Quantized scalar field in rotating coordinates

John R. Letaw
Center for Particle Theory, The University of Texas at Austin, Austin, Texas 78712

Jonathan D. Pfautsch
Center for Theoretical Physics, The University of Texas at Austin, Austin, Texas 78712
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Second quantization of the free scalar field is carried out in rotating coordinates and the spectrum of vacuum fluctuations is calculated for an orbiting observer using these coordinates. Normal-mode decomposition is identical to that in Minkowski coordinates except for the definition of positive-frequency modes. Unlike the uniformly accelerating observer, the orbiting observer predicts that the Minkowski vacuum will contain no particles as he would define them. The spectrum of vacuum fluctuations is composed of the usual zero-point energy plus a contribution arising from the observer's acceleration. The latter is not, as with uniformly accelerated motion, thermal. The peak energy appears to be dependent only on the torsion of the observer's world line.

I. INTRODUCTION

For several years it has been known that the quantization of scalar fields in flat space is not unique.[1] In particular, what one observer identifies as particles in his reference frame may be considered vacuum by an observer in another frame. An example is found in the comparison of a uniformly accelerated observer with an inertial observer. Straightforward generalization of the Klein-Gordon equation to the Rindler coordinate system[2] of an observer undergoing constant linear acceleration κ leads to what appears to be a unique quantization procedure. However, the accelerated observer encounters a thermal spectrum[3] of massless scalar particles with temperature $kT = \kappa/2\pi$ in what the inertial observer describes as vacuum.

Important questions concerning the validity of canonical quantum field theory in general coordinate systems and the meaning of "particles" arise from these results. Apart from those considerations, though, is the problem of the origin or cause of the particles. They will be found by static observers using either Rindler coordinates in flat space or exterior Schwarzschild coordinates in the static curved space surrounding a black hole, both of which have event horizons. This suggests that either incomplete covering of the manifold or some physical process[4] associated with the horizon may explain the presence of particles. Furthermore, energy must be supplied to keep these observers static in their coordinate systems. Unruh[5] maintains that this work will be perceived as the source of energy necessary to create particles in the case of uniformly accelerated motion.

In order to explore these questions we study the quantized scalar field in rotating coordinates as an observer orbiting about a point would view it. Two characteristics of this system make it of interest: (i) There is no event horizon and (ii) no work is required to maintain the observer in this state of motion. One also finds that the exact mode functions can be found in four dimensions and that these solutions are well defined throughout space.

Second quantization of the field is performed as in Minkowski coordinates. The results are identical with ordinary Minkowski quantization except for one surprise. Because an orbiting observer will define positive-frequency modes in a different manner than an inertial observer, the algebra defining creation and annihilation operators requires that these operators be defined in an unfamiliar fashion. The number operator constructed from them, however, is identical to the usual Minkowski number operator. Thus, the Minkowski vacuum contains no particles as the oribiting observer would define them.

Finally, the spectrum of vacuum fluctuations of the massless scalar field for an orbiting observer is calculated. It consists of a term equal to the spectrum of an inertial observer plus an additional finite portion. The latter is nonthermal. The maximum energy seems to depend only on the torsion τ of the observer's world line. This calculation is independent of the results of second quantization in the orbiting frame.

II. ROTATING COORDINATES AND THE ORBITING OBSERVER

In cylindrical Minkowski coordinates (t', r', ϕ', z') the Killing vector

$$\xi = \frac{\partial}{\partial t'} + \Omega \frac{\partial}{\partial \phi'} \tag{1}$$

is tangent to the world line of an observer traveling in a circle with constant angular velocity Ω. A transformation to rotating coordinates

$$t = t', \quad r = r', \quad \phi = \phi' - \Omega t', \quad z = z' \tag{2}$$

provides a system adapted to this observer in the sense that

$$\xi = \frac{\partial}{\partial t} . \tag{3}$$

In this coordinate system the line element is

$$ds^2 = -(1 - \Omega^2 r^2) dt^2 + 2\Omega r^2 d\phi\, dt + r^2 d\phi^2 + dr^2 + dz^2 . \tag{4}$$

