{4}IIDR(-)$	$+\ -\ -\ +$	$\frac{1}{4}IIDI$
	$I\ I\ I\ I$	

$\Xi = \Omega$

Observable \ Bicom	$\frac{R\,R\,R\,R}{\xi u\ \xi u\ \xi v\ \xi v}$ $\frac{\Xi U\ \Xi V\ \Xi U\ \Xi V}{\omega v\ \omega v\ \omega u\ \omega u}$ $\Xi V\ \Xi U\ \Xi V\ \Xi U$	
$\frac{1}{4}RRRD$	$+\ +\ +\ +$	$(-)\frac{1}{4}RRID$
$\frac{1}{4}RIID$	$+\ -\ +\ -$	$\frac{1}{4}RIRD$
$\frac{1}{4}IRID$	$+\ +\ -\ -$	$\frac{1}{4}IRRD$
$\frac{1}{4}IIRD(-)$	$+\ -\ -\ +$	$\frac{1}{4}IIID$
	$I\ I\ I\ I$	

TABLE I. (*Continued.*)



$$u=v,\quad U=V,\quad \xi=\omega,\quad \Xi=\Omega$$
$$\tfrac{1}{16}DDDD=+R[\xi u,\ \Xi U,\ \xi u,\ \Xi U]=|\,D(\xi u,\ \Xi U)\,|^2$$

lism as a special case. All these formalisms look exactly the same in the above formulas, and can be differentiated only by the way we interpret the meaning of the rows and columns of the matrices in Eq. (2.5).

III. POLARIZATION STRUCTURE WITH ALL-POLARIZED OBSERVABLES

The first order of business is to write out, explicitly, the relationships implied in Eq. (2.6). This is done in Table I. In it, a compacted notation

is used for the bilinear combination of amplitudes (bicoms) and for the observables:

$$\begin{Bmatrix} R \\ I \end{Bmatrix}(\xi v, \Xi U \,|\, \omega u, \Omega V) \equiv \begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix}[D(\xi v, \Xi U) D^*(\omega u, \Omega V)], \tag{3.1}$$

$$IRIR \equiv \mathcal{L}(uvI, UVR; \xi\omega I, \Xi\Omega R). \tag{3.2}$$

In the observables, D means "diagonal", that is, repeated indices ($u = v$, or $\xi = \omega$, or $U = V$, or $\Xi = \Omega$). Since all elements in the matrix relating the bicoms to the observables are $+1$ or -1, the tables give only the signs. Furthermore, the matrices of a certain set of Re-type bicoms and that of the corresponding set of Im-type bicoms are identical, and so they are written in one table, so that the headings above the matrix relate to the labels on the left-hand margin, and the headings below the matrix relate to the labels at the right-hand margin. To give an example, from the first of the 4×4 matrices we get

$$\begin{aligned} \mathcal{L}(uu, UVR; \xi\omega I, \Xi\Omega R) = &-\mathrm{Im}[D(\xi u, \Xi U) D^*(\omega u, \Omega V)] \\ &+ \mathrm{Im}[D(\omega u, \Xi U) D^*(\xi u, \Omega V)] \\ &- \mathrm{Im}[D(\xi u, \Xi V) D^*(\omega u, \Omega U)] \\ &+ \mathrm{Im}[D(\omega u, \Xi V) D^*(\xi u, \Omega U)]. \end{aligned} \tag{3.3}$$

From Table I we see that there are definite and simple rules about which two amplitudes are combined into bicoms in what type of matrix. In particular, let us consider a subset of 16 amplitudes out of the total number of amplitudes. These 16 are chosen to be those in which each of the four indices has one of two values. For example, if in $D(\xi u, \Xi U)$ we specify that $\xi = +3$ or -2, $u = +1$ or -4, $\Xi = -1$ or -2, and $U = +3$ or $+2$, then the set consists of the amplitudes $D(+3, +1; -1, +3)$, $D(+3, +1; -1, +2)$, $D(+3, +1; -4, -1)$, $D(+3, +1; -4, -2), \ldots, D(-2, -4; -2, +2)$. To keep this notation general, we will call the two possible values of the first index ξ and ω, those of the second index u and v, those of the third index Ξ and Ω, and those of the fourth index U and V.

The rule then is that a bicom consisting of two amplitudes with only one of the four indices being different appears in a 1×1 matrix, those with two indices different in a 2×2 matrix, those with three indices different in a 4×4 matrix, and those with all four indices different in an 8×8 matrix. Furthermore, which bicoms appear together in a matrix can be also seen easily. In Fig. 1, the 16 amplitudes are pictured in a certain array, with lines connecting those which differ in only one index. In this array, therefore, we can define a "distance" between amplitudes, depending on how many straight lines one has to traverse (i.e., how many "steps" one has to take) to reach from one

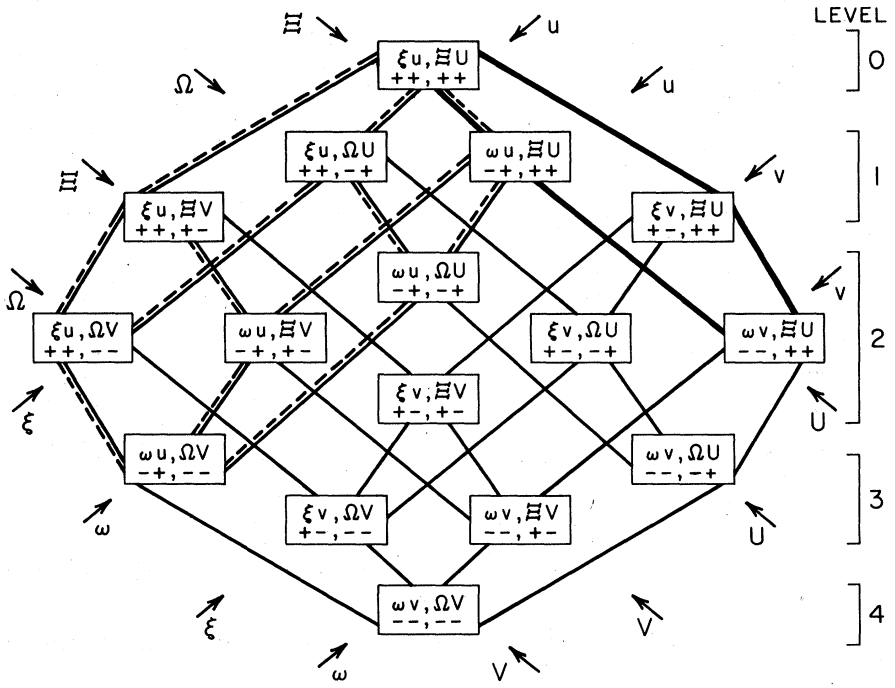


FIG. 1. Amplitude structure for a 16-set. The lines connect amplitudes the set of indices of which differ in only one index. For an explanation of levels and arrows, and heavy and dotted lines, see the text. In each amplitude two sets of labels are shown: the one pertaining to a general 16-set, and the one which hold for reaction (1.1).

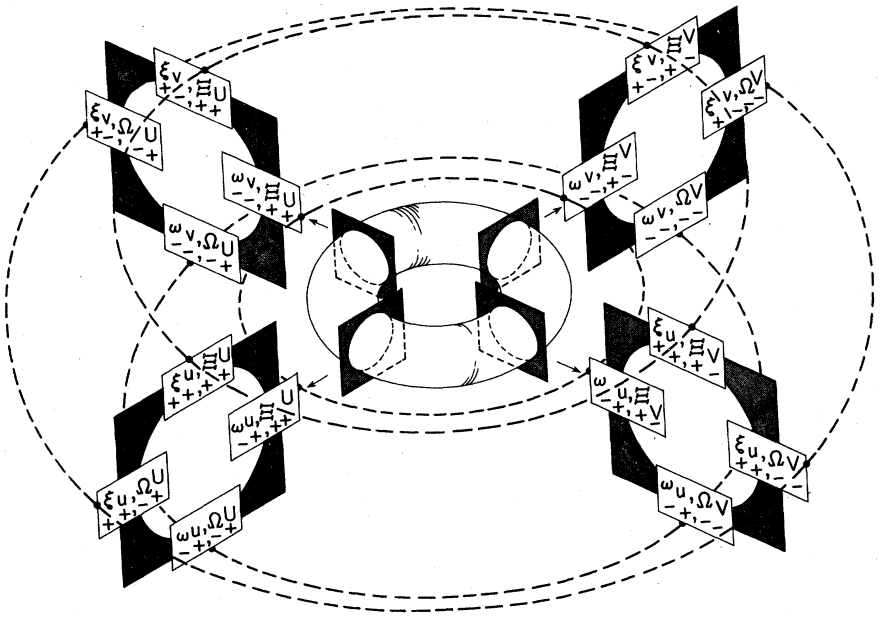


FIG. 2. The three-dimensional diagram of the amplitude structure of a 16-set, showing also the labels in the case of reaction (1.1). The center of the picture shows the overall view of the torus with the four planes. The surrounding figure is a blow-up of the four planes with labels, showing also (through the dotted lines) the connections between neighboring amplitudes not situated in the same plane. In addition, the two closest-lying amplitudes in the same plane are neighbors to a given amplitude. Thus each amplitude has four neighbors.

amplitude to another. If we consider only distances from the top amplitude $(\xi u, \Xi U)$, we can even define "levels" which contain amplitudes equally distant from the top amplitude. In this two-dimensional figure, however, that top amplitude appears to be treated on an unequal footing from the others, and hence the distances from some other amplitude do not appear to show an easily recognizable pattern.

This can be remedied and the full symmetry of the situation exhibited if the amplitudes are visualized in a three-dimensional diagram (see Fig. 2). Here the 16 amplitudes are placed equidistantly onto four circles which in turn are equidistantly located on a torus (doughnut) and then each amplitude is connected to four neighboring ones on that diagram. Throughout the remainder of the paper, the reader may want to consider, in addition to the two-dimensional figures I will explicitly use, the corresponding three-dimensional diagrams which, in some cases, may bring additional simplification to the visualization of the relationships among amplitudes.

In the upper right-hand corner of the array in Fig. 1, a subarray involving four amplitudes contains heavy solid lines. That subarray corresponds to the fifth 2×2 submatrix in Table I $(U = V, \Xi = \Omega)$, and the bicoms in it are those combinations of amplitudes which can be reached from each other by two steps along the lines [e.g., $(\xi v, \Xi U)$ can be reached from $(\omega u, \Xi U)$ by the two-step progression of $(\omega u, \Xi U) \to (\xi u, \Xi U) \to (\xi v, \Xi U)$.]

Similarly, in the upper left-hand side of the large array, a subarray is connected with heavy broken lines. That subarray corresponds to the first 4×4 matrix in Table I $(u = v)$ and the bicoms in it are combinations of two amplitudes which can be reached from each other by a three-step progression.

Let us now count the number of matrices of various sizes we have for such a set of 16 amplitudes (which henceforth we will call a 16-set). The results of this count are given in Table II, and can

TABLE II. Characteristics of observable-bicom matrices in one 16-set.

Matrix size	Number of matrices Re	Im	Total	Number of bicoms or observables in each	Total
8×8	1	1	2	8	16
4×4	8	8	16	4	64
2×2	24	24	48	2	96
1×1 (not magnitude squares)	32	32	64	1	64
1×1 (magnitude squares)	16		16	1	16
Total	81	65	146		256

be obtained from Table I and from the number of values the identical indices can have.

From this we can then construct the total number of matrices of various sizes in an arbitrary reaction of type (2.1). Using $\eta_i \equiv 2s_i + 1$, we see that among the amplitudes of a general reaction we can form $\prod_{i=1}^{4}(\eta_i)$ 16 sets.

If we then want to count up the total number of 8×8 matrices, we note that there are two per 16-set, there are $\prod_{i=1}^{4}(\eta_i)$ 16-sets, and an 8×8 matrix appears in only one of the 16-sets. Thus the total number of 8×8 matrices is

$$2\prod_{i=1}^{4}\binom{\eta_i}{2} = \tfrac{1}{8}\Phi \prod_{i=1}^{4}(\eta_i - 1), \tag{3.4}$$

where $\Phi \equiv \prod_{i=1}^{4}\eta_i$. We see that (3.4) agrees with (2.35) of Ref. 1.

Similarly, there are 16 matrices of 4×4 size in each 16-set. The one with, say, $u = v$, however, appears in other 16-sets also in which the ξ, U, and Ξ are the same but v is anything but u. There are $\eta_1 - 1$ of such 16-sets. Similarly, the 4×4 matrix with $U = V$ appears in $\eta_2 - 1$ other 16-sets, etc. Thus the number of 4×4 matrices will altogether

$$16\prod_{i=1}^{4}\binom{\eta_i}{2}\left[\tfrac{1}{4}\sum_{i=1}^{4}\frac{1}{\eta_i - 1}\right] = \tfrac{1}{4}\Phi \prod_{i=1}^{4}(\eta_i - 1)\sum_{i=1}^{4}(\eta_i - 1)^{-1} \tag{3.5}$$

which agrees with (2.34) of Ref. 1.

A similar argument for the 2×2 matrices gives for their total number

$$48\prod_{i=1}^{4}\binom{\eta_i}{2}\left[\tfrac{1}{6}\sum_{i<j}(\eta_i - 1)(\eta_j - 1)\right] = \tfrac{1}{2}\Phi\left[\sum_{i<j}\sum(\eta_i - 1)(\eta_j - 1)\right] \tag{3.6}$$

which agrees with (2.33) of Ref. 1. Likewise we can show that the remaining two types of 1×1 matrices correspond to the second and first terms, respectively, of (2.32) in Ref. 1.

Thus we see that the amplitudes of an arbitrary reaction can be thought of, in an optimal formalism, to be composed of (partially overlapping) 16-sets of amplitudes, and that therefore the bicom-observable structure of an arbitrarily complicated reaction can be understood in terms of the bicom-observable structure in a single 16-set. It is for this reason that even an arbitrarily complicated reaction has at most only 8×8 matrices in its bicom-observable structure, and never larger ones.

This insight into the reducibility of the bicom-observable structure of an arbitrarily complicated reaction to the relatively simple structure of a 16-set will be fully exploited in a later paper dealing with such a general reaction. In the present paper, however, we want to confine ourselves to the reaction containing four spin-$\frac{1}{2}$ particles and any number of spinless particles, i.e., to a reaction of the type (1.1).

The crucial remark to make now is that *a 16-set is exactly the description of a (1.1)-type reaction.* Thus the (1.1)-type reaction becomes not only an interesting one in its own right, but it also forms the fundamental building block in the analysis of *any* more complicated reaction.

The identification of the 16-set with reaction (1.1) can, of course, be made because the amplitudes of the latter have four indices, each of which can take on two values, just as the indices in the 16-set. We can therefore identify u and v, ξ and ω, U and V, and Ξ and Ω, with these two values which we will denote by + and −.

It is worth emphasizing again that throughout all of the discussion in the remainder of this paper, we need not commit ourselves on the choice of the quantization axes with respect to which the two values + and − represent $s_z = \frac{1}{2}$. The optimal formalism will be the same for any quantization direction. In fact, what axes of quantization we select should be determined by the experimental possibilities and by the particular dynamical theory we are considering being simply expressable. Such a freedom will allow us to assure a maximal interfacing[4] of theory with experiment.

In the rest of this section we will exhibit the structure of (1.1) in terms of unaveraged observables, i.e., in terms of $\mathfrak{L}$'s in which each of the four arguments consists of ++, +−, −+, or −−. The structure is simple in terms of these observables, but the observables themselves are experimentally not simple since we have to measure each of the four particles in a definite polarization state. In Sec. IV we will discuss the (more complicated) structure in terms of the (experimentally less complicated) observables in which one or several particles are unpolarized.

The structure involving the unaveraged observables can be read off directly from Table I. It is shown in Table III. That table also lists the averaged observables, so for the moment attention should be concentrated only on the bottom of each section of that table.

In terms of these observables, the systematic determination of the amplitudes is very simple. To start with, for example, we measure one of the $\mathfrak{L}(uu, UU; \xi\xi, \Xi\Xi)$'s which directly gives the magnitude of $D(\xi u, \Xi U)$. Since this first amplitude can be taken to be real, we have now determined it.

TABLE III. Observable-bicom relationship for the reaction described in Eq. (1.1), using unaveraged and averaged (A) observables, with and without the constraint of identical particles. The notation is described in the text, but is summarized here as follows: the observables $\mathcal{L}(uv, UV; \xi\omega, \Xi\Omega)$ appear on the left-hand edge, with R standing for "real" and I for "imaginary." The bilinear products of amplitudes ("bicoms") appear at the top of each table, both for the general case (only Lorentz or rotation invariance assumed) and for the additional constraint of identical particles. Each table contains two sets of observables, and hence in one of the four symbols in the arguments the notation of K's is used (see below). Correspondingly, a K also appears at the top for the bicoms. These two sets of observables and bicoms correspond to the two sets that appear in each of the tables of Table I also. The +'s and −'s in the tables themselves indicate the coefficients +1 or −1 that multiply the particular bicom at the top in the expression for the particular observable on the left-hand edge. The names of the tables (8_1, $4_1, \cdots, 1_M$) correspond to the various tables in Table I, where there are one 8×8, four 4×4's, six 2×2's, four 1×1's, and one featuring only absolute-value squares (1_M, where M indicates magnitudes). The notation for the K's is as follows:

$$K_1=\begin{pmatrix}R\\I\end{pmatrix},\quad K_2=\begin{pmatrix}R\\-I\end{pmatrix},\quad K_3=\begin{pmatrix}I\\R\end{pmatrix},\quad K_4=\begin{pmatrix}I\\-R\end{pmatrix},\quad K_{i+4}=-K_i.$$

An example might help. Using the twelfth entry of the table called 2_2 we have

$$\mathcal{L}(++,+-\text{Im};++,+-\text{Im})=\text{Re}[D(++,++)D^*(++,--)-D(++,+-)D^*(++,-+)]$$

and

$$\mathcal{L}(++,+-\text{Im};++,+-\text{Re})=2m[D(++,++)D^*(++,--)-D(++,+-)D^*(++,-+)]\,.$$

The averaging A is defined as $(++)-(--)$. For the case of identical particles all signs in the columns marked at the top by α must be reversed.