This metric is stationary but not static, i.e., ξ is not orthogonal to a family of spacelike hypersurfaces. The adaptation of the Killing vector to an observer orbiting with radius R is completed by replacing ξ with

$$\zeta = (1 - \Omega^2 R^2)^{-\frac{1}{2}} \frac{\partial}{\partial t} , \tag{5}$$

which is normalized along that observer's world line. ζ is timelike for $\Omega R < 1$ and therefore for any real observer because ΩR is his velocity in the Minkowski frame.

The world line of this observer is contained within a hypersurface of constant z. Any nonnull curve in a flat three-dimensional space may be characterized by two parameters: curvature and torsion.[6] These can be defined in terms of an orthonormal triad composed of the tangent vector, the normal, and the binormal. In rectangular Minkowski coordinates the world line, parametrized by arc length s, is

$$x^\mu = (\gamma s, R \cos\Omega\gamma s, R \sin\Omega\gamma s, z_0), \tag{6}$$

with $\gamma = (1 - v^2)^{-1/2}$ and $v = \Omega R$. The tangent , normal, and binormal are, respectively,

$$T^\mu = \gamma(1, -v \sin\Omega\gamma s, v \cos\Omega\gamma s, 0), \tag{7}$$

$$N^\mu = (0, -\cos\Omega\gamma s, -\sin\Omega\gamma s, 0) , \tag{8}$$

$$B^\mu = \gamma(-v, \sin\Omega\gamma s, -\cos\Omega\gamma s, 0) . \tag{9}$$

The curvature and torsion may then be defined by

$$\kappa = N_\mu \frac{dT^\mu}{ds} = \Omega\gamma^2 v , \tag{10}$$

$$\tau = -N_\mu \frac{dB^\mu}{ds} = \Omega\gamma^2 . \tag{11}$$

Physically, the curvature is identified with the observer's acceleration and the torsion with his angular velocity.[7]

III. QUANTIZATION OF THE SCALAR FIELD

In rotating coordinates the Klein-Gordon equation is

$$\left[-\left(\frac{\partial}{\partial t} - \Omega \frac{\partial}{\partial \phi} \right)^2 + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right] \psi = M^2 \psi . \tag{12}$$

Mode functions satisfying this equation are

$$\psi = \frac{1}{2\pi |2\omega|^{1/2}} e^{ikz} e^{im\phi} e^{-i(\omega - m\Omega)t} J_m(qr) , \tag{13}$$

with the restriction

$$\omega^2 = k^2 + q^2 + M^2 . \tag{14}$$

Here m is an integer, $|\omega| \geq M$, and $q \geq 0$. These mode functions are identical in appearance to the ordinary Minkowski mode functions transformed to rotating coordinates. The distinction between the two arises when defining energy for these modes (i.e., defining positive-frequency modes). In general, energy may be defined by

$$E\left(i \int_\Sigma \psi'^* \overleftrightarrow{f}^\mu \psi\, d\Sigma_\mu \right) = -\tfrac{1}{2} \int_\Sigma (\pounds_K \psi'^*) \overleftrightarrow{f}^\mu \psi\, d\Sigma_\mu , \tag{15}$$

where $\pounds_K$ is the Lie derivative with respect to some timelike Killing vector K (which must be present), $\overleftrightarrow{f}^\mu$ is the Wronskian operator

$$\overleftrightarrow{f}^\mu = g^{1/2} g^{\mu\nu} \frac{\overrightarrow{\partial}}{\partial x^\nu} - \frac{\overleftarrow{\partial}}{\partial x^\nu} g^{1/2} g^{\mu\nu} , \tag{16}$$

Σ is some spacelike hypersurface, and the coefficient of E is a normalization factor. Both an orbiting observer and an observer at rest in the Minkowski frame will naturally prefer a t = constant surface for Σ. For K, the inertial observer will employ the Killing vector $\partial/\partial t'$, which is tangent to his world line and orthogonal to Σ, while the orbiting observer will employ ζ of Eq. (5). Evaluation of Eq. (15) leads the inertial observer to positive-energy modes for $\omega > 0$, as usual, while the orbiting observer finds that positive-energy modes are defined by $\omega - m\Omega > 0$. Thus, in the rotating frame $\omega < 0$ modes are perfectly permissible positive-frequency modes. For such modes the normalization factor in Eq. (15) is unavoidably negative (actually, the negative of a collection of δ functions). Furthermore, since k and q only specify $|\omega|$ in Eq. (14), we must also specify the *sign* of ω in addition to k, q, and m in order to uniquely specify a mode. We will therefore specify a positive-frequency mode explicitly, when necessary, by subscripts $(k,\ q,\ m;\ \omega)$. (The same notation will be used for Minkowski modes, even though the label ω is superfluous.) Figure 1 describes the domains of definition for each observer's positive-frequency modes.

The field is quantized in the customary fashion by defining the field operator Φ and its conjugate momentum Π, which is the projection of the vector density

$$\Pi^\mu = g^{1/2} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \Phi \tag{17}$$

onto the normal to Σ, and invoking the commutation relations

$$[\Phi(x,t), \Pi(x',t)] = i\delta^3(x-x'), \tag{18a}$$

$$[\Phi(x,t), \Phi(x',t)] = 0, \tag{18b}$$

$$[\Pi(x,t), \Pi(x',t)] = 0. \tag{18c}$$

If the field operator is expanded in terms of rotating positive-frequency modes,

$$\Phi = \sum_{\substack{m=-\infty \\ \omega - m\Omega > 0}}^{\infty} \int_{k,q} \sum_{\omega} (a_{kqm;\omega}\psi_{kqm;\omega} + a^\dagger_{kqm;\omega}\psi^*_{kqm;\omega}) q\, dq\, dk \tag{19}$$

($\sum_\omega$ means the sum over all values of ω allowed by the pair $k,\ q$), then the commutation relations for the operators a and $a^\dagger$ which follow from Eq. (18) are

$$[a_{kqm;\omega}, a^\dagger_{k'q'm';\omega'}] = \frac{\omega}{|\omega|} \delta_{\omega\omega'} \delta_{mm'} \delta(k-k) \frac{1}{q} \delta(q-q'). \tag{20}$$

This mandates that the rotating vacuum state be defined by

$$a|0\rangle_R = 0, \quad \omega > 0 \tag{21a}$$

$$a^\dagger|0\rangle_R = 0, \quad \omega < 0. \tag{21b}$$

In light of this result, we regard a as a creation operator for modes with $\omega < 0$, while for $\omega > 0$ it is an annihilation operator as usual. The operators a and $a^\dagger$ may be expressed in terms of the field by

$$a_{kqm;\omega} = \frac{\omega}{|\omega|} i \int_\Sigma \psi^*_{kqm;\omega} \overleftrightarrow{f}^\mu \Phi\, d\Sigma_\mu, \tag{22a}$$

$$a^\dagger_{kqm;\omega} = -\frac{\omega}{|\omega|} i \int_\Sigma \psi_{kqm;\omega} \overleftrightarrow{f}^\mu \Phi\, d\Sigma_\mu. \tag{22b}$$

Quantization of the field in Minkowski coordinates proceeds uneventfully with $\tilde{\psi}_{kqm;\omega}$ defined for $\omega > 0$ to yield

$$\Phi = \sum_{\substack{m=-\infty \\ \omega > 0}}^{\infty} \int_{k,q} (\tilde{a}_{kqm;\omega}\tilde{\psi}_{kqm;\omega} + \tilde{a}_{kqm;\omega}\tilde{\psi}^*_{kqm;\omega}) q\, dq\, dk, \tag{23}$$