8_1

		$D(\lambda l,\Lambda L)D^*(\lambda' l',\Lambda' L')$	K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
	General	λl	++	−−	+−	−+	++	−−	+−	−+
		ΛL	++	++	++	++	+−	+−	+−	+−
		$\lambda' l'$	−−	++	−+	+−	−−	++	−+	+−
		$\Lambda' L'$	−−	−−	−−	−−	−+	−+	−+	−+
	Identical	λl	++	++	++	++	++	+−	+−	−+
	particles	ΛL	++	−−	+−	−+	+−	−−	+−	+−
$\frac{1}{2}\mathcal{L}(uv,UV;\xi\omega,\Xi\Omega)$		$\lambda' l'$	−−	++	−+	+−	−+	++	−+	−+
uv UV $\xi\omega$ $\Xi\Omega$		$\Lambda' L'$	−−	−−	−−	−−	−−	−+	−+	+−
						α	α	α	α	
R R R K_2			+	+	+	+	+	+	+	+
R R I K_7			+	−	+	−	+	−	+	−
R I R K_3			+	+	+	+	−	−	−	−
R I I K_2			+	−	+	−	−	+	−	+
I R R K_3			+	−	−	+	+	−	−	+
I R I K_2			+	+	−	−	+	+	−	−
I I R K_6			+	−	−	+	−	+	+	−
I I I K_3			+	+	−	−	−	−	+	+

4_1

		$D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$	K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
	General	λl	++	−+	++	−+	+−	−−	+−	−−
		ΛL	++	++	+−	+−	++	++	+−	+−
		$\lambda' l'$	−+	++	−+	++	−−	+−	−−	+−
		$\Lambda' L'$	−−	−−	−+	−+	−−	−−	−+	−+
	Identical	λl	++	++	++	−+	++	++	+−	+−
	particles	ΛL	++	+−	+−	+−	+−	−−	+−	−−
$\frac{1}{4}\mathcal{L}(uv,UV;\xi\omega,\Xi\Omega)$		$\lambda' l'$	−+	++	−+	++	−−	+−	−+	−+
uv UV $\xi\omega$ $\Xi\Omega$		$\Lambda' L'$	−−	−−	−+	−+	−−	−−	−−	+−
				α			α		α	α
A R R K_2			+	+	+	+	+	+	+	+
A R I K_7			+	−	+	−	+	−	+	−
A I R K_3			+	+	−	−	+	+	−	−
A I I K_2			+	−	−	+	+	−	−	+
$++$ R R K_2			+	+	+	+				
$++$ R I K_7			+	−	+	−				
$++$ I R K_3			+	+	−	−				
$++$ I I K_2			+	−	−	+				
$--$ R R K_2							+	+	+	+
$--$ R I K_7							+	−	+	−
$--$ I R K_3							+	+	−	−
$--$ I I K_2							+	−	−	+

4_2

		$D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$	K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
	General	λl	++	−−	+−	−+	++	−−	+−	−+
		ΛL	++	++	++	++	+−	+−	+−	+−
		$\lambda' l'$	−−	++	−+	+−	−−	++	−+	+−
		$\Lambda' L'$	−+	−+	−+	−+	−−	−−	−−	−−
	Identical	λl	++	++	++	++	++	+−	+−	−+
	particles	ΛL	++	−−	+−	−+	+−	−−	+−	+−
$\frac{1}{4}\mathcal{L}(uv,UV;\xi\omega,\Xi\Omega)$		$\lambda' l'$	−+	++	−+	−+	−−	++	−+	+−
uv UV $\xi\omega$ $\Xi\Omega$		$\Lambda' L'$	−−	−+	−+	+−	−−	−−	−−	−−
			α		α	α		α		
R A R K_2			+	+	+	+	+	+	+	+
R A I K_7			+	−	+	−	+	−	+	−
I A I K_2			+	+	−	−	+	+	−	−
I A R K_3			+	−	−	+	+	−	−	+
R $++$ R K_2			+	+	+	+				
R $++$ I K_7			+	−	+	−				
I $++$ I K_2			+	+	−	−				
I $++$ R K_3			+	−	−	+				
R $--$ R K_2							+	+	+	+
R $--$ I K_7							+	−	+	−
I $--$ I K_2							+	+	−	−
I $--$ R K_3							+	−	−	+

TABLE III. (*Continued.*)

4_3 $D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$

				K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
			General	λl	++	+−	++	+−	−+	−−	−+	−−
				ΛL	++	++	+−	+−	++	++	+−	+−
				$\lambda' l'$	+−	++	+−	++	−−	−+	−−	−+
				$\Lambda' L'$	−−	−−	−+	−+	−−	−−	−+	−+
			Identical	λl	++	++	++	+−	++	++	−+	+−
			particles	ΛL	++	+−	+−	+−	−+	−−	+−	−−
$\frac{1}{4}\mathfrak{L}(uv,UV;\xi\omega,\Xi\Omega)$				$\lambda' l'$	+−	++	−+	++	−−	−+	−+	−+
uv	UV	$\xi\omega$	$\Xi\Omega$	$\Lambda' L'$	−−	−−	+−	−+	−−	−−	−−	−+
						α			α		α	α
R	R	A	K_2		+	+	+	+	+	+	+	+
I	R	A	K_3		+	−	+	−	+	−	+	−
R	I	A	K_3		+	+	−	−	+	+	−	−
I	I	A	K_6		+	−	−	+	+	−	−	+
R	R	++	K_2		+	+	+	+				
I	R	++	K_3		+	−	+	−				
R	I	++	K_3		+	+	−	−				
I	I	++	K_6		+	−	−	+				
R	R	−−	K_2						+	+	+	+
I	R	−−	K_3						+	−	+	−
R	I	−−	K_3						+	+	−	−
I	I	−−	K_6						+	−	−	+

4_4 $D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$

				K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
			General	λl	++	++	+−	+−	++	++	+−	+−
				ΛL	++	+−	++	+−	−+	−−	−+	−−
				$\lambda' l'$	−−	−−	−+	−+	−−	−−	−+	−+
				$\Lambda' L'$	+−	++	+−	++	−−	−+	−−	−+
			Identical	λl	++	++	++	+−	++	++	−+	+−
			particles	ΛL	++	+−	+−	+−	−+	−−	+−	−−
$\frac{1}{4}\mathfrak{L}(uv,UV;\xi\omega,\Xi\Omega)$				$\lambda' l'$	+−	++	−+	++	−−	−+	−+	−+
uv	UV	$\xi\omega$	$\Xi\Omega$	$\Lambda' L'$	−−	−−	+−	−+	−−	−−	−−	−+
					α		α	α		α		
R	R	K_2	A		+	+	+	+	+	+	+	+
R	I	K_3	A		+	−	+	−	+	−	+	−
I	R	K_3	A		+	+	−	−	+	+	−	−
I	I	K_6	A		+	−	−	+	+	−	−	+
R	R	K_2	++		+	+	+	+				
R	I	K_3	++		+	−	+	−				
I	R	K_3	++		+	+	−	−				
I	I	K_6	++		+	−	−	+				
R	R	K_2	−−						+	+	+	+
R	I	K_3	−−						+	−	+	−
I	R	K_3	−−						+	+	−	−
I	I	K_6	−−						+	−	−	+

2_1 $D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$

				K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
			General	λl	++	−+	++	−+	+−	−−	+−	−−
				ΛL	++	++	+−	+−	++	++	+−	+−
				$\lambda' l'$	−+	++	−+	++	−−	+−	−−	+−
				$\Lambda' L'$	−+	−+	−−	−−	−+	−+	−−	−−
			Identical	λl	++	++	++	−+	++	++	+−	+−
			particles	ΛL	++	−+	+−	+−	+−	−−	+−	−−
$\frac{1}{4}\mathfrak{L}(uv,UV;\xi\omega,\Xi\Omega)$				$\lambda' l'$	−+	++	−+	++	−+	−+	−−	+−
uv	UV	$\xi\omega$	$\Xi\Omega$	$\Lambda' L'$	−+	−+	−−	−−	−−	+−	−−	−−
						α						α
A	A	R	K_2		+	+	+	+	+	+	+	+
A	A	I	K_7		+	−	+	−	+	−	+	−
A	++	R	K_2		+	+			+	+		
A	++	I	K_7		+	−			+	−		
A	−−	R	K_2				+	+			+	+
A	−−	I	K_7				+	−			+	−
++	A	R	K_2		+	+	+	+				
++	A	I	K_7		+	−	+	−				
−−	A	R	K_2						+	+	+	+
−−	A	I	K_7						+	−	+	−
++	++	R	K_2		+	+						
++	++	I	K_7		+	−						
++	−−	R	K_2			+	+					
++	−−	I	K_7			+	−					
−−	++	R	K_2						+	+		
−−	++	I	K_7						+	−		
−−	−−	R	K_2								+	+
−−	−−	I	K_7								+	−

2_2 $D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$

				K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
			General	λl	++	++	−+	−+	+−	+−	−−	−−
				ΛL	++	+−	++	+−	++	+−	++	+−
				$\lambda' l'$	++	++	−+	−+	+−	+−	−−	−−
				$\Lambda' L'$	−−	−+	−−	−+	−−	−+	−−	−+
			Identical	λl	++	++	++	−+	++	+−	++	+−
			particles	ΛL	++	+−	−+	+−	+−	+−	−−	−−
$\frac{1}{4}\mathfrak{L}(uv,UV;\xi\omega,\Xi\Omega)$				$\lambda' l'$	++	++	−+	−+	+−	−+	−−	−+
uv	UV	$\xi\omega$	$\Xi\Omega$	$\Lambda' L'$	−−	−+	−−	−+	−−	+−	−−	−−
							α			α		
A	R	A	K_2		+	+	+	+	+	+	+	+
A	I	A	K_3		+	−	+	−	+	−	+	−
A	R	++	K_2		+	+			+	+		
A	I	++	K_3		+	−			+	−		
A	R	−−	K_2				+	+			+	+
A	I	−−	K_3				+	−			+	−
++	R	A	K_2		+	+	+	+				
++	I	A	K_3		+	−	+	−				
−−	R	A	K_2						+	+	+	+
−−	I	A	K_3						+	−	+	−
++	R	++	K_2		+	+						
++	I	++	K_3		+	−						
++	R	−−	K_2				+	+				
++	I	−−	K_3				+	−				
−−	R	++	K_2						+	+		
−−	I	++	K_3						+	−		
−−	R	−−	K_2								+	+
−−	I	−−	K_3								+	−

Next we determine the amplitudes in which *one* of the indices is different from those in $D(\xi u,\Xi U)$. Let us take, for example, $D(\omega u,\Xi U)$. To determine it we measure $\mathfrak{L}(uu,UU;\omega\omega,\Xi\Xi)$ which fixes its magnitude, and then $\mathfrak{L}(uu,UU;\xi\omega R,\Xi\Xi)$ or $\mathfrak{L}(uu,UU;\xi\omega I,\Xi\Xi)$, either of which measures the relative phase between the two amplitudes.

Once $D(\omega u,\Xi U)$, $D(\xi v,\Xi U)$, etc., are determined, we can use them as "anchor points" and repeat the procedure outlined in the previous paragraph, thus determining the amplitudes which differ from the original $D(\xi u,\Xi U)$ by two indices being different, etc. Thus, by this successive procedure, we can determine all the amplitudes by measuring $2\times16-1=31$ quantities.

Note that throughout this procedure we always used only the 1×1 matrices in the bicom-observable relationship, and hence a given experiment *directly* determines a particular bicom. This allows us to use the actual minimum number of experiments (31 in our case) to determine all amplitudes unambiguously (except for a discrete set of ambiguities).

Note also that the above procedure holds equally well for a general reaction the amplitudes of which, we showed above, were a link of partially overlapping 16-sets. Thus we obtain the following theorem:

TABLE III. (*Continued.*)

2_3 $D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$

		K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
General	λl	++	++	++	++	+−	+−	+−	+−
	ΛL	++	+−	−+	−−	++	+−	−+	−−
	$\lambda' l'$	−+	−+	−+	−+	−−	−−	−−	−−
	$\Lambda' L'$	+−	++	−−	−+	+−	++	−−	−+
Identical particles	λl	++	++	++	++	++	+−	−+	+−
	ΛL	++	+−	−+	−−	+−	+−	+−	−−
	$\lambda' l'$	−+	−+	−+	−+	+−	−−	−−	−+
	$\Lambda' L'$	+−	++	−−	−+	−−	++	−−	−−
$\frac{1}{8}\mathcal{L}(uv,UV;\xi\omega,\Xi\Omega)$			α						α
uv UV $\xi\omega$ $\Xi\Omega$									
A R K_2 A		+	+	+	+	+	+	+	+
A I K_3 A		+	−	+	−	+	−	+	−
A R K_2 ++		+	+			+	+		
A I K_3 ++		+	−			+	−		
A R K_2 −−				+	+			+	+
A I K_3 −−				+	−			+	−
++ R K_2 A		+	+	+	+				
++ I K_3 A		+	−	+	−				
−− R K_2 A						+	+	+	+
−− I K_3 A						+	−	+	−
++ R K_2 ++		+	+						
++ I K_3 ++		+	−						
++ R K_2 −−				+	+				
++ I K_3 −−				+	−				
−− R K_2 ++						+	+		
−− I K_3 ++						+	−		
−− R K_2 −−								+	+
−− I K_3 −−								+	−

2_4 $D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$

		K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
General	λl	++	+−	−+	−−	++	+−	−+	−−
	ΛL	++	++	++	++	+−	+−	+−	+−
	$\lambda' l'$	+−	++	−−	−+	+−	++	−−	−+
	$\Lambda' L'$	−+	−+	−+	−+	−−	−−	−−	−−
Identical particles	λl	++	++	++	++	++	+−	−+	+−
	ΛL	++	+−	−+	−−	+−	+−	+−	−−
	$\lambda' l'$	−+	++	−+	−+	++	++	−−	−+
	$\Lambda' L'$	+−	−+	−−	−+	+−	−−	−−	−−
$\frac{1}{8}\mathcal{L}(uv,UV;\xi\omega,\Xi\Omega)$			α						α
uv UV $\xi\omega$ $\Xi\Omega$									
R A A K_2		+	+	+	+	+	+	+	+
I A A K_3		+	−	+	−	+	−	+	−
R A ++ K_2		+	+			+	+		
I A ++ K_3		+	−			+	−		
R A −− K_2				+	+			+	+
I A −− K_3				+	−			+	−
R ++ A K_2		+	+	+	+				
I ++ A K_3		+	−	+	−				
R −− A K_2						+	+	+	+
I −− A K_3						+	−	+	−
R ++ ++ K_2		+	+						
I ++ ++ K_3		+	−						
R ++ −− K_2				+	+				
I ++ −− K_3				+	−				
R −− ++ K_2						+	+		
I −− ++ K_3						+	−		
R −− −− K_2								+	+
I −− −− K_3								+	−

2_5 $D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$

		K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
General	λl	++	+−	++	+−	++	+−	++	+−
	ΛL	++	++	−+	−+	+−	+−	−−	−−
	$\lambda' l'$	−−	−+	−−	−+	−−	−+	−−	−+
	$\Lambda' L'$	++	++	−+	−+	+−	+−	−−	−−
Identical particles	λl	++	++	++	−+	++	+−	++	+−
	ΛL	++	+−	−+	+−	+−	−+	−−	−−
	$\lambda' l'$	++	++	−+	−+	+−	−+	−−	−+
	$\Lambda' L'$	−−	−+	−−	−+	−−	+−	−−	−−
$\frac{1}{8}\mathcal{L}(uv,UV;\xi\omega,\Xi\Omega)$				α		α			
uv UV $\xi\omega$ $\Xi\Omega$									
R A K_2 A		+	+	+	+	+	+	+	+
I A K_3 A		+	−	+	−	+	−	+	−
R A K_2 ++		+	+			+	+		
I A K_3 ++		+	−			+	−		
R A K_2 −−				+	+			+	+
I A K_3 −−				+	−			+	−
R ++ K_2 A		+	+	+	+				
I ++ K_3 A		+	−	+	−				
R −− K_2 A						+	+	+	+
I −− K_3 A						+	−	+	−
R ++ K_2 ++		+	+						
I ++ K_3 ++		+	−						
R ++ K_2 −−				+	+				
I ++ K_3 −−				+	−				
R −− K_2 ++						+	+		
I −− K_3 ++						+	−		
R −− K_2 −−								+	+
I −− K_3 −−								+	−

2_6 $D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$

		K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
General	λl	++	+−	++	+−	−+	−−	−+	−−
	ΛL	++	++	−+	−+	++	++	−+	−+
	$\lambda' l'$	+−	++	+−	++	−−	−+	−−	−+
	$\Lambda' L'$	+−	+−	−−	−−	+−	+−	−−	−−
Identical particles	λl	++	++	++	−+	++	++	−+	−+
	ΛL	++	+−	−+	+−	−+	−+	−−	−+
	$\lambda' l'$	+−	++	+−	++	+−	−+	−−	−+
	$\Lambda' L'$	+−	+−	−−	−−	−−	+−	−−	−−
$\frac{1}{8}\mathcal{L}(uv,UV;\xi\omega,\Xi\Omega)$			α						α
uv UV $\xi\omega$ $\Xi\Omega$									
R K_1 A A		+	+	+	+	+	+	+	+
I K_8 A A		+	−	+	−	+	−	+	−
R K_1 A ++		+	+			+	+		
I K_8 A ++		+	−			+	−		
R K_1 A −−				+	+			+	+
I K_8 A −−				+	−			+	−
R K_1 ++ A		+	+	+	+				
I K_8 ++ A		+	−	+	−				
R K_1 −− A						+	+	+	+
I K_8 −− A						+	−	+	−
R K_1 ++ ++		+	+						
I K_8 ++ ++		+	−						
R K_1 ++ −−				+	+				
I K_8 ++ −−				+	−				
R K_1 −− ++						+	+		
I K_8 −− ++						+	−		
R K_1 −− −−								+	+
I K_8 −− −−								+	−

The amplitudes of any arbitrary reaction can be determined completely (except for a set of discrete ambiguities) through the one-by-one bicomobservables matrices alone, and hence by exactly $2n-1$ *measurements where n is the number of amplitudes.*

Note also that a *partial* completion of our procedure provides *complete* information on a *partial* set of amplitudes, and not, as in general we would have expected, *partial* information about the *entire* set of amplitudes. In particular, the first $2k-1$ steps in the procedure determine k amplitudes completely.

There are, of course, other sets of measurements also which determine the amplitudes completely. For example, we may not want to mea-

TABLE III. (*Continued*.)

$\frac{1}{16}\mathcal{L}(uv, UV; \xi\omega, \Xi\Omega)$

1_1 $D(\lambda l; \Lambda L)D^*(\lambda' l'; \Lambda' L')$

uv	UV	$\xi\omega$	$\Xi\Omega$			K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
				General	λl	+ +	− +	+ +	− +	+ −	− −	+ −	− −
					ΛL	+ +	+ +	+ −	+ −	+ +	+ +	+ −	+ −
					$\lambda' l'$	+ +	− +	+ +	− +	+ −	− −	+ −	− −
					$\Lambda' L'$	− +	− +	− −	− −	− +	− +	− −	− −
				Identical	λl	+ +	+ +	+ +	− +	+ +	+ +	+ −	+ −
				particles	ΛL	+ +	− +	+ −	+ −	− +	− −	+ −	− −
					$\lambda' l'$	+ +	− +	+ +	− +	− +	− +	+ −	− −
					$\Lambda' L'$	− +	− +	− −	− −	+ −	− −	− −	− −
								α			α	α	
A	A	A	K_2				+	+	+	+	+	+	+	+	α
+ +	A	A	K_2				+	+	+	+					
− −	A	A	K_2								+	+	+	+	
A	+ +	A	K_2				+	+			+	+			
A	− −	A	K_2						+	+			+	+	
A	A	+ +	K_2				+		+		+		+		
A	A	− −	K_2					+		+		+		+	
+ +	+ +	A	K_2				+	+							
+ +	− −	A	K_2						+	+					
− −	+ +	A	K_2								+	+			
− −	− −	A	K_2										+	+	
+ +	A	+ +	K_2				+		+						
+ +	A	− −	K_2					+		+					
− −	A	+ +	K_2								+		+		
− −	A	− −	K_2									+			+
A	+ +	+ +	K_2				+				+				
A	+ +	− −	K_2					+				+			
A	− −	+ +	K_2						+				+		
A	− −	− −	K_2							+					+
+ +	+ +	+ +	K_2				+								
+ +	+ +	− −	K_2					+							
+ +	− −	+ +	K_2						+						
+ +	− −	− −	K_2							+					
− −	+ +	+ +	K_2								+				
− −	+ +	− −	K_2									+			
− −	− −	+ +	K_2										+		
− −	− −	− −	K_2												+