$$\tilde{a} = i \int_\Sigma \tilde{\psi} \overleftrightarrow{f}^\mu \Phi\, d\Sigma_\mu, \tag{24a}$$

$$\tilde{a}^\dagger = -i \int_\Sigma \tilde{\psi}^* \overleftrightarrow{f}^\mu \Phi\, d\Sigma_\mu, \tag{24b}$$

with the Minkowski vacuum defined for all $\omega > 0$ by

$$\tilde{a}|0\rangle_M = 0. \tag{25}$$

If Φ is expanded in Eqs. (24) in terms of the positive-frequency modes of the orbiting observer [Eq. (19)], the result is

$$\tilde{a}_{kqm;\omega} = \begin{cases} a_{kqm;\omega}, & \omega > m\Omega \quad (26a) \\ (-1)^m a^\dagger_{-kq-m;-\omega}, & \omega < m\Omega \quad (26b) \end{cases}$$

$$\tilde{a}^\dagger_{kqm;\omega} = \begin{cases} a^\dagger_{kqm;\omega}, & \omega > m\Omega \quad (26c) \\ (-1)^m a_{-kq-m;-\omega}, & \omega < m\Omega. \quad (26d) \end{cases}$$

These may be readily inverted to yield

$$a_{kqm;\omega} = \begin{cases} \tilde{a}_{kqm;\omega}, & \omega > 0 \quad (27a) \\ (-1)^m \tilde{a}^\dagger_{-kq-m;-\omega}, & \omega < 0 \quad (27b) \end{cases}$$

$$a^\dagger_{kqm;\omega} = \begin{cases} \tilde{a}^\dagger_{kqm;\omega}, & \omega > 0 \quad (27c) \\ (-1)^m \tilde{a}_{-kq-m;-\omega}, & \omega < 0. \quad (27d) \end{cases}$$

Thus, creation and annihilation of particles as defined by each observer are identical in region I of Fig. 1. However, creation in the rotating frame of a particle with momentum (k,q,m) and $\omega < 0$ (region II of Fig. 1) corresponds to creation of a particle in the inertial observer's frame of a particle in region IV with momentum $(-k,q,-m)$. We

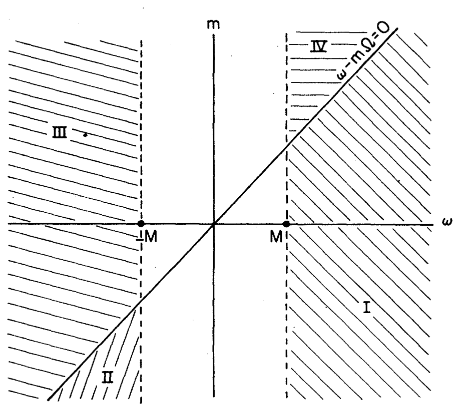

FIG. 1. ω–m plane. Regions I and II are the domain for positive-frequency mode functions defined by an orbiting observer. Regions I and IV are the domain for the usual Minkowski positive-frequency modes.

might say that the effect of the observer's rotation is to replace the modes in region IV with the time-reversed solutions of region II.

Quantized scalar fields in rotating and Minkowski coordinates are very similar; however, there is an important difference. Equation (12), and hence the field, is not invariant under time reversal. The violation is easily seen in the mode expansion [Eq. (19)]. Under time reversal, the explicitly time-dependent part of the mode function

$$e^{\pm i(\omega - m\Omega)t}$$

is not transformed into its complex conjugate because the sign of m, in addition to t, changes. The time-reversed field differs from the original by this sign. Physically it corresponds to the field of an observer with the opposite sense of rotation.