1_2 $D(\lambda l; \Lambda L)D^*(\lambda' l'; \Lambda' L')$

uv	UV	$\xi\omega$	$\Xi\Omega$			K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
				General	λl	+ +	+ +	+ +	+ +	+ −	+ −	+ −	+ −
					ΛL	+ +	− +	+ −	− −	+ +	− +	+ −	− −
					$\lambda' l'$	− +	− +	− +	− +	− −	− −	− −	− −
					$\Lambda' L'$	+ +	− +	+ −	− −	+ +	− +	+ −	− −
				Identical	λl	+ +	+ +	+ +	+ +	+ +	− +	+ −	+ −
				particles	ΛL	+ +	− +	+ −	− −	+ −	+ −	+ −	− −
					$\lambda' l'$	+ +	− +	− +	− +	+ +	− +	+ −	− −
					$\Lambda' L'$	− +	− +	+ −	− −	− −	− −	− −	− −
							α			α	α	α	
A	A	K_2	A				+	+	+	+	+	+	+	+	
+ +	A	K_2	A				+	+	+	+					
− −	A	K_2	A								+	+	+	+	
A	+ +	K_2	A				+	+			+	+			
A	− −	K_2	A						+	+			+	+	
A	A	K_2	+ +				+		+		+		+		
A	A	K_2	− −					+		+		+		+	
+ +	+ +	K_2	A				+	+							
+ +	− −	K_2	A						+	+					
− −	+ +	K_2	A								+	+			
− −	− −	K_2	A										+	+	
+ +	A	K_2	+ +				+		+						
+ +	A	K_2	− −					+		+					
− −	A	K_2	+ +								+		+		
− −	A	K_2	− −									+			+
A	+ +	K_2	+ +				+				+				
A	+ +	K_2	− −					+				+			
A	− −	K_2	+ +						+				+		
A	− −	K_2	− −							+					+
+ +	+ +	K_2	+ +				+								
+ +	+ +	K_2	− −					+							
+ +	− −	K_2	+ +						+						
+ +	− −	K_2	− −							+					
− −	+ +	K_2	+ +								+				
− −	+ +	K_2	− −									+			
− −	− −	K_2	+ +										+		
− −	− −	K_2	− −												+

$\frac{1}{16}\mathcal{L}(uv, UV; \xi\omega, \Xi\Omega)$

1_3 $D(\lambda l; \Lambda L)D^*(\lambda' l'; \Lambda' L')$

uv	UV	$\xi\omega$	$\Xi\Omega$			K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
				General	λl	+ +	+ +	− +	− +	+ −	+ −	− −	− −
					ΛL	+ +	− +	+ +	− +	+ +	− +	+ +	− +
					$\lambda' l'$	+ +	+ +	− +	− +	+ −	+ −	− −	− −
					$\Lambda' L'$	+ −	− −	+ −	− −	+ −	− −	+ −	− −
				Identical	λl	+ +	+ +	+ +	− +	+ +	− +	+ +	− +
				particles	ΛL	+ +	− +	− +	− +	+ −	+ −	− −	− −
					$\lambda' l'$	+ +	+ +	− +	− +	+ −	+ −	+ −	− −
					$\Lambda' L'$	+ −	− −	+ −	− −	+ −	− −	− −	− −
								α			α	α	
A	K_1	A	A				+	+	+	+	+	+	+	+	
+ +	K_1	A	A				+	+	+	+					
− −	K_1	A	A								+	+	+	+	
A	K_1	+ +	A				+	+			+	+			
A	K_1	− −							+	+			+	+	
A	K_1	A	+ +				+		+		+		+		
A	K_1	A	− −					+		+		+		+	
+ +	K_1	+ +	A				+	+							
+ +	K_1	− −	A						+	+					
− −	K_1	+ +	A								+	+			
− −	K_1	− −	A										+	+	
+ +	K_1	+ +	− −				+		+						
+ +	K_1	− −	+ +					+		+					
− −	K_1	+ +	+ +						+						
− −	K_1	− −	− −							+					
− −	K_1	+ +	− −								+				
− −	K_1	− −	+ +									+			
− −	K_1	+ +	+ +										+		
− −	K_1	− −	− −												+

1_4 $D(\lambda l; \Lambda L)D^*(\lambda' l'; \Lambda' L')$

uv	UV	$\xi\omega$	$\Xi\Omega$			K_1	K_1	K_1	K_1	K_1	K_1	K_1	K_1
				General	λl	+ +	+ +	− +	− +	+ +	+ +	− +	− +
					ΛL	+ +	− +	+ +	− +	+ −	− −	+ −	− −
					$\lambda' l'$	+ −	+ −	− −	− −	+ −	− −	− −	− −
					$\Lambda' L'$	+ +	− +	+ +	− +	+ −	− −	+ −	− −
				Identical	λl	+ +	+ +	+ +	− +	+ +	+ +	− +	− +
				particles	ΛL	+ +	− +	− +	− +	+ −	− −	+ −	− −
					$\lambda' l'$	+ +	− +	+ +	− +	+ −	+ −	+ −	− −
					$\Lambda' L'$	+ −	+ −	− −	− −	+ −	− −	− −	− −
							α		α	α			α
K_1	A	A	A				+	+	+	+	+	+	+	+	
K_1	+ +	A	A				+	+	+	+					
K_1	− −	A	A								+	+	+	+	
K_1	A	+ +	A				+	+			+	+			
K_1	A	− −	A						+	+			+	+	
K_1	A	A	+ +				+		+		+		+		
K_1	A	A	− −					+		+		+		+	
K_1	+ +	+ +	A				+	+							
K_1	+ +	− −	A						+	+					
K_1	− −	+ +	A								+	+			
K_1	− −	− −	A										+	+	
K_1	+ +	A	+ +				+		+						
K_1	+ +	A	− −					+		+					
K_1	− −	A	+ +								+		+		
K_1	− −	A	− −									+			+
K_1	+ +	+ +	+ +				+				+				
K_1	+ +	+ +	− −					+				+			
K_1	+ +	− −	+ +						+				+		
K_1	+ +	− −	− −							+					
K_1	− −	+ +	+ +								+				
K_1	− −	+ +	− −									+			
K_1	− −	− −	+ +										+		
K_1	− −	− −	− −												+

TABLE III. (*Continued.*)

				$^1_M D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$															
				λl	++	++	−+	−+	++	++	−+	−+	+−	+−	−−	−−	+−	+−	−−	−−
				ΛL	++	−+	++	−+	+−	−−	+−	−−	++	−+	++	−+	+−	−−	+−	−−
				$\lambda' l'$	++	++	−+	−+	++	++	−+	−+	+−	+−	−−	−−	+−	+−	−−	−−
				$\Lambda' L'$	++	−+	++	−+	+−	−−	+−	−−	++	−+	++	−+	+−	−−	+−	−−
$\frac{1}{16}\mathcal{L}(uv,UV;\xi\omega,\Xi\Omega)$				See footnote [a]															
uv	UV	$\xi\omega$	$\Xi\Omega$			*	*		†	≠	§	¶	†	§	≠	¶		‖	‖	
A	A	A	A		+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
++	A	A	A		+	+	+	+	+	+	+	+								
−−	A	A	A										+	+	+	+	+	+	+	+
A	++	A	A		+	+	+	+					+	+	+	+				
A	−−	A	A						+	+	+	+					+	+	+	+
A	A	++	A		+	+			+	+			+	+			+	+		
A	A	−−	A				+	+			+	+			+	+			+	+
A	A	A	++		+		+		+		+		+		+		+		+	
A	A	A	−−			+		+		+		+		+		+		+		+
++	++	A	A		+	+	+	+												
++	−−	A	A						+	+	+	+								
−−	++	A	A										+	+	+	+				
−−	−−	A	A														+	+	+	+
++	A	++	A		+	+			+	+										
++	A	−−	A				+	+			+	+								
−−	A	++	A										+	+			+	+		
−−	A	−−	A												+	+			+	+
++	A	A	++		+		+		+		+									
++	A	A	−−			+		+		+		+								
−−	A	A	++										+		+		+		+	
−−	A	A	−−											+		+		+		+
A	++	++	A		+	+							+	+						
A	++	−−	A				+	+							+	+				
A	−−	++	A						+	+							+	+		
A	−−	−−	A								+	+							+	+
A	++	A	++		+		+						+		+					
A	++	A	−−			+		+						+		+				
A	−−	A	++						+		+						+		+	
A	−−	A	−−							+		+						+		+
A	A	++	++		+				+				+				+			
A	A	++	−−			+				+				+				+		
A	A	−−	++				+				+				+				+	
A	A	−−	−−					+				+				+				+

sure all of the $\mathcal{L}(uu,UU;\xi\xi,\Xi\Xi)$'s (i.e., all the absolute values), but may instead want to add to the set of measurements some more of the observables connecting amplitudes on neighboring levels in Fig. 1. It can be shown easily, however, that with no absolute values at all these latter types of observables by themselves cannot determine the amplitudes completely. This is so because any array of such amplitudes would form a diagram in which there are no loops and in which one can return to the same vertex only in an even number of steps. Such diagrams, according to the rules given in Ref. 5, correspond to sets of observables that cannot completely determine the amplitudes.