The number operator as defined by the orbiting observer is

$$N = \sum_{m=-\infty}^{\infty} \int_{\substack{k,q \\ \omega - m\Omega > 0}} \sum_{\omega} [\theta(\omega)a^\dagger a + \theta(-\omega)aa^\dagger] q \, dq \, dk \, . \quad (28)$$

If Eqs. (27) are used to reexpress this operator in terms of the Minkowski creation and annihilation operators, one finds with a little relabeling that

$$N = \sum_{m=-\infty}^{\infty} \int_{\substack{k,q \\ \omega < 0}} \tilde{a}^\dagger a \, q \, dq \, dk \, , \quad (29)$$

which is the same as the inertial observer's number operator, so

$$N|0\rangle_M = 0 \quad (30)$$

and there are *no* particles present as the orbiting observer would define them in the Minkowski vacuum. This differs fundamentally from the corresponding result for uniformly accelerated motion, where a continuous spectrum of particles as defined in the accelerating frame is present in the Minkowski vacuum. Also, because creation (annihilation) operators in the rotating frame are identified with creation (annihilation) operators in the Minkowski frame, and since the two sets of mode functions are identical in appearance and in one-to-one correspondence, it follows that the vacuum expectation value of the stress tensor $\langle T_{\mu\nu}\rangle$ *must* be the same in each frame.

Finally, we emphasize that in this procedure it has not been possible to satisfy the following three criteria, which are normally desired: (i) define positive- and negative-frequency modes relative to the Killing vector ξ; (ii) employ a canonical quantization procedure; and (iii) regard a and $a^\dagger$ as strictly annihilation and creation operators, respectively (i.e., satisfy the commutation relation $[a, a^\dagger]=1$).

IV. SPECTRUM OF VACUUM FLUCTUATIONS

If the metric is static and flat, the spectrum of vacuum fluctuations is the Fourier transform of the autocorrelation function[8] multiplied by the appropriate density of states,

$$S_{\text{static}}(\omega) = \frac{\rho(\omega)}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega s} A(x(s) - x(0), M) ds \, , \quad (31)$$

where s is the proper-time interval for the observer between space-time points x and x_0. For a massless scalar field the autocorrelation function is related to the geodetic interval between the points

$$A(x - x_0, 0) = |x - x_0|^{-2} \, . \quad (32)$$

The spectrum has a minimum value at all frequencies for an inertial observer

$$S_{\text{inertial}}(\omega) = \omega^3/2\pi \, . \quad (33)$$

When the metric is stationary the energy, and hence the generator of time translations, is no longer proportional to ω. Equation (31) is generalized to stationary metrics by expressing the spectrum as a function of energy:

$$S_{\text{stationary}}(E) = \frac{E^2}{4\pi^3} \int_{-\infty}^{\infty} e^{-iEs} |x(s) - x(0)|^{-2} ds \, . \quad (34)$$

The spectrum of fluctuations for an orbiting observer is

$$S_{\text{orbiting}}(E) = S_{\text{inertial}}(E) + \frac{E^2}{4\pi^3} \times \int_{-\infty}^{\infty} e^{-iEs} \left[\frac{1}{s^2} - \frac{1}{(\gamma s)^2 - 4R^2 \sin^2(\Omega\gamma s/2)}\right] . \quad (35)$$

In order to eliminate the singularity in the integrand, the spectrum of the inertial observer has been factored out. The remainder yields an additional contribution due to the acceleration of the observer.

Two parameters such as angular velocity Ω and radius R are needed to specify the motion of an orbiting observer. There will therefore be a two-parameter set of spectra. This is reduced to one by rescaling the energy variable. We have numerically calculated two sets of spectra. The first is parametrized by velocity at constant acceleration, the second by velocity at constant torsion.

Case I: *constant acceleration*. The first set of spectra (Fig. 2) is plotted with constant acceleration and variable velocity ($0.05c \leq v \leq 0.95c$). Fix-

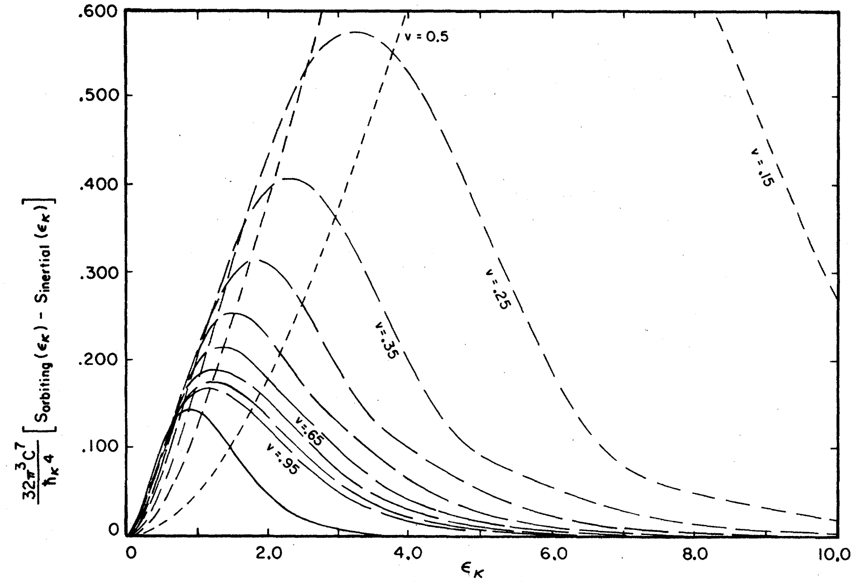

FIG. 2. Vacuum fluctuation spectra at constant acceleration. The solid line represents a uniformly accelerated observer. The dashed lines represent orbiting observers with velocities ranging from $0.05c$ to $0.95c$ in steps of $0.1c$.

ing the acceleration allows comparison with the thermal spectrum of fluctuations predicted for a uniformly accelerated motion. The acceleration of a rotating observer is

$$\kappa = \Omega v \gamma^2 . \tag{36}$$

We define a dimensionless energy by

$$\epsilon_\kappa = 2E/\kappa . \tag{37}$$

Using Eqs. (36) and (37) the spectrum at constant acceleration may be expressed as

$$S_{\text{orbiting}}(\epsilon_\kappa) = S_{\text{inertial}}(\epsilon_\kappa) + \frac{\kappa^4 v \epsilon_\kappa^{\;2}}{32\pi^3\gamma} \times \int_0^\infty \cos(v\gamma\epsilon_\kappa x)\left[\frac{x^2 - \sin^2 x}{x^2(x^2 - v^2\sin^2 x)}\right] dx , \tag{38}$$

with velocity v being the only free parameter. Figure 2 is plotted disregarding the inertial contribution and overall constants. In comparison, the spectrum found by a uniformly accelerated observer is

$$S_{\text{unif. accel.}}(\epsilon_\kappa) = S_{\text{inertial}}(\epsilon_\kappa) + \frac{\kappa^4 \epsilon_\kappa^{\;3}}{32\pi^2(e^{\pi\epsilon_\kappa} - 1)} . \tag{39}$$

Several characteristics of these spectra are of interest. At constant acceleration the figure shows that higher-energy fluctuations are attributed to the lower-velocity observer. This occurs because the angular velocity is inversely proportional to the velocity. As v approaches one, the spectrum does not become thermal. This might be expected because the torsion in the observer's world line is always nonvanishing; in other words, there is no limit in which a rotating observer's world line approaches that of a uniformly accelerated observer. Finally, the spectra always have a region with lower energy than the thermal spectrum.

Case II: *constant torsion.* The spectra of Fig. 3 are at fixed torsion with varying velocity. The torsion of the orbiting observer is

$$\tau = \Omega\gamma^2 . \tag{40}$$

A dimensionless energy is defined in this case by

$$\epsilon_\tau = 2E/\tau . \tag{41}$$

The spectrum at constant torsion may be written

$$S_{\text{orbiting}}(\epsilon_\tau) = S_{\text{inertial}}(\epsilon_\tau) + \frac{\tau^4 v^2 \epsilon_\tau^{\;2}}{32\pi^3\gamma}\int_0^\infty \cos(\gamma\epsilon_\tau x) \times \left[\frac{x^2 - \sin^2 x}{x^2(x^2 - v^2\sin^2 x)}\right] dx . \tag{42}$$