Observables appearing in 2×2, 4×4, and 8×8 matrices can, of course, be utilized toward the determination of amplitudes, but in that case it is not clear that one can accomplish the task with no more than the minimum number (namely, $2n-1$) of observables.

IV. POLARIZATION STRUCTURE WITH AVERAGED OBSERVABLES

In present-day experiments it is much simpler to carry out measurements in which one or several of the four particles are unpolarized. Unfortunately, this simplicity can be attained only at the cost of increasing the complexity of structure in the relationship between the experimental observables and the bicoms.

It is easy to see how this increasing complexity comes about. Consider, for example, $\mathcal{L}(uu,UV;\xi\xi,A)$, where the A in the fourth index denotes "averaged." This averaging process means $\sum\mathcal{L}(uu,UV;\xi\xi,\Xi\Xi)$. Since $\mathcal{L}(uu,UV;\xi\xi,\Xi\Xi)$ is, for each value of Ξ, in a different 1×1 matrix, $\sum\mathcal{L}(uu,UV;\xi\xi,\Xi\Xi)$ will be a linear combination of the $2s_4+1$ bicoms $D(\xi u,\Xi U)D^*(\xi u,\Xi V)$. In our case of four spin-$\frac{1}{2}$ particles, $2s_4+1=2$.

Thus, in the reaction we are considering now, each averaging doubles the number of bicoms on

TABLE III. (*Continued.*)

					1_M (*Continued*) $D(\lambda l;\Lambda L)D^*(\lambda' l';\Lambda' L')$															
					λl	++	++	−+	−+	++	++	−+	−+	+−	+−	−−	−−	+−	+−	−−	−−
					ΛL	++	−+	++	−+	+−	−−	+−	−−	++	−+	++	−+	+−	−−	+−	−−
					$\lambda' l'$	++	++	−+	−+	++	++	−+	−+	+−	+−	−−	−−	+−	+−	−−	−−
					$\Lambda' L'$	++	−+	++	−+	+−	−−	+−	−−	++	−+	++	−+	+−	−−	+−	−−
$\frac{1}{16}\mathcal{L}(uv,UV;\xi\omega,\Xi\Omega)$					See footnote [a]															
uv	UV	$\xi\omega$	$\Xi\Omega$			*	*		†	≠	§	¶	†	§	≠	¶		‖	‖	
++	++	++	A		+	+														
++	++	−−	A				+	+												
++	−−	++	A						+	+										
++	−−	−−	A								+	+								
−−	++	++	A										+	+						
−−	++	−−	A												+	+				
−−	−−	++	A														+	+		
−−	−−	−−	A																+	+
++	++	A	++		+		+													
++	++	A	−−			+		+												
++	−−	A	++						+		+									
++	−−	A	−−							+		+								
−−	++	A	++										+		+					
−−	++	A	−−											+		+				
−−	−−	A	++														+		+	
−−	−−	A	−−															+		+
++	A	++	++		+				+											
++	A	++	−−			+				+										
++	A	−−	++				+				+									
++	A	−−	−−					+				+								
−−	A	++	++										+				+			
−−	A	++	−−											+				+		
−−	A	−−	++												+				+	
−−	A	−−	−−													+				+
A	++	++	++		+								+							
A	++	++	−−			+								+						
A	++	−−	++				+								+					
A	++	−−	−−					+								+				
A	−−	++	++						+								+			
A	−−	++	−−							+								+		
A	−−	−−	++								+								+	
A	−−	−−	−−									+								+
++	++	++	++		+															
++	++	++	−−			+														
++	++	−−	++				+													
++	++	−−	−−					+												
++	−−	++	++						+											
++	−−	++	−−							+										
++	−−	−−	++								+									
++	−−	−−	−−									+								
−−	++	++	++										+							
−−	++	++	−−											+						
−−	++	−−	++												+					
−−	++	−−	−−													+				
−−	−−	++	++														+			
−−	−−	++	−−															+		
−−	−−	−−	++																+	
−−	−−	−−	−−																	+

[a] For identical particles the columns marked with the same sign are equal.

which an experimental observable depends. The actual number of bicoms will depend on what size matrix the unaveraged observable belongs to, and on how many averagings have been performed. The situation is summarized in Table IV. It should be remembered that Table IV holds for the case when no symmetry other than Lorentz invariance is assumed. Additional symmetries reduce these numbers.

A number of remarks can be made on the basis of Table IV. First, we see that the completely unpolarized differential cross section is the clumsiest observable imaginable, even in the optimal formalism, since it contains the magnitude-squares of *all* amplitudes. Second, comparing Table IV with the results of the description of this reaction in the more conventional formalism,[6] we see that in the latter *all* observables (regardless of how many averagings they contain) are in 16×16 or 8×8 matrices, while in the optimal formalism only 1 observable depends on 16 bicoms, 88 observables depend on 8 each, 232 on 4 each, 224 on 2 each, and 80 only on 1 each. Incidentally, the reason why the total number of observables listed in the optimal formalism exceeds the $4^4=256$ is that listing the averaged as

TABLE IV. The number of bicoms an observable depends on in the reaction given by Eq. (1.1), as a function of the size of the matrix the unaveraged observables belong to, and of the number of averagings in the observable, is given by the numbers not in parentheses. The numbers in parentheses give the total number of observables in that category.

The number of averagings in the observable	Size of the unaveraged matrix	1×1 with magnitude squares	1×1 other	2×2	4×4	8×8
4		16 (1)	8 (8)			
3		8 (8)	4 (48)			
2		4 (24)	2 (96)	8 (24)		
1		2 (32)	1 (64)	4 (96)	8 (32)	
0		1 (16)		2 (96)	4 (64)	8 (16)

well as the unaveraged observables makes the set highly linearly dependent. In fact, the averaging includes also a fifth value of an index (namely A), and hence we should have (and in fact do have) $5^4 = 625$ observables, of which only 256 are linearly independent.

We now turn to the question of which sets of observables are sufficient to fix all amplitudes unambiguously (except for a *discrete* set of ambiguities). We see that the set of observables in which four or three particles are averaged over is insufficient since the total number of such observables is less than 31. This result also follows from the Simonius theorem.[7]

Whether the set of observables in which 4, 3, or 2 observables are averaged over can determine the amplitudes is not obvious. The Simonius theorem allows this possibility but does not demand it. Looking at the explicit tables of the relationship between observables and bicoms does not help, since a single observable does not uniquely determine a single bicom, and hence the results of Ref. 5 cannot be used directly. On the other hand, this does not necessarily mean that the observables fail to determine the amplitudes themselves, but a general theory of how to tell whether they do or not still does not exist, except for the explicit computation of the Jacobian in each case, which is exceedingly tedious even for our relatively simple reaction. The formulation of an easy, practical, complete, and yet general criterion for when a set of observables uniquely determines the amplitudes remains the most important unsolved problem in polarization physics.

The problem of course vanishes if each observable we use depends on only one single bicom, because in that case the criteria of Ref. 5 can be directly applied. This allowed us to state the theorem in Sec. III. On the other hand, in such observables the polarization of all four particles must be specified, which makes the experiments more difficult.

In testing dynamical models, however, we need not necessarily determine *all* of the amplitudes. For a partial check of such models, it may be sufficient to determine only a few amplitudes or even just a few bicoms. We saw in the previous section that the optimal formalism is very much more accommodating in this respect than the conventional formalisms, and this remains to be the case also for averaged observables. This is not only so because in the optimal formalism the structure itself of the relationships between observables and bicoms is maximally simple, but also because we have the additional freedom of choosing the quantization direction of polarization at will, and hence maximally conveniently for the particular dynamical model in question.

V. POLARIZATION STRUCTURE WITH IDENTICAL PARTICLES

The general consequences of having identical particles in a reaction have been outlined in Ref. 8. It is evident from there that while some simplification might result also in the case when only the initial or only the final particles are identical, the much more interesting case is the one when both the initial and the final particles are identical. It is, however, not necessary that the initial particles be the same as the final particles. In this section I will demonstrate the simplifications that arise in (1.1) when the two initial particles are the same and the two spin-$\frac{1}{2}$ particles in the final state are the same.

Our notation is ideally suited to accommodate this situation. Our amplitudes are indexed in such a way that one of the particles with spin has lower-case indices, while the other one capital indices. If the two particles are identical (both in the initial and final states), the indistinguishability principle requires that two amplitudes, which differ only by the exchange of lower-case and capital indices, be equal (with a possible minus sign). Thus, for (1.1), we have

$$(++,-+)=-(-+,++),\quad (++,--)=(--,++),\quad (-+,--)=-(--,-+),$$
$$(++,+-)=-(+-,++),\quad (-+,+-)=(+-,-+),\quad (+-,--)=-(--,+-). \tag{5.1}$$

In our figure showing the amplitudes (Fig. 1), this constraint simply means that the figure must be symmetric around a vertical axis passing through $(++,++)$, $(-+,-+)$, $(+-,+-)$, and $(--,--)$.

The modification of the results of the previous sections in the case of identical particles is therefore trivially easy: All diagrams are to be reflected around this vertical axis. Some examples are shown in Fig. 3. The number of different amplitudes decreases from 16 to 10, and correspondingly the observable-amplitude structure also simplifies. In some cases two submatrices simply become identical, while in other cases they decrease in size or partially overlap. Since the modifications are obvious, they are simply exhibited in Table III. It should be noted again how much simpler the situation is in the optimal formalism than in the more conventional ones, for example in Sec. VII of Ref. 8.