An important observation which we cannot explain is that the maxima of these spectra appear to be determined solely by the torsion of the observer's world line. Thus

$$E_{\max} \cong 0.4\tau . \tag{43}$$

The total energy density in the spectrum is pro-

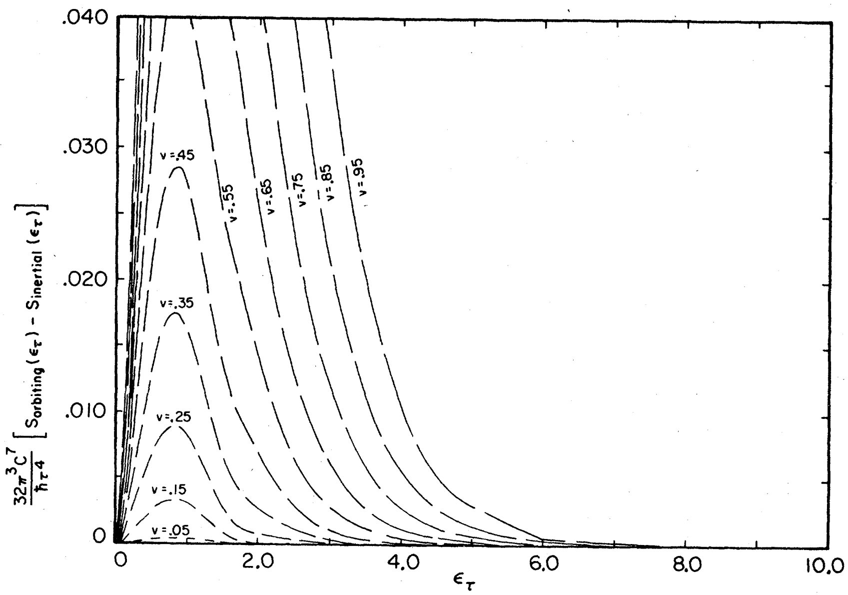

FIG. 3. Vacuum fluctuation spectra at constant torsion.

portional to τ^4 for any fixed velocity. Note that if the velocity is fixed, the acceleration is fixed and we expect the result to depend only on the other curvature invariant, the torsion.

V. CONCLUSIONS

We have calculated the occupation number of the vacuum in Sec. III and the spectrum of vacuum fluctuations in Sec. IV by independent methods. Our results differ significantly from the same calculations done for a linearly accelerating observer using Rindler coordinates. In that case, the occupation number and vacuum fluctuations show identical thermal spectra in the vacuum state. One is therefore led to identify the two, that is, a detector linearly accelerating through the vacuum will receive quanta of energy from the field which are assumed to be just those quanta which the number operator predicts. This interpretation is false as our results on the orbiting observer illustrate. The orbiting detector will receive quanta from the field yet the number operator predicts *no* quanta in the vacuum of rotating coordinates. We must at least conclude that the occupation number and the spectrum of vacuum fluctuations refer to different aspects of the field and cannot be identified.

The origin of scalar particles in the vacuums of non-Minkowski coordinate systems in flat space may be explained in a way which is consistent with these results. A spectrum of vacuum fluctuations different from that of the inertial observer will be found by any detector not moving along a geodesic.[8] This may be calculated by standard methods.[9] The energy measured by the detector will be derived from this motion[5] and will therefore impede this motion. We must conclude that the motion is impeded since no energy would be required to maintain the orbiting detector's motion otherwise, and the energy causing the detector's response would be unaccounted for. Real particles—defined as those mode excitations counted by the number operator—occur only if there is an event horizon. This is strongly suggested by the available results of quantum field theory in static and stationary systems. They exist because virtual pairs can be separated at an event horizon[4] while the number operator counts particles only in a restricted portion of the manifold. Whether these "real" particles are perceived as particles is unknown.

ACKNOWLEDGMENT

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