VI. AN EXAMPLE

The content of the previous sections may appear to the uninitiated as very abstract and of little use in practical situations. To counteract this impression, in this section I will consider a specific example on which the use of the results obtained so far can be demonstrated.

In particular, I will assume that some dynamical model predicted the amplitudes for the process (1.1), in which the identical particle constraints hold. Thus, at some specific set of values of the kinematic variables (energies and angles), we are given ten complex numbers, as shown in Fig. 4.

The first step, which I will *not* perform here explicitly, would be the use of the results of Ref. 4 to choose a set of quantization directions for the particles so that the ten numbers are in a most convenient form. For example, we might want to make one or several of the amplitudes zero (see explanation in Ref. 4). In general, we want to make some amplitudes large and some of them very small compared to the former, since in that case an approximate determination of the large amplitudes is easier and may be followed by an iterative approximation scheme to determine the other amplitudes also. Details on this point will be given in a forthcoming paper. Similarly, we may want to choose a set of quantization directions such that the relative phases of the amplitudes also become convenient.

Since this first step depends only on the content of Ref. 4, it will not be carried out here, but I will assume that it has already been done, and that the numbers given in Fig. 4 are in fact the results of this first step. Thus, the quantization directions have already been fixed, and so the various polarization observables have to be interpreted and the corresponding experiments carried out with those quantization directions in mind.

The next step in the determination of the amplitudes (or in the testing of the theoretical predictions) depends on what type of experiments we are able to perform. If *all* conceivable experiments are at our disposal, the task is easy: We are then able to use exclusively those observables which are related to bicoms by one-by-one matrices and hence each measurement directly determines one bicom. In giving values for the experiments, I will use units determined by the overall constant that connects each bicom with the actual experimental value. It depends on flux factors, etc., and is a standard and routine factor. In those units, therefore, we have to check whether $\mathfrak{L}(++,++;++,++)$ is $(2.1)^2=4.42$, whether $\mathfrak{L}(++,--;++,++)$ is 0, whether $\mathfrak{L}(++,--;--,--)$ is $(1.2)^2=1.44$, etc., and then whether $\mathfrak{L}(++,+-R;++,++)$ is $2.1\times 0.5\times\cos 30^\circ=0.91$, etc. As mentioned, in this case the procedure is a very simple and straightforward one, and each measurement provides a definite piece of information. In fact, if we wish, we may choose to determine only a subset of the ten amplitudes, and we can do this with a minimum number of measurements without mixing in any particle information about the other amplitudes. It should be emphasized again that such simplicity occurs only in the optimal formalism. In most conventional formalisms the determination of amplitudes is very much more involved even if we allow the use of *all* possible experiments, including those in which all four particles are polarized.

In practice, however, such unconstrained sets of experiments seldom are at our disposal. So let us assume now that we are constrained to experiments in which no more than two particles are polarized in each observable. Such a constraint is eminently realistic in terms of present-day experimental techniques.

On the amplitude figures I indicated, by arrows and indices on the margins, the diagonal rows which "belong" to a certain index. Observables in which two polarizations are averaged over will contain *all* the amplitudes which are at the inter-

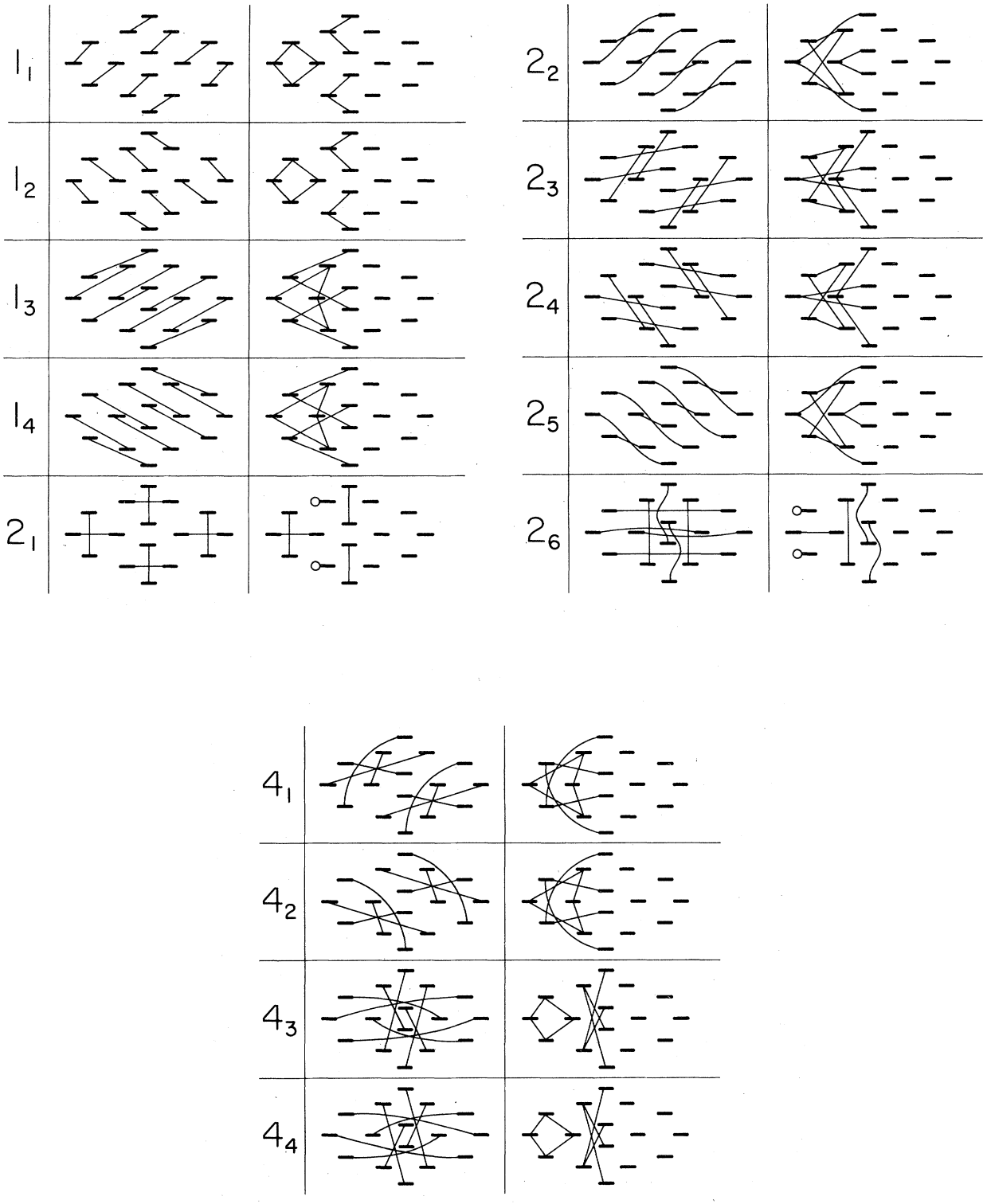


FIG. 3. Amplitude structures for the various submatrices, indicating (by connecting lines) the bicoms that occur. These relations are given for both with (on the right) and without (on the left) the constraints of identical particles. The figures for the one-by-one submatrices pertaining to magnitude-squares are trivial and are not shown.

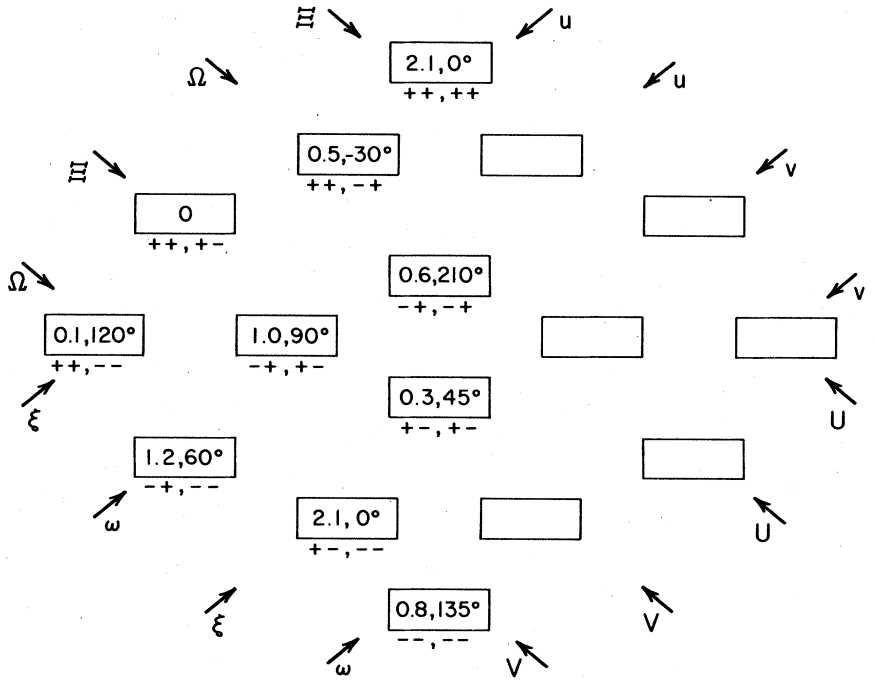


FIG. 4. Amplitude structure for the example given in Sec. VI.

sections of the rows corresponding to the other two (nonaveraged) indices. For example, if U and Ξ are averaged over, we get the row with $(++,++)$, $(++,-+)$, $(++,+-)$, and $(++,--)$. If, instead, U and ξ are averaged over, we get the $(++,-+)$, $(-+,-+)$, $(-+,--)$, and $(++,--)$, which form the corners of a rectangle.

With the imposition of the constraints of identical particles, the above prescription still holds, but after picking out the four amplitudes corresponding to the two averagings, we must also reflect the figure around its vertical symmetry axis, as indicated in Sec. V.

So far we considered only *one* of the amplitudes in each bicom when we talked about averaging. It turns out that doing that is sufficient. The amplitudes are coupled into bicoms in such a way that when we average over indices in one of the two amplitudes in the bicom, we automatically average also over the indices in the other.

Armed with this information, let us now look at Fig. 4. Suppose we want to get information on the magnitude of $(++,++)$. With our constraint of only two particles polarized, *any* measurement related to the magnitude-squares will yield the sum of at least *four* magnitude-squares. We have, however, a choice about *which* three unwanted magnitudes we want to lump into our measurements, together with the magnitude-square of $(++,++)$. Our choice will be guided by the desire of making the (expected values of the) unwanted magnitudes as small as possible so that our measurement yields, to a good approximation, the magnitude-square of the wanted amplitude only. The best procedure, therefore, in our example appears to be to average over u and U, thus dealing with $(++,++)$, $(+-,+-)$, and (twice) $(++,+-)$. Since the latter is supposed to be zero, and the square of $(+-,+-)$ is supposed to be only 0.09, measuring $\mathfrak{L}(A,A;++,++)$ will give (if the predictions are correct) 4.50 instead of 4.41 which is the square of the actual magnitude of $(++,++)$.

Similarly, suppose we want to measure the *phase* between $(++,++)$ and $(+-,--)$, and we already have evidence from previous experiments that the two magnitudes are 2.1 each. The distance of the two amplitudes (see Fig. 1) is three steps, and so the suitable observables will be found in the 4×4 observable-bicom matrices. In this case, therefore, there is not much choice. In fact, it is evident from our tables that there is *no* observable in which two or more particles are unpolarized and which involves the phase between $(++,++)$ and $(+-,--)$. A moment's reflection convinces us that this is true for the phase between *any* two amplitudes which are three steps apart.

We obtain, therefore, the general rule that *when measuring the relative phases of amplitudes, it is advisable and in fact often necessary to do this "step by step," measuring always the phase difference between two amplitudes the distance between which is zero or one step.*

So let us, for example, determine the phase between $(-+,+-)$ and $(-+,--)$, assuming that we have already determined the two magnitudes. The bicom involving these two amplitudes occurs in a one-by-one matrix (since the two amplitudes are one step apart). If we then consider these observables which have two particles unpolarized, we see that we still have three choices, with three different sets of three bicoms lumped in with the one we are interested in:

(a) $(++,++)(++,-+)$, $(-+,++)(-+,-+)$, $(++,+-)(++,--)$;

(b) $(++,+-)(++,--)$, $(+-,+-)(+-,--)$, $(--,+-)(--,--)$;

(c) $(-+,++)(-+,+-)$, $(--,++)(--,-+)$, $(--,+-)(--,--)$.

A brief inspection of the expected values for these amplitudes shows that none of these measurements separates the phase between $(-+,+-)$ and $(-+,--)$ very well, but that the sum (a)+(b) accomplishes the task very well, since, of the eight bicoms appearing in that sum, two are identical and are the bicom we are directly interested in, three others vanish, and the three remaining ones together cancel almost completely.

The above example illustrates the situation when a theoretical prediction is available for the amplitudes and we want to verify this prediction. If no such prediction is available, the same procedure can also be used to determine the amplitudes phenomenologically. The best procedure is to start with observables which measure combinations of magnitude-squares, and thereby fix the larger amplitudes first, followed by their phases and then by the smaller amplitudes.

VII. CONCLUSIONS

The aim of this paper has been to demonstrate, on a practical, often used example, the very substantial advantages of shaping the study of particle reactions through polarization measurements in terms of a so-called optimal type of formalism. In particular, the following points were made:

(1) In the optimal-type formalism, the relationship between the experimental observables and the (bilinear combination of) amplitudes is the simplest possible, and in fact very much simpler than in the customarily used formalism. In particular, the matrix connecting the observables with the bicoms is as diagonal as possible, i.e., it contains as few nonzero coefficients as possible, and furthermore all these coefficients are $+1$ or -1. This simple matrix gives a large amount of additional power in determining amplitudes phenomenologically, in checking dynamical models, and in planning new experiments.

(2) Since the optimal-type formalism is a whole infinite class of formalism, because the choice of the directions of the quantization axes is still entirely at our disposal, this formalism offers an additional amount of power and flexibility in that these quantization directions can be chosen specifically to interface smoothly with the use to which the results of the polarization experiments will be put. In particular, the class of optimal formalisms includes manifestly covariant as well as explicitly three-dimensional formalisms, "helicity"- or "transversity"-type descriptions, etc. The most interesting and practical instances of optimal-type formalisms may, however, be ones which do not coincide with those used in the past.

(3) In addition to the observable-bicom matrix being maximally diagonalized, its structure is also very transparent and easy to handle. It is no longer necessary, as it was in past formalisms, to carry out the computation of the whole matrix to ascertain which observables are related to which bicoms and how. Here there are very simple rules for constructing the observable-bicom structure. In particular, there are easy prescriptions for

(a) which observable is related to which bicom,

(b) how large the submatrix is which connects a given observable with bicoms, and

(c) in terms of the number of particles in the reaction that need to be polarized, how complex the experiment has to be in order to yield information on the magnitude or relative phase of a given amplitude. In particular, one can define the concept of "distance" between amplitudes and then show that in order to minimize the complexity of experiments to be performed, one should measure the phases between "neighboring" amplitudes and thus, step by step, map out the whole amplitude structure.

These rules constitute a *systematic* procedure to obtain any specific type and amount of full or partial information about the reaction amplitudes of the process under investigation.

(4) The constraints on the polarization structure when identical particles are present in the reaction can be imposed very easily, and these constraints do not complicate the rules mentioned under point (3).

To take maximal advantage of these results, cooperation must occur between theorists and experimentalists so that the theoretical results are expressed in terms of a set of amplitudes that are convenient also for the designing of the experiments and vice versa.

The present paper contains the detailed discussion of this optimal structure only for the specific reaction (1.1), with occasional pointers to the way the results would generalize to an arbitrary reaction. The complete outline of the method in the case of an arbitrary reaction will be given in a forthcoming paper.

ACKNOWLEDGMENTS

I am indebted to Lori Gomes for the imaginative and artistic drawing of Fig. 2. This work was supported in part by the U. S. Department of Energy under Contract No. EY-76-S-06-2230, TA 4, Mod. A-08.

APPENDIX A: "REALISTIC" OBSERVABLES

When, in the past, theorists have suggested new formalisms to be used for polarization experiments, the question of "realistic" observables *versus* "theoretical" observables was invariably raised by those who actually perform the experiments. The point is that whereas the observables suggested by theorists generally satisfy the obvious requirements (they are real, correspond to eigenvalues of measurable operators, etc.), they still may not be identical with what the experimentalist measures directly, using a partially polarized beam and target, and setting up his equipment subject to both traditions in principle and physical limitations (bulky magnets, beam lines, etc.) in practice. Thus, for any formalism, in order to obtain the actual quantities used in reporting data, the experimentalist has to take various linear combinations of the readings obtained directly from his experiments.

Yet the situation can be improved, even if probably not to an unlimited extent. First, on the part of the experimentalist, the setting up of experiments involves more freedom than is usually used, and it should be the aim to utilize this additional freedom to smooth the interface between theory and experiment. Second, on the part of the theoretical formalism, the experimental observables should be chosen so as to be as close as possible to what is actually measured in experiments. This question is discussed in some detail in Ref. 4, where it is pointed out that with the "optimal" formalism it should be possible to eliminate altogether the "intermediate" phenomenological polarization quantities and make a direct connection between the theoretical observables and the experimental, "realistic" quantities. This point will be further discussed in a forthcoming paper which will apply optimal formalisms to the extensive set of polarization data on 6-GeV proton-proton scattering soon to be entirely released.

APPENDIX B: POSITIVITY CONSTRAINTS

In all formalisms, there are certain constraints which arise from the physical interpretation of certain quantities as experimental observables. For example, in the customary form of the density-matrix formalism, the constraint manifests itself as the "positivity condition," that is, the diagonal elements of the density matrix (which represent probabilities of finding a system in pure polarization states) must be positive. In the pure-optimal formalism these constraints are expressed as positivity constraints on various sums of certain observables. We will demonstrate this with an example. For the reaction $0+\frac{1}{2}\to 0+\frac{1}{2}$, Table I of Ref. 4 gives the relationship between observables and bicoms in terms of the "modified" optimal observables, namely observables labeled by two specific orientation axes. Reference 4 showed that using two such axes is always sufficient to determine the amplitudes completely. It can be seen from Table I of Ref. 4 that all the observables listed are positive, since they all are absolute-value squares of some linear combination of the four amplitudes. In this case, therefore, we have a simple analog for the positivity constraint in the density-matrix formalism. In the *pure* optimal formalism the table for the same reaction is given in Table I of Ref. 1. Although in that case a single observable need not be positive, certain linear combinations of the observables (corresponding to the modified optimal observables in Table I of Ref. 4) must be positive. Thus we have, for example,

$$L(Q_2, Q_4) = \tfrac{1}{4}\mathcal{T}(11;11) + \tfrac{1}{4}\mathcal{T}(11;22) + \tfrac{1}{2}\mathcal{T}(11;12I) \geq 0\,, \tag{B1}$$

or

$$L(Q_4, Q_4) = \tfrac{1}{4}[\mathcal{T}(11;11) + \mathcal{T}(22;11) + \mathcal{T}(11;22) + \mathcal{T}(22;22)] + \mathcal{T}(12I;12I) + \tfrac{1}{2}[\mathcal{T}(12I,11) + \mathcal{T}(11,12I) + \mathcal{T}(22;12I) + \mathcal{T}(12I,22)] \geq 0\,. \tag{B2}$$

